Semilinear Dirichlet problem for the fractional Laplacian

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The semilinear equation

Let $d \in \{1, 2, \dots\}$, $0 < \alpha < 2$,

$$\nu(z) = \frac{2^\alpha \Gamma((d+\alpha)/2)}{\pi^{d/2} |\Gamma(-\alpha/2)|} |z|^{-d-\alpha}, \quad z \in \mathbb{R}^d,$$

$$\nu(x, y) = \nu(y - x),$$

and $\nu(x, A) = \int_A \nu(x, y) \mathrm{d}y$. Define (the fractional Laplacian)

$$\Delta^{\alpha/2} u(x) = -(-\Delta)^{\alpha/2} u(x) = \lim_{\epsilon \to 0^+} \int_{|x-y|>\epsilon} (u(y) - u(x)) \nu(x, y) \, \mathrm{d}y.$$

Let $\emptyset \neq D \subset \mathbb{R}^d$ be open. We ponder the existence, representation and uniqueness of solutions $u : \mathbb{R}^d \mapsto \mathbb{R}$ of the semilinear problem

$$-\Delta^{\alpha/2} u(x) = F(x, u(x)) \quad \text{on } D,$$

with Dirichlet-type conditions for u **on** $D^c = \mathbb{R}^d \setminus D$ and **at** ∂D.

Motivations

[Aba15] Large solutions for $\Delta^{\alpha/2}$.

[BKK08] Representation of nonnegative α-harmonic functions.

[BH86] Complete maximum principle.

[BC17] Large solutions in Lipschitz domains. (?!)

[MV14] Classical semilinear problems.

Geometry

In examples we consider, e.g., balls, half-spaces, cones, Lipschitz sets, $C^{1,1}$ sets, or arbitrary open sets D. Let

$$B_r(x) := \left\{ y \in \mathbb{R}^d : |x - y| < r \right\} \text{ (ball)},$$

$$\delta_D(x) := \mathrm{dist}(x, D^c) \text{ (distance)}.$$

Let $\partial_* D$ be the set of limit points of D:

$$\partial_* D = \partial D \text{ if } D \text{ is bounded, or}$$
$$\partial_* D = \partial D \cup \{\infty\} \text{ if } D \text{ is unbounded.}$$

Let $D^* = D \cup \partial_* D$. Thus, $D^* = \overline{D}$ or $D^* = \overline{D} \cup \{\infty\}$.

The isotropic α-stable Lévy process

We have (the Lévy-Khintchine exponent)

$$|\xi|^\alpha = \int_{\mathbb{R}^d} (1 - \cos(\xi x))\nu(x)\, dx, \qquad \xi \in \mathbb{R}^d.$$

Let $p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp(-i\xi x) \exp(-t|\xi|^\alpha)\, d\xi,\ t > 0, x \in \mathbb{R}^d$.
Let $(X_t, \mathbb{P}^x)$ be the standard rotation invariant α-stable Lévy process in $\mathbb{R}^d$ with the characteristic function

$$\mathbb{E}^x[\exp(i\xi(X_t - x))] = \exp(-t|\xi|^\alpha), \quad x, \xi \in \mathbb{R}^d, t \geq 0.$$

(X_t) is strong Markov w/trans. prob. $\mathbb{P}^x(X_t \in A) = \int\limits_{A-x} p_t(y)dy$,

$\Delta^{\alpha/2}$ is the generator of (the semigroup of) the process.

The α-harmonic functions

For open $U \subset \mathbb{R}^d$, we define the first exit time of U:

$$\tau_U = \inf \{t \geq 0 : X_t \in U^c\}.$$

Function $h : \mathbb{R}^d \to \mathbb{R}$ is called α-harmonic in D ($h \in \mathcal{H}^\alpha(D)$) if

$$h(x) = \mathbb{E}^x h(X_{\tau_U}), \quad x \in U \subset\subset D.$$

We call h *regular α-harmonic* in D ($h \in \mathcal{H}^\alpha_{\mathrm{reg}}(D)$) if

$$h(x) = \mathbb{E}^x h(X_{\tau_D}), \quad x \in D.$$

If $h \in \mathcal{H}^\alpha(D)$ and $h = 0$ on D^c, then h is called singular α-harmonic on D.

