Noise sensitivity of Lévy driven SDE's: estimates and applications

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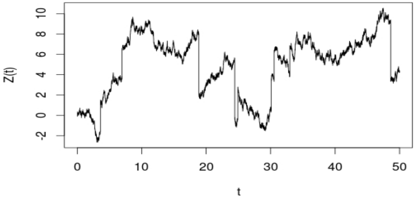
Motivation: Where is it useful?

The model selection problem for Lévy driven SDEs with additive noise

- Modelling phenomena that exhibit jump behaviour: in climatology, reliability theory, finance etc. ...
- Assumption: The observed data comes from the dynamics, which follows the SDE with Lévy noise, i.e. the process of the following type

$$X(t) = x + \int_0^t V(X(s))\, ds + Z(t) \quad t \geq 0,\ x \in \mathbb{R}, \tag{1}$$

with a "good" function V and a Lévy process $(Z(t))_{t \geq 0}$.



A realisation of Lévy process[1]

[1] L. Schreiter

Lévy process: definition and characterisation

Definition A standard Lévy process $(Z(t)), t > 0$ is a real-valued stochastic cádlág process with $Z(0) = 0$ and **independent and stationary increments**.

Lévy-Khinchin characterisation of the law
Every Lévy process $(Z(t))_{t\geq 0}$ is uniquely determined by a characteristic triplet (a, b^2, Π)

- drift $a \in \mathbb{R}$, covariance $b^2 \in \mathbb{R}^+$
- Lévy measure Π, a σ-finite measure such that $\int_{\mathbb{R}\setminus\{0\}}(|u|^2 \wedge 1)\Pi(du) < \infty$

by means of its *cumulant function* ψ

$$\mathbb{E}e^{iuZ(t)} = e^{t\psi(u)}, \ u \in \mathbb{R}$$

linked to the characteristic triplet via the Lévy-Khinchin formula

$$\psi(z) = iaz - \frac{1}{2}b^2z^2 + \int_{\mathbb{R}\setminus\{0\}} \left[e^{izu} - 1 - izu\mathbf{1}_{|u|\leq 1}\right] \Pi(du), \quad z \in \mathbb{R}. \tag{2}$$

Example (symmetric α-stable measure): $\Pi_{\alpha,c}(du) = \frac{\alpha c}{|u|^{\alpha+1}}du, \quad \alpha \in (0,2), \quad c > 0.$
Cumulant function is $\psi(z) = |\alpha cz|^{\alpha}$.

The model selection problem for Lévy driven SDEs with additive noise

- Modelling phenomena that exhibit jump behaviour: in climatology, reliability theory, finance etc. ...
- Assumption: The observed data comes from the dynamics, which follows the SDE with Lévy noise, i.e. the process of the following type

$$X(t) = x + \int_0^t V\left(X(s)\right) ds + Z(t) \quad t \geq 0,\ x \in \mathbb{R}, \tag{3}$$

with a "good" function V and a Lévy process $(Z(t))_{t\geq 0}$, which is uniquely determined by a characteristic triplet (a, b^2, Π) by means of its *cumulant function* ψ linked to the characteristic triplet via the Lévy-Khinchin formula.

- How to quantify the distance between the model and the data?
- What is the distance between two processes of this type?

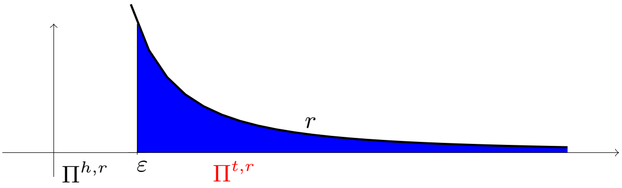
$$\text{“}d(X_1, X_2) \leq C(d(a_1, a_2) + d(b_1, b_2) + d(\Pi_1, \Pi_2))\text{''}$$

Coupling distance

Construction

For any Lévy measure Π and given $r > 0$ there exists $\varepsilon = \varepsilon(r)$ such that

$$\Pi = \Pi^{h,r} + \Pi^{t,r}, \quad \Pi^{t,r}(\mathbb{R}) = r$$



Define the **probability measure**

$$\pi^r(du) = \frac{1}{r}\Pi^{t,r}(du).$$

Coupling distance

- Recall that on a metric space (S, d) the *Wasserstein-Kantorovich-Rubinstein metric* of order 2, between two probability measures μ, ν on (S, d) is defined by

$$W_{2,d}(\mu, \nu) := \inf_{(\xi,\eta)\in\mathcal{C}(\mu,\nu)} \left(\mathbf{E}\, d^2(\xi,\ \eta)\right)^{1/2},$$

where $\mathcal{C}(\mu, \nu)$ denotes the set of all (μ, ν)-couplings.

