Online Self-Assessment for M.Sc. Mathematics Program

1 Basic mathematical concepts and rigorous proofs

References: [2, 4, 10]

1. Prove that, for all integers $n \geq 1$, the term $n^3 - n$ is divisible by 3.
2. Let L_n be defined by $L_1 = 1$, $L_2 = 3$ and

$$L_n = L_{n-1} + L_{n-2}, \quad \text{for} \quad n \geq 3.$$

   Use induction to show that

$$L_n \geq \frac{1}{2}\left(\frac{3}{2}\right)^n, \quad \text{for all} \quad n \geq 1.$$

3. Prove that the set of prime numbers is not a finite set.
4. Prove that the set of rational numbers $\mathbb{Q}$ is countably infinite. Then prove that the set $\mathbb{Q}^d$ of d-tuples of rationals is countably infinite, for any $d \in \mathbb{N}$.

2 Analysis

References: [8, 11, 13, 14, 15]

1. Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, give the definition of its infimum, supremum, limit inferior, limit superior, and limit. Give an example of a sequence that has no limit.
2. Given a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers, give the definition of the quantity $\sum_{i\in\mathbb{N}} a_i$. Describe sufficient conditions on the sequence such that the quantity $\sum_{i\in\mathbb{N}} a_i$ exists and is finite.
3. Give the definition for a function $f : \mathbb{R} \to \mathbb{R}$ to have the following properties at a point $x \in \mathbb{R}$: (i) left-continuity; (ii) right-continuity; (iii) continuity. Give the definition of a uniformly continuous function.
4. Give the definition of the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ and give the geometric interpretation of the derivative. Generalise this to the case where $f : \mathbb{R}^d \to \mathbb{R}$ for $d \in \mathbb{N}$, $d \geq 2$.

5. Define the Riemann integral of a function $f : \mathbb{R} \to \mathbb{R}$ in terms of a limit.

6. State the following theorems (including their hypotheses): (i) the intermediate value theorem; (ii) the mean value theorem; (iii) the fundamental theorem of calculus.

7. Let $d \in \mathbb{N}$, $d > 1$, and $f : \mathbb{R}^d \to \mathbb{R}$. Let $k \in \{1, \ldots, d\}$ and define the k-th partial derivative $D_k f$ of f. Define the gradient of a scalar-valued, continuously differentiable function on $\mathbb{R}^d$, and the divergence of a continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}^d$.

8. State a theorem about the existence and uniqueness of solutions to ordinary differential equations. Describe the main points of the proof of this theorem.

9. Consider the following differential equation in $\mathbb{R}^2$:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} 2 & -8 \\ -8 & -17 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

   - State whether there exists a solution to the differential equation, and whether this solution is unique.
   - Describe the long-time behaviour of solutions and their dependence on the initial condition $(x(0), y(0))^\top \in \mathbb{R}^2$. Justify your answers.
   - Draw the phase portrait of the differential equation.

10. Consider the flow associated to the following vector field on $\mathbb{R}^2$:

$$F(x, y) = \begin{pmatrix} x(y-1) \\ -y^2 \end{pmatrix}.$$

    Determine whether the flow has any critical points. If there are critical points, identify all of them.

11. State the Cauchy integral formula for holomorphic functions. Under what assumptions is this formula valid?

12. Use one or more theorems to justify why holomorphic functions are infinitely differentiable.

3 Linear Algebra

References: [5, 6, 12]

1. Let V, W be vector spaces and $\Phi : V \to W$ a linear map.
   (a) What does it mean to say that a subset $U \subseteq V$ is a subspace?
   (b) Define the kernel $\mathrm{Ker}(\Phi)$ and show that it is a subspace of V.
   (c) Prove that $\Phi : V \to W$ is injective if and only if $\mathrm{Ker}(\Phi) = \{0\}$

2. Let $\Phi : V \to W$ be a linear map between finite dimensional vector spaces.

(a) Define the rank of Φ and the nullity Φ.

(b) Find the rank of Φ_A for the linear map $\Phi_A : \mathbb{R}^4 \to \mathbb{R}^3 : x \to Ax$ given by the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 2 & 2 \\ 3 & 2 & 6 & 8 \end{bmatrix}.$$

(c) How many vectors must there be in a basis for the null space of A? Briefly justify your answer.

3. Let V be a finite dimensional vector space over a field $\mathbb{F}$.

   (a) Define the span$\{v_1, \ldots, v_n\}$ of a list of vectors $v_i \in V$.

   (b) What does it mean to say that the list $v_1, \ldots, v_n$ is (i) linearly independent (ii) a basis of V.

4. Let Φ be a linear operator on a finite-dimensional complex inner product space V with adjoint Φ^*, and let U be a subspace of V.

   (a) Define what it means to say that U is a Φ-invariant subspace, and define the orthogonal space $U^\perp$.

   (b) Show that if U is Φ-invariant, then $U^\perp$ is Φ^*-invariant.

   (c) State the Spectral Theorem for normal operators, using orthogonal projections.

5. In this question A, B and C are sets, $f : A \to B$ and $g : B \to C$ are maps and $h = g \circ f$ is the composed map so $h : A \to C$

   (a) Suppose that h is surjective. Does it follow that f is surjective? Justify your answer.