Harmonic majorization

We say that u is harmonically majorized on D if there exists $h \geq 0$ on $\mathbb{R}^d$ which is α-harmonic on D, and $|u| \leq h$ on $\mathbb{R}^d$.

For functions $\psi \geq 0$ and ϕ on D we write "$\phi = o(\psi)$ on D" if for every $\varepsilon > 0$ there is compact $F \subset D$ such that $|\phi| \leq \varepsilon\psi$ on $D \setminus F$.

We say that u is harmonically small on D if there is $h \geq 0$ on $\mathbb{R}^d$ which is α-harmonic on D, $|u| \leq h$ on $\mathbb{R}^d$, and $u = o(h)$ on D.

Weak fractional Laplacian

For $u \in \mathcal{L}^1 := L^1(\mathbb{R}^d, (1+|x|)^{-d-\alpha}\mathrm{d}x)$, we define ([BB99])

$$\langle \widetilde{\Delta}^{\alpha/2} u, \phi \rangle = \langle u, \Delta^{\alpha/2}\phi \rangle = \int_{\mathbb{R}^d} u(x)\Delta^{\alpha/2}\phi(x)\, dx, \quad \phi \in C_c^\infty(\mathbb{R}^d).$$

If $u \in \mathcal{H}^\alpha(D)$, then $u \in \mathcal{L}^1$, $u \in C^\infty(D)$, $\Delta^{\alpha/2}u = 0$ on D and $\widetilde{\Delta}^{\alpha/2}u = 0$ on D.
Conversely, if $u \in \mathcal{L}^1$ and $\widetilde{\Delta}^{\alpha/2}u = 0$ on D, then $u \in \mathcal{H}^\alpha(D)$, after a modification on a set of Lebesgue measure zero.

Thus, weakly α-harmonic and α-harmonic functions coincide *a.e.*

Integral kernels

Let $G_D(x, y)$, $x, y \in \mathbb{R}^d$, be the **Green function**. For instance if $\alpha < d$, then $G_{\mathbb{R}^d}(x, y) = c\,|y - x|^{\alpha-d}$ (the Riesz kernel). We have $\int G_D(x, v)\Delta^{\alpha/2}\varphi(v)dv = -\varphi(x)$ if $x \in \mathbb{R}^d$, $\varphi \in C_c^\infty(D)$. Also,

$$\int_{\mathbb{R}^d} G_D(x, y)f(y)\mathrm{d}y = \mathbb{E}^x \int_0^{\tau_D} f(X_t)dt, \quad x \in D, z \in D^c.$$

The **Poisson kernel** of D is given by Ikeda-Watanabe formula

$$P_D(x, z) := \int_D G_D(x, y)\nu(y, z)\mathrm{d}y, \quad x \in D, z \in D^c.$$

Let $\omega_D^x(A) = \mathbb{P}^x(X_{\tau_D} \in A)$–**harmonic measure** of D for $\Delta^{\alpha/2}$.
If $x \in D$ and $\mathrm{dist}(A, D) > 0$, then $\omega_D^x(A) = \int_A P_D(x, y)\mathrm{d}y$.
We have $\omega_D^x(\partial D) = 0$ if, e.g., D is Lipschitz.

Martin kernel

Fix (any) $x_0 \in D$. We say that $y \in D^c$ is accessible from D if

$$P_D(x_0, y) = \int_{\mathbb{R}^d} G_D(x_0, z)\nu(z, y)dz = \infty .$$

The point at infinity is called accessible from D if

$$\int_{\mathbb{R}^d} G_D(x_0, z)dz = \infty.$$

We define the Martin boundary as the set of accessible points:

$$\partial_M D = \{y \in \partial_* D \ : \ y \text{ is accessible from } D\}.$$

We define the Martin kernel,

$$M_D(x, y) = \lim_{D \ni z \to y} \frac{G_D(x, z)}{G_D(x_0, z)}, \quad x \in \mathbb{R}^d, y \in \partial_* D.$$

The limit always exits. It is α-harmonic iff $y \in \partial_M D$ ([BKK08]).