- Let the metric ρ on $\mathbb{R}$ be defined by

$$\rho(x, y) = |x - y| \wedge 1.$$

- Define

$$T_r(\Pi_1, \Pi_2) := r^{1/2} W_{2,\rho}(\pi_1^r, \pi_2^r), r > 0,$$
$$T(\Pi_1, \Pi_2) := \sup_{r>0} T_r(\Pi_1, \Pi_2).$$

We shall call T_r and T *coupling (semi)distances* on the set of Lévy measures.

- **Proposition** The function (T_r) T is a (semi)metric on the set of Lévy measures. (Gairing, Högele, K., Kulik'15)

Example: α-stable Lévy measure

Let us consider the symmetric α-*stable measure* on $\mathbb{R}$

$$\Pi_{\alpha,c}(du) = \frac{\alpha c}{|u|^{\alpha+1}}du, \quad \alpha \in (0,2), \quad c > 0.$$

Proposition Let Π_{α,c_1}, Π_{α,c_2} have the same parameter α and different $c_1 \neq c_2$. Then there exists a constant $C > 0$ such that

$$\mathbf{T}(\Pi_{\alpha,c_1}, \Pi_{\alpha,c_2}) \leq C \left| c_1^{1/\alpha} - c_2^{1/\alpha} \right|^{\alpha/2}.$$

If $\Pi_{\alpha_1,c}$, $\Pi_{\alpha_2,c}$ have the same scale parameter c, but different shape parameters $\alpha_1 \neq \alpha_2$ and $0 < \alpha_1 < \alpha_2 < 2$, there exists a constant C such that

$$\mathbf{T}(\Pi_{\alpha_1,c}, \Pi_{\alpha_2,c}) \leq C \left(\alpha_2 - \alpha_1\right)^{\alpha_2/2}.$$

Proof of the first statement through quantile functions see blackboard.

Sensitivity bounds for the solutions of the Lévy driven SDEs with additive noise

Theorem (Gairing, Högele, K., Kulik'15)
Let (a_j, b_j^2, Π_j) be two Lévy characteristics and $x_j \in \mathbb{R}$ given initial values, $j = 1, 2$. And the function $V \in \mathcal{C}^2$ and satisfies for some constant $L > 0$ the condition $(V(x) - V(y))(x - y) \leqslant L(x - y)^2, \quad x, y \in \mathbb{R}$. Then for any two solutions X_j of equation (3) and any $r > 0$ there exists a constant $C > 0$ such that the following estimate holds true on $\mathbb{D}(0,1)$ with a metric $\zeta(x, y) := \sup_{t\in[0,1]} \rho(x(t), y(t))$

$$W_{2,\zeta}^2\Big(\mathrm{Law}(X_1), \mathrm{Law}(X_2)\Big) \leqslant C\Big(\rho^2(x_1, x_2) + |a_1 - a_2|^2 + (b_1 - b_2)^2 + U_r(\Pi_1) + U_r(\Pi_2) + T_r^2(\Pi_1, \Pi_2)\Big), \tag{4}$$

and

$$U_r(\Pi_j) = \int_{|u|\leqslant \varepsilon_j^r} u^2 \Pi_j(du), \qquad j = 1, 2.$$

Key ideas of the proof on the blackboard

Lévy-Itô representation of the path for Lévy process

For any Lévy process $(Z(t))_{t\geqslant 0}$ with the characteristic triplet (a, b^2, Π) and a fixed $\varepsilon > 0$

$$Z(t) = at + bW(t) + \underbrace{\int_0^t \int_{|u|\leq\varepsilon} u\,[\nu(du,dt) - \Pi(du)dt]}_{Z^{head}} + \underbrace{\int_0^t \int_{|u|>\varepsilon} u\,\nu(du,dt)}_{Z^{tail}} \quad \text{a.s. for all } t > 0, \tag{5}$$

where

- W is a standard Brownian motion in $\mathbb{R}$
- ν is a Poisson point measure on $\mathbb{R} \times \mathbb{R}^+$ with an intensity measure $\Pi(du) \times dt$,
- Z^{tail} is a Compound Poisson Process in $\mathbb{R}$ with the jump intensity $r(\varepsilon) = \Pi(|u| > \varepsilon)$,
- Z^{head} is a pure jump process of possibly infinite intensity with jumps bounded by one.

