Collective motion of living organisms: the Vicsek model

Stochastic processes and Machine learning I

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Introduction

Introduction

The object of this talk is the Vicsek model.

- → It is an individual-based model, where the particles are self-propelled, and have same constant velocity;
- → The interaction is one of alignment, in the presence of noise;
- → The discrete-time dynamics is given by

$$x_i(t+\Delta t) = x_i(t) + v_i(t)\Delta t,$$

where the speed is of the form

$$v = |v| e^{i\vartheta(t)},$$

with constant norm, and direction given by the angle

$$\vartheta(t+\Delta t) = \langle \vartheta(t) \rangle + \Delta\vartheta.$$

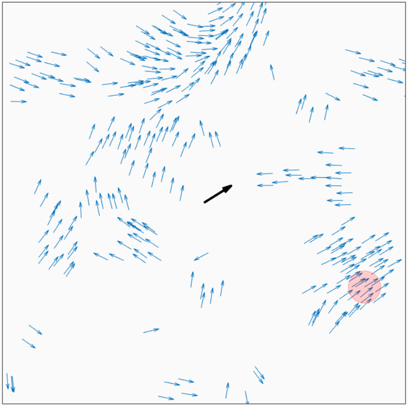

Figure 1: Simulation of 300 individuals after 200 time steps of the Vicsek model, with parameters L $= 25$, noise $\eta = 0.1$.

Once we have provided the time-continuous version of this model, we are going to discuss its mean-field limit, as well as its large-scale behaviour.
The time-continuous dynamics is given by N particles, the position and velocity of which evolve according to

$$\begin{cases} dX_t^{i,N} = V_t^{i,N}\, dt, \\ dV_t^{i,N} = \sqrt{2d}\, (\mathbb{I} - V_t^{i,N} \otimes V_t^{i,N}) \circ dB_t^i + (\mathbb{I} - V_t^{i,N} \otimes V_t^{i,N})\, J_t^{i,N}\, dt, \end{cases}$$

where

$$J^{i,N} := \frac{1}{N} \sum_{j=1}^{N} K\big(|X^{j,N} - X^{i,N}|\big) V^{j,N}$$

is the K-weighted momentum of the i-th particle.

Mean-field limit

Let $f^N = \frac{1}{N}\sum_{j=1}^{N}\delta_{(X_t^{j,N},V_t^{j,N})}(x,v)$ be the **empirical distribution**. Its limit as $N\to\infty$ is given by a probability density function f satisfying

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot\big((\mathbb{I}-v\otimes v)\,\bar{J}_f\, f\big) = d\Delta_v f, \tag{1}$$

where

$$\bar{J}_f(x) := \int_{\mathbb{R}^n\times\mathbb{S}} K(|x-y|)f(y,v)v\,dydv,\ \text{ for all } x\in\mathbb{R}^n.$$

There exists a pathwise unique global solution to

$$\begin{cases} d\bar{X}_t^i = \bar{V}_t^i\,dt \\ d\bar{V}_t^i = \sqrt{2d}\,(\mathbb{I}-\bar{V}_t^i\otimes\bar{V}_t^i)\circ dB_t^i + (\mathbb{I}-\bar{V}_t^i\otimes\bar{V}_t^i)\bar{J}_{f_t}(\bar{X}_t^i)\,dt \\ (\bar{X}_0^i,\bar{V}_0^i) = (X_0^{i,N},V_0^{i,N}),\ f_t = \mathrm{Law}(\bar{X}_t^i,\bar{V}_t^i), \end{cases}$$

with initial data $(X_0^{i,N},V_0^{i,N})$ for $i=1,\ldots,N$.

Moreover, we can prove the following

Theorem (Propagation of chaos)

There exist N independent processes $(\bar{X}_t^i, \bar{V}_t^i)_{t\geq 0}$ with law f, such that

$$\mathbb{E}\left[|X_t^{i,N} - \bar{X}_t^i|^2 + |V_t^{i,N} - \bar{V}_t^i|^2\right] \leq \frac{C}{N},$$

for all $0 \leq t \leq T$, $N \geq 1$, $1 \leq i \leq N$.

Macroscopic scaling

We perform a **hydrodynamic scaling**.