Green, Poisson and Martin integrals

We define

$$G_D[f](x) = \int_D G_D(x,y)f(y)dy, \quad x \in \mathbb{R}^d,$$

$$P_D[\lambda](x) = \int_{D^c} P_D(x,y)\lambda(dy) \text{ on } D \text{ and } P_D[\lambda] = \lambda \text{ on } D^c,$$

$$M_D[\mu](x) = \int_{\partial_M D} M_D(x,y)\,\mu(dy), \quad x \in \mathbb{R}^d.$$

Theorem (Martin representation [BKK08])

Let $h \geq 0$. Then $h \in \mathcal{H}^\alpha(D)$ if and only if $h = P_D[\lambda] + M_D[\mu]$ with nonnegative measures λ and μ.

Hint for the semilinear problem:

Consider $u = G_D[f] + P_D[\lambda] + M_D[\mu] = G_D[f] + h$.

The idea of the proof of the Martin representation

Let $u \geq 0$ be singular α-harmonic on D. Let $x \in D_n \uparrow D$ be nice;

$$
\begin{aligned}
u(x) &= \int_{D \setminus D_n} P_{D_n}(x,y)u(y)dy \\
&= \int_{D_n} \frac{G_{D_n}(x,v)}{G_{D_n}(x_0,v)} \left(G_{D_n}(x_0,v) \int_{D \setminus D_n} \nu(v,y)u(y)dy \right) dv\,. \\
&:= \int_{D_n} M_{D_n}(x,v)\mu_n(dv)dv\,.
\end{aligned}
$$

Here $\mu_n(\mathbb{R}^d) = \int\limits_D G_{D_n}(x_0,v) \int\limits_{D \setminus D_n} \nu(v,y)u(y)dydv = u(x_0) < \infty$.

The measures weakly converge on $\partial_* D$.

$M_{D_n}(x,v) = G_{D_n}(x,v)/G_{D_n}(x_0,v) \to M_D(x,z)$ on $\partial_* D$ by BHP.

The boundary condition

For $U \subset\subset D$ we define

$$\eta_U[u](A) = \int_A G_U(x_0, z) \int_{D\setminus U} \nu(z, y) u(y)\, dy\, dz, \quad A \subset \mathbb{R}^d.$$

We let

$$W_D[u] = \lim_{U \uparrow D} \eta_U[u].$$

If u has an α-harmonic majorant w, then

$$\eta_U[|u|] \leq w(x_0).$$

Moreover, harmonic smallness yields $W_D[u] = 0$.

Lemma

$$W_D\,[\,G_D[f] + P_D[\lambda] + M_D[\mu]\,] = \mu.$$

The boundary condition, II

Proof.

Assume that $f \geq 0$ and take nice $U \uparrow D$. For $x \in D$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} G_U(x,z) \int_{D\setminus U} \nu(z,y) G_D[f](y)\, dydz \\
&= \int_{D\setminus U} P_U(x,y) G_D[f](y)\, dy \\
&= \mathbb{E}^x \left[G_D[f](X_{\tau_U}) \right] \\
&= \mathbb{E}^x \left[\mathbb{E}^{X_{\tau_U}} \left[\int_0^{\tau_D} f(X_t)\, dt \right] \right] \\
&= \mathbb{E}^x \left[\int_{\tau_U}^{\tau_D} f(X_t)\, dt \right] \leq G_D f(x).
\end{aligned}$$

□

The boundary condition, III

Our semilinear problem is finally formulated as follows:

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2} u(x) = F(x, u(x)) & \text{on } D, \\ u = \lambda & \text{on } D^c, \\ W_D[u] = \mu & \text{on } \partial D \text{ (on } \partial_M D). \end{cases} \tag{1}$$

Here we assume: $P_D[|\lambda|](x) + M_D[|\mu|](x) < \infty$ for some (all) $x \in D$; u is harmonically majorized; $F_u(x) := F(x, u(x))$ is locally integrable on D; $\widetilde{\Delta}^{\alpha/2} u$ and $F_u(x)dx$ are equal as distributions on D. Note: u is a measure on D^c.