$$
\begin{aligned}
dY(t) = d(X_1(t) - X_2(t)) = \\
& (V(X_1)(t) - V(X_2)(t))dt \\
& + (\bar{a}_1 - \bar{a}_2)dt + (b_1 - b_2)dW(t) \\
& + d(Z_1^{head,r} - Z_2^{head,r}) \\
& + d(Z_1^{tail,r} - Z_2^{tail,r})
\end{aligned}
$$

$$G(Y(t)) = G(Y(0)) + \int_0^t h(X_1(s), X_2(s))\, ds + M_t, \quad \text{where}$$

$$
\begin{aligned}
h(z_1, z_2) = \\
& (V(z_1) - V(z_2))G'(z_1 - z_2) + (a_1 - a_2)G'(z_1 - z_2) + \frac{1}{2}(b_1 - b_2)^2 G''(z_1 - z_2) \\
& + \int_{\mathbb{R}^2} \Big[G(z_1 - z_2) + (u_1 - u_2)) - G(z_1 - z_2) - G'(z_1 - z_2)(u_1 - u_2)\Big]\hat{\Pi}(du) \\
& + \int_{\mathbb{R}^2} \Big[G((z_1 - z_2) + (u_1 - u_2)) - G(z_1 - z_2) - G'(z_1 - z_2)\Big(\tau(u_1) - \tau(u_2)\Big)\Big]\Pi^{T,r}(du)
\end{aligned}
\qquad (6)
$$

Modelling of the phenomena with state dependent characteristics

- Energy balance models in climatology
- The dynamics of the particle in a heterogeneous media
- Local volatility models in finance etc. ...
  $\Rightarrow$ jump characteristics of the model show state dependent behaviour.
  It leads to a process with a state dependent characteristic triplet
  $(a(x), b^2(x), \Pi(x, \cdot))$.

Lévy-type process

Such process is called a **Lévy-type process**. It is defined as a solution to the martingale problem associated to the following integro-differential operator A acting on $\varphi \in \mathcal{C}_c^2(\mathbb{R})$

$$A[\varphi](x) = a(x)\varphi'(x) + \frac{1}{2}b^2(x)\varphi''(x) + \int_{\mathbb{R}\setminus\{0\}} \left(\varphi(x+u) - \varphi(x) - \varphi'(x)u\mathbf{1}_{\{|u|\leqslant 1\}}\right)\Pi(x,du), \tag{7}$$

where $a, b : \mathbb{R} \to \mathbb{R}$ and $x \mapsto \Pi(x,\cdot)$ is a Lévy kernel, which associates to each $x \in \mathbb{R}$ the Lévy measure $\Pi(x,\cdot)$.

- Under what conditions on the coefficients of the triplet the process will be uniquely characterised?
- Can we recover the results about noise sensitivity as in the additive case?

p-distance

- **Definition**
  An *admissible transport plan* between two Lévy measures Π_1 and Π_2 is any positive Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that $\Gamma(\{0\} \times \{0\}) = 0$ and for any $A \in \mathbb{R}^d \setminus \{0\}$

$$\Gamma(A \times \mathbb{R}^d) = \Pi_1(A), \quad \Gamma(\mathbb{R}^d \times A) = \Pi_2(A).$$

  The set of all admissible transport plans will be denoted by $\mathrm{Adm}(\Pi_1, \Pi_2)$.