We set $\hat{x} = \varepsilon x,\ \hat{t} = \varepsilon t$, and define

- $f^{\varepsilon}(\hat{x}, v, \hat{t}) := f(x, v, t); \quad K^{\varepsilon}(\hat{x}) := \frac{1}{\varepsilon^n} K(x);$
- $\bar{J}^{\varepsilon}_{f^{\varepsilon}}(x, t) = \int_{\mathbb{S}} (K^{\varepsilon} * f^{\varepsilon})(x, v, t) v \, dv,$

Equation (1)

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left((\mathbb{I} - v \otimes v)\, \bar{J}_f\, f \right) = d\Delta_v f,$$

can then be rewritten as

$$\varepsilon \left(\partial_t f^{\varepsilon} + v \cdot \nabla_x f^{\varepsilon} \right) = -\nabla_v \cdot \left((\mathbb{I} - v \otimes v) \bar{J}_{f^{\varepsilon}} f^{\varepsilon} \right) + \Delta_v f^{\varepsilon}, \tag{2}$$

Studying its limit as $\varepsilon \to 0$ means observing the large-scale behaviour of the model.

The first thing to notice is that, as $\varepsilon \to 0$, we get the following expansion:

$$\overline{J}^{\varepsilon}_{f^{\varepsilon}}(x,t) := \int_{\mathbb{S}} (K^{\varepsilon} * f^{\varepsilon})(x,v,t)v \, dv = \int_{\mathbb{S}} f^{\varepsilon}(x,v,t)v \, dv + O(\varepsilon^2).$$

We write $J_f(x,t) := \int_{\mathbb{S}} f(x,v,t)v \, dv$. Ignoring the $O(\varepsilon^2)$ term, we can then rewrite (2) as

$$\varepsilon(\partial_t f^{\varepsilon} + v \cdot \nabla_x f^{\varepsilon}) = Q(f^{\varepsilon}), \tag{3}$$

where $Q(f) := -\nabla_v \cdot \big((\mathbb{I} - v \otimes v) \, J_f \, f\big) + \Delta_v f$ is the **collision operator**.

Since $Q(f)$ is the only term of order zero in ε, particular interest lies in the **equilibria** of this operator, i.e. the functions f such that $Q(f) = 0$.

Since Q acts only on the v variable, in the following we consider (x,t) fixed, so we study the case of equilibria $f = f(v)$.

Definition (Von Mises-Fischer distribution)

We introduce the Von Mises-Fischer distribution with concentration parameter $\kappa \geq 0$ and orientation $\Omega \in \mathbb{S}$ as the probability density on the sphere defined by

$$M_{\kappa\Omega}(v) := \frac{e^{\kappa\, v\cdot\Omega}}{\int_{\mathbb{S}} e^{\kappa\, w\cdot\Omega} dw}, \quad v \in \mathbb{S}.$$

Notice that

$$\int_{\mathbb{S}} Q(f) \frac{f}{M_{J_f}}\, dv = -\int_{\mathbb{S}} \left| \nabla_v \left(\frac{f}{M_{J_f}} \right) \right|^2 M_{J_f}\, dv \leq 0,$$

so that an equilibrium $f_{eq}(v)$ has to be of the form

$$f_{eq}(v) = \varrho M_{\kappa\Omega}(v),$$

for $\varrho > 0$, $\Omega \in \mathbb{S}$, and $\kappa \geq 0$ that has to satisfy an implicit condition.

In fact, since the following has to hold

$$\kappa\Omega = J_{f_{eq}} = \int_{\mathbb{S}} v\, f_{eq}(v)\, dv = \int_{\mathbb{S}} v \varrho\, M_{\kappa\Omega}\, dv = \varrho\, J_{M_{\kappa\Omega}} = \varrho\, c(\kappa)\Omega,$$

we get $f_{eq}(v) = \varrho M_{\kappa(\varrho)\Omega}(v)$, with $\kappa = \kappa(\varrho)$ such that the following **compatibility condition** holds:

$$\varrho c(\kappa) = \kappa, \tag{CC}$$

where $c(\kappa) = \langle v \cdot \Omega \rangle_{M_{\kappa\Omega}} = \langle \cos\vartheta \rangle_{M_\kappa} = \dfrac{\int_0^\pi \cos\vartheta e^{\kappa\cos\vartheta} \sin^{n-2}\vartheta d\vartheta}{\int_0^\pi e^{\kappa\cos\vartheta} \sin^{n-2}\vartheta d\vartheta}$.

Equilibria

Since $\frac{c(\kappa)}{\kappa} \to \frac{1}{n}$ as $\kappa \to 0$, we find that the compatibility condition leads to the following phase transition:

Proposition (Phase transition)

- $\varrho \le n$. **Uniqueness of the equilibrium**.
  $\kappa = 0$ is the unique solution of (CC). The only equilibria are $h = \varrho$ for an arbitrary $0 \le \varrho \le n$.
- $\varrho > n$. **Two equilibria**.
  (CC) has 2 roots, $\kappa = 0$ and $\kappa(\varrho) > 0$. The equilibria are: $h = \varrho > n$; $\varrho M_{\kappa(\varrho)\Omega}$, for $\varrho > n$ and $\Omega \in \mathbb{S}$ arbitrary, which form a manifold of dimension n.