We say that $r : [0, \infty) \to [0, \infty)$ is sublinear increasing if it is nondecreasing and $\lim_{v \to \infty} r(v)/v = 0$.

Uniform integrability and Vitali's theorem

Definition

$q : D \to [-\infty, \infty]$ is in the Kato class $\mathcal{J}^\alpha(D)$, if the functions $G_D(x, y)|q(y)|$ are uniformly integrable with respect to $\mathrm{d}y$ on D.

$$q \in \mathcal{J}^\alpha(D) \text{ if } |D| < \infty, \lim_{\varepsilon \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<\varepsilon} |q(y)||y-x|^{\alpha-d} dy = 0.$$

Example

If D is a bounded open set with the outer cone property and $-\infty < \beta < \alpha$, then $\delta_D(x)^{-\beta} \in \mathcal{J}^\alpha(D)$.

Lemma

Let D be regular. Then $q \in \mathcal{J}^\alpha(D)$ if and only if $G_D|q| \in C_0(D)$, and in this case $G_D q \in C_0(D)$.

Existence

Denote $h = P_D[\lambda] + M_D[\mu]$ and $H = P_D[|\lambda|] + M_D[|\mu|]$.

Theorem (A)

Let D be regular. Let $P_D[|\lambda|] + M_D[|\mu|] < \infty$ on D. Let $F : D \times \mathbb{R} \to \mathbb{R}$ and $|F(x,t)| \le q(x)r(|t|)$ for all $x \in D$, $t \in \mathbb{R}$, where $r : [0,\infty) \to [0,\infty)$ is nondecreasing. Let r be sublinear, or $m > 0$ be small. If $q, qr(2H) \in \mathcal{J}^\alpha(D)$, then there is a solution u harmonically majorized by $H + const$ for

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2} u(x) = mF(x, u(x)) & \text{on } D, \\ u = \lambda & \text{on } D^c, \\ W_D[u] = \mu & \text{on } \partial D. \end{cases}$$

Theorem (B)

Under the assumptions of Theorem (A), suppose that u is a solution to (1) harmonically majorized by $H + const$. Then after a modification on a set of Lebesgue measure zero, u is continuous on D and $u = G_D[F_u] + P_D[\lambda] + M_D[\mu]$ *on D.*

Theorem (C)

In addition to the assumptions of Theorem (B) suppose that $v \mapsto F(x, v)$ *is nonincreasing for each* $x \in D$. *If the solution of (1) is continuous on D, then it is unique.*

Corollary

Let D be regular. Let $0 \le q \in \mathcal{J}^\alpha(D)$ *and* $|F(x, v)| \le q(x)$. *If* $P_D[|\lambda|] + M_D[|\mu|] < \infty$ *on D, then there is harmonically majorized continuous solution to (1), unique if* $v \mapsto F(x, v)$ *is nonincreasing.*

Linear Dirichlet problem

The semilinear problem builds on the linear case, as in [MV14].

Lemma

Let $f \in L^1_{loc}(D)$. Suppose $P_D[|\lambda|] + M_D[|\mu|] < \infty$ on D. There is at most one (unique a.e.) harmonically majorized solution u of

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2} u = f & \text{on } D, \\ u = \lambda & \text{on } D^c, \\ W_D[u] = \mu & \text{on } \partial D. \end{cases}$$

Proof of the existence (Theorem A)

Recall $h = P_D[\lambda] + M_D[\mu]$ and $H = P_D[|\lambda|] + M_D[|\mu|]$.
We define the operator T on $C_0(D)$:

$$Tv(x) = m \int_D G_D(x,y) F(y, v(y) + h(y)) dy, \quad x \in \mathbb{R}^d.$$

T satisfies the assumptions of the Schauder Fixed Point Theorem.
Thus there is $v_0 \in K$ such that $Tv_0 = v_0$. Then,

$$u := v_0 + h = G_D[mF_u] + P_D[\lambda] + M_D[\mu]$$

is a solution to (1) continuous on D. Indeed, by [BB00],

$$-\widetilde{\Delta}^{\alpha/2} u := -\widetilde{\Delta}^{\alpha/2}(v_0 + h) = -\widetilde{\Delta}^{\alpha/2} v_0 = -\widetilde{\Delta}^{\alpha/2} G_D[mF_u] = mF_u.$$

□

Proof of representation (Theorem B)

Let $\tilde{u} = G_D[F_u] + P_D[\lambda] + M_D[\mu]$.
We have $|F_u| \leq cq + qr(2H) \in \mathcal{J}^\alpha(D)$.
Hence $\tilde{u}$ is continuous and harmonically majorized by $H +$ const.
Uniqueness of the linear problem implies that $u = \tilde{u}$ *a.e.* on $\mathbb{R}^d$.