- **Definition** (N. Guillen, C. Mou, A. Świech'18)
  Let $1 \leq p \leq 2$. The p-distance between measures $\Pi_1, \Pi_2 \in \mathcal{M}_p(\mathbb{R}^d)$ with finite p-th moment is defined by

$$\mathrm{d}_{\mathrm{L}_p}(\Pi_1, \Pi_2) := \left(\inf_{\Gamma \in \mathrm{Adm}(\Pi_1, \Pi_2)} \int_{\mathbb{R}^d \setminus \{0\}} |u - v|^p \Gamma(du, dv) \right)^{1/p}.$$

- **Theorem** There exists at least one admissible plan that achieves the minimum value of the above integral.
  Remark Lipschitz condition for a Lévy kernel in terms of $\mathrm{d}_{\mathrm{L}_p}(\Pi(x, \cdot), \Pi(y, \cdot))$ is highly inexplicit.

Transportation distance

Transportation distance

- The main idea of the construction is to present any Lévy measure Π as a transformation of the fixed Lévy measure Π_0 by means of the function c.
- **Proposition** Let $\Pi_0(du) = du/u^2$ be a reference measure and Π be an arbitrary Lévy measure. Then there exists a unique non-decreasing "**transport**" function $c : \mathbb{R} \to \mathbb{R}$, such that

$$\Pi = \Pi_0 \circ c^{-1}.$$

- **Definition** (Kulik, K.'14)
  For two Lévy measures $\Pi_1,\ \Pi_2$ with transportation functions c_1 and c_2 on $\mathbb{R}$ we will call the quantity

$$\mathcal{T}(\Pi_1, \Pi_2) = \sqrt{\int_{\mathbb{R}} (|c_1(u) - c_2(u)|^2 \wedge 1)\, \Pi_0(du)}$$

  a **transportation distance**.

- Now we can represent the Lévy kernel as follows

$$\Pi(x, A) = \Pi_0(\{v : c(x, v) \in A\}), \quad A \in \mathfrak{B}.$$

This idea together with the first studies of the existence and uniqueness for Lévy driven SDE's is back to Ito'51 and Skorokhod'65.

- Now we can write down the SDE for the Lévy-type process explicitly

$$\begin{aligned} dX(t) &= a(X(t))dt + b(X(t))dW(t) \\ &+ \int_{\mathbb{R}} c(X(t-), v)\mathbf{1}_{|c(X(t-),v)|\leqslant 1}\left[\nu_0(ds, dv) - \Pi_0(dv)dt\right] \\ &+ \int_{\mathbb{R}} c(X(t-), v)\mathbf{1}_{|c(X(t-),v)|>1}\nu_0(ds, dv), \end{aligned} \tag{8}$$

where ν_0 is a Poisson point measure with the intensity measure $\Pi_0 \otimes dt$, W is independent of ν_0 Wiener process.

Noise sensitivity estimates in terms of $\mathcal{T}$

Theorem

Under Lipschitz conditions on the functions $a_i(x), b_i^2(x),\ i = 1, 2$ and *Lipschitz-type conditions in terms of* $\mathcal{T}$ *for the kernels* $\Pi_i(x, \cdot),\ i = 1, 2$ it holds that:

1. (Kulik, K.'14) There exist strong solutions $X_i,\ i = 1, 2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the corresponding SDE's with respective initial conditions $x_i \in \mathbb{R},\ i = 1, 2$
2. (Gairing, Högele, K.'18) There exists a constant $K > 0$ such that for $G(x) = \max\{\sqrt{x}, x\}, x \geqslant 0$ the following estimate holds

$$\mathbb{E} \sup_{t\in[0,T]} \rho^2(X_1(t), X_2(t)) \leqslant KG(\Delta), \tag{9}$$

where

$$\Delta = \rho(x_1, x_2) + \|a_1 - a_2\|_\infty^2 + \|b_1 - b_2\|_\infty^2 + \sup_{x\in\mathbb{R}} \mathcal{T}(\Pi_1(x, \cdot), \Pi_2(x, \cdot)).$$

Summary

Two explicit constructions of the distances for Lévy measures in $\mathbb{R}$:

- Coupling distance $\Rightarrow$ Noise sensitivity estimates in the case of an SDE's with additive noise
- Transportation distance $\Rightarrow$ Noise sensitivity estimates in the case of an SDE's with multiplicative noise
- On-going projects:
  - Explicit constructions of the distances for time-inhomogeneous and state dependent Lévy measures in $\mathbb{R}^d$ (with M. Högele)
  - Lower bounds for Wasserstein distance between the solutions of Lévy driven SDE's (with E. Mariucci)

References

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