Moreover, in the first case, the equilibria are stable while, if $\varrho > n$, then the equilibria $h = \varrho$ become unstable and there is exponentially fast convergence to the Von Mises-Fischer ones.

Fix $\varepsilon > 0$, and come back to

$$\varepsilon(\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon) = Q(f^\varepsilon). \tag{4}$$

Assuming **space-homogeneity**, and considering $g^\varepsilon = f^\varepsilon / \varrho^\varepsilon$, we get

Theorem (Existence of a solution)

Suppose g_0 is a probability measure, belonging to $H^s(\mathbb{S})$. Then there exists a unique weak solution g to

$$\varepsilon \partial_t (g^\varepsilon) = -\varrho^\varepsilon \nabla_\omega \cdot \left((\mathbb{I} - \omega \otimes \omega) J_{g^\varepsilon} g^\varepsilon \right) + \Delta_\omega g^\varepsilon, \tag{5}$$

with initial condition $g(0) = g_0$. This solution is a classical one, is positive for all time $t > 0$, and belongs to $C^\infty\left((0, +\infty) \times \mathbb{S}\right)$.

Theorem (Convergence rates)

The long time behaviour of the solution g depends on the value of J_{g_0}, in fact:

- If $J_{g_0} = 0$ then (5) reduces to the heat equation on the sphere, and g converges exponentially fast to the uniform distribution in any H^p form.
- If $J_{g_0} \neq 0$ then we have 3 possibilities:
  - → $\varrho^\varepsilon < n$: g converges exponentially fast in any H^p norm to the uniform distribution.
  - → $\varrho^\varepsilon = n$: g converges to the uniform distribution in any H^p norm, with algebraic asymptotic rate 1/2.
  - → $\varrho^\varepsilon > n$: there exists $\Omega \in \mathbb{S}$ such that g converges exponentially fast to $M_{\kappa(\varrho^\varepsilon)\Omega}$ in any H^p norm.

Let's get back to the **space-inhomogeneous** case. We need to specify the dependence of ϱ and Ω on (x, t).
What we have just seen inspires us to consider two distinct regions:

- $\rightarrow$ A "disordered" one
  $\mathcal{R}_d = \big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ :\ n - \varrho^\varepsilon(x,t) \gg \varepsilon \text{ as } \varepsilon \downarrow 0\big\}$,
- $\rightarrow$ And an "ordered" one
  $\mathcal{R}_o = \big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}_+ :\ \varrho^\varepsilon(x,t) - n \gg \varepsilon \text{ as } \varepsilon \downarrow 0\big\}$.

We assume that

$$\lim_{\varepsilon \to 0} f^\varepsilon(x,v,t) = \varrho(x,t), \text{ for all } (x,t) \in \mathcal{R}_d;$$
$$\lim_{\varepsilon \to 0} f^\varepsilon(x,v,t) = \varrho(x,t) M_{\kappa(\varrho)\Omega(x,t)}, \text{ for all } (x,t) \in \mathcal{R}_o,$$

where the convergence is as smooth as needed.

In the disordered region, where $\varrho \leq n$, we have that $J_{f^\varepsilon} \to J_h = 0$, and $\partial_t \varrho = 0$.

Theorem

For $\varepsilon \to 0$, the formal first order approximation to the solution of the rescaled mean-field system (3) in the disordered region $\mathcal{R}_d$ is given by

$$f^\varepsilon(x, \omega, t) = \varrho^\varepsilon(x, t) - \varepsilon \frac{n\omega \cdot \nabla_x \varrho^\varepsilon(x, t)}{(n-1)\big(n - \varrho^\varepsilon(x, t)\big)},$$

where the density ϱ^ε satisfies the following diffusion equation:

$$\partial_t \varrho^\varepsilon = \frac{\varepsilon}{n-1}\left(\nabla_x \cdot \frac{\nabla_x \varrho^\varepsilon}{n - \varrho^\varepsilon}\right).$$

If $\varrho > n$, then the following holds

Theorem

For $\varepsilon \to 0$, the formal limit of the solution $f^\varepsilon(x, v, t)$ of the rescaled mean-field system (3) in the ordered region $\mathcal{R}_o$ is given by

$$\varrho(x,t) M_{\kappa(\varrho(x,t))\Omega(x,t)}(v),$$

where $\kappa = \kappa(\varrho)$ is the unique positive solution to $\varrho c(\kappa) = \kappa$. Moreover, the local density ϱ and the