$\square$

Proof of uniqueness (Theorem C)

Suppose that u_1, u_2 satisfy (1). By Theorem B and assumed continuity, $u_i = G_D[F_{u_i}] + P_D[\lambda] + M_D[\mu]$ on D for $i = 1, 2$.
Thus $u_1 - u_2 = G_D[F_{u_1} - F_{u_2}]$.
Fix $x \in D$ and assume that $F(x, u_1(x)) - F(x, u_2(x)) > 0$.
By the monotonicity of F, $u_2(x) > u_1(x)$.
Then $G_D[F_{u_1} - F_{u_2}](x) = u_1(x) - u_2(x) < 0$ for this x.
By the complete maximum principle [BH86],
$u_1 - u_2 = G_D[F_{u_1} - F_{u_2}] \leq 0$ everywhere $\mathbb{R}^d$.
By symmetry, $u_2 - u_1 \leq 0$, too, and so $u_1 = u_2$. □

Examples: the ball

Let $D = B_r = \left\{x \in \mathbb{R}^d : |x| < r\right\}$ and $x_0 = 0$. Here is the M. Riesz' formula for the Poisson kernel of B_r:

$$P_{B_r}(x,y) = \mathcal{C}_{d,\alpha} \left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^{\alpha/2} |x-y|^{-d}, \quad x \in B_r,\, y \in B_r^c,$$

with $\mathcal{C}_{d,\alpha} = \Gamma(d/2)\pi^{-1-d/2}\sin(\pi\alpha/2)$. The Green function is

$$G_{B_r}(x,y) = \mathcal{B}_{d,\alpha}|x-y|^{\alpha-d} \int_0^\omega \frac{s^{\alpha/2}}{(s+1)^{d/2}} \frac{ds}{s}, \quad x,y \in B_r;$$

$\omega = \dfrac{(r^2 - |x|^2)(r^2 - |y|^2)}{|x-y|^2}$ and $\mathcal{B}_{d,\alpha} = \Gamma(d/2)/(2^\alpha \pi^{d/2}[\Gamma(\alpha/2)]^2)$.

Examples: the ball, II

Let $r = 1$ and $B = B_1$. The Martin kernel of the ball B is

$$M_B(x, y) = \frac{(1 - |x|^2)^{\alpha/2}}{|y - x|^d}, \quad x \in B, \ y \in \partial B,$$

and $(1 - |x|^2)_+^{\alpha/2-1} = c \int_{\partial B} M_B(x, y)\sigma(dy)$ is α-harmonic on B.

Lemma

Suppose that $f \in L^1_{loc}(B)$ and λ is a measure on B^c such that $P_D[|\lambda|] < \infty$ on B. Up to a modification on a set of Lebesgue measure zero there is at most one solution of

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2} u = f & \text{on } B, \\ u = \lambda & \text{on } B^c, \end{cases}$$

harmonically small on B with respect to $w(x) = (1 - |x|^2)^{\alpha/2-1}$.

The function $h(x) := (1-|x|^2)_+^{\alpha/2-1} = c\int_{\partial B} M_B(x,y)\sigma(dy)$ is (singular) α-harmonic on B. For (as large as possible) $p>0$ and (sufficiently small) $m>0$ we look for solutions to

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2}u(x) = m\ u(x)^p & \text{on } D,\\ u=0 & \text{on } D^c,\\ W_D[u] = c\sigma & \text{on } \partial D. \end{cases}$$

In the setting of Theorem (A) we have $H=h$, $q\equiv 1$ and $r(t)=t^p$. Let $0<p<2\alpha/(2-\alpha)$. Since $G_B\delta_B^{-\alpha}\le const.$, $h^p\in\mathcal{J}^\alpha(B)$. Indeed, $(1-\alpha/2)p<\alpha$. This allows for (superlinear) $p>1$ if $\alpha>2/3$. The critical exponent $p^*=2\alpha/(2-\alpha)$ is smaller (=worse) than in [Aba15], where $p^*=(1+\alpha/2)(1-\alpha/2)$.