   (b) Suppose that h is injective. Does it follow that f is surjective? Justify your answer.

4 Measure theory

References: [7]

1. Define a sigma-algebra $\mathcal{F}$ on a nonempty set Ω. What does it mean for a sigma-algebra to be countably generated? What does it mean for $\mathcal{G}$ to be a sub-sigma algebra of $\mathcal{F}$?

2. Let $d \in \mathbb{N}$ be arbitrary. Define the Borel sigma-algebra on $\mathbb{R}^d$.

3. Define what a measurable space is and what a measure on a measurable space is.

4. Let $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$ be two measurable spaces. What does it mean for a function $f : S_1 \to S_2$ to be $(\mathcal{S}_1, \mathcal{S}_2)$-measurable?

5. State the invariance properties of d-dimensional Lebesgue measure on $\mathbb{R}^d$.

6. Define the Lebesgue integral on $\mathbb{R}$. For a subset S of $\mathbb{R}$, let $\mathbb{I}_S$ denote the indicator function of S, and let $\mathbb{Q}$ denote the set of rational numbers. Calculate the value of the Lebesgue integral $\int_{\mathbb{R}} \mathbb{I}_{\mathbb{Q}}(x)\ \mathrm{Leb}(\mathrm{d}x)$, where 'Leb' denotes Lebesgue measure.

7. State the monotone convergence theorem, the dominated convergence theorem, and Fatou's lemma.

8. Let μ, ν be two measures on a measurable space $(S, \mathcal{S})$. Define what it means for μ to be absolutely continuous with respect to ν and what it means for μ and ν to be mutually equivalent. Give an example of two measures μ and ν such that μ is absolutely continuous with respect to ν but not mutually equivalent to ν.

5 Geometry / topology

References: [9]

1. Give the definition of a topology on a nonempty set and define all the terms that you use in the definition.

2. What does it mean for a topology to be Hausdorff?

3. Define what is a homeomorphism of topological spaces.

4. Give the definition of a connected set, a path connected set, and a simply connected set. Give an example of a set that is connected but not path connected.

5. Give the definition of a d-dimensional real manifold. Give an example of a 2-dimensional real manifold.

6. What is the tangent space to a manifold M at a point $p \in M$? What is the tangent space to the example of the 2-manifold you gave for the previous question?

7. A non-empty intersection of two planes in $\mathbb{R}^3$ is a line. Prove that a non-empty inter-section of two spheres in $\mathbb{R}^3$ is a circle (considering a point as a circle of radius 0).

8. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a regular unit-speed smooth curve such that all tangent lines to γ intersect in a point. Show that γ is a straight line.

9. Let S be a compact connected surface which is not homeomorphic to a sphere. Show that there are points on S where the Gaussian curvature is positive, negative and zero.

10. Let α be a space curve with an arbitrary regular parametrization. Show that its curvature is given by
$$\kappa(t) = \frac{||\dot{\alpha} \times \ddot{\alpha}||}{||\dot{\alpha}||^3}.$$

11. Define the *catenoid* as the surface S_f parametrized by

$$f(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$

and the *helicoid* as the surface S_g parametrized by

$$g(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

(a) Show that, for any point $(u, v) \in \mathbb{R}^2$, the Gaussian curvature of S_f at $f(u, v)$ coincides with the Gaussian curvature of S_g at $g(u, v)$.

(b) Show that the mean curvature of both S_f and S_g is identically zero.

6 Probability theory

References: [1, 3]

1. Define a probability space and define every object involved in the definition.

2. State what sigma-additivity or countable additivity of a probability measure means.

3. Let E be a nonempty set. Give the definition of an E-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, define the cumulative distribution function (c.d.f.) of X and state all the properties that imply a function $F : \mathbb{R} \to \mathbb{R}$ is the cumulative distribution function of a $\mathbb{R}$-valued random variable.

4. Give the definition of mutual independence of a collection of measurable events, and of a collection of random variables. Let X and Y be two $\mathbb{R}$-valued random variables on a probability space. Define such that the covariance of X and Y is zero. Does this imply that X and Y are independent? If yes, then give a proof. If not, then give a counterexample.

5. If two random variables are equal in distribution, does this imply they are equal almost surely? If yes, then give a proof. If not, then give a counterexample.

6. Let (X, Y) be a $\mathbb{R}^2$-valued random variable with joint p.d.f. $f_{X,Y}$. Define the marginal p.d.f. f_X and the conditional p.d.f. $f_{X|Y}$.

7. Let X be a $\mathbb{R}^d$-valued random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $p \geq 1$. Define what it means for X to belong to $L^p(\mathbb{P})$.

8. Give the definition of a probability density function (p.d.f.) of a $\mathbb{R}^d$-valued random variable. Write down the formula for the p.d.f. of a $\mathbb{R}^d$-valued normal random variable with mean $m \in \mathbb{R}^d$ and strictly positive definite covariance matrix $C \in \mathbb{R}^{d\times d}$. Make sure to specify all the necessary assumptions on m and C for the equation to be valid.

References

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