Example: the half-space and other cones

Suppose that $D = \{x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_d > 0\} =: \mathbb{H}_+$.
The Poisson kernel for the half-space is

$$P_{\mathbb{H}_+}(x,y) = c_{\alpha,d} \frac{x_d^{\alpha/2}}{|y_d|^{\alpha/2}} |x-y|^{-d}, \quad x \in \mathbb{H}_+, \; y \in \mathrm{int}(\mathbb{H}_+^c),$$

where $c_{\alpha,d} = \sin(\pi\alpha/2)\Gamma(d/2)\pi^{-1-d/2}$. The Green function is

$$G_{\mathbb{H}_+}(x,y) = \mathcal{B}_{d,\alpha}|x-y|^{\alpha-d} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\alpha/2}}{(t+1)^{d/2}} \frac{dt}{t}.$$

The Martin kernel for $y \in \partial\mathbb{H}_+$ is

$$M_{\mathbb{H}_+}(x,y) = \frac{x_d^{\alpha/2}}{|x-y|^d}(1+|y|^2)^{d/2}, \quad x_d > 0,$$

and, for the point at infinity,

$$M_{\mathbb{H}_+}(x,\infty) = x_d^{\alpha/2}, \quad x_d > 0.$$

Example: the half-space and other cones, II

Lemma

Suppose that $f \in L^1_{loc}(\mathbb{H}_+)$ and λ is a measure on $\mathbb{H}_+^c$ such that $P_D[|\lambda|] < \infty$ on $\mathbb{H}_+$. Up to a modification on a set of Lebesgue measure zero there is at most one solution to the problem

$$\begin{cases} -\widetilde{\Delta}^{\alpha/2} u = f & \text{on } \mathbb{H}_+, \\ u = \lambda & \text{on } \mathbb{H}_+^c \end{cases}$$

harmonically small on $\mathbb{H}_+$ with respect to $w(x) = x_d^{\alpha/2-1} + x_d^{\alpha/2}$.

The Martin kernel with the pole at infinity for arbitrary open cones is discussed in [BB04].

Further results of this type depend on detailed asymptotics of Martin and Green kernels for specific domains.

Existing results on Lipschitz sets are not satisfactory.

N. Abatangelo.
Large s-harmonic functions and boundary blow-up solutions for the fractional Laplacian.
Discrete Contin. Dyn. Syst., 35(12):5555–5607, 2015.

K. Bogdan and T. Byczkowski.
Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains.
Studia Math., 133(1):53–92, 1999.

K. Bogdan and T. Byczkowski.
Potential theory of Schrödinger operator based on fractional Laplacian.
Probab. Math. Statist., 20(2, Acta Univ. Wratislav. No. 2256):293–335, 2000.

R. Bañuelos and K. Bogdan.
Symmetric stable processes in cones.
Potential Anal., 21(3):263–288, 2004.

M. Ben Chrouda.
Existence and nonexistence of positive solutions to the fractional equation $\Delta^{\frac{\alpha}{2}} u = -u^{\gamma}$ in bounded domains.
Ann. Acad. Sci. Fenn. Math., 42(2):997–1007, 2017.

J. Bliedtner and W. Hansen.
Potential theory.
Universitext. Springer-Verlag, Berlin, 1986.
An analytic and probabilistic approach to balayage.

K. Bogdan, T. Kulczycki, and M. Kwaśnicki.
Estimates and structure of α-harmonic functions.
Probab. Theory Related Fields, 140(3-4):345–381, 2008.

Moshe Marcus and Laurent Véron.
Nonlinear second order elliptic equations involving measures, volume 21 of *De Gruyter Series in Nonlinear Analysis and Applications*.
De Gruyter, Berlin, 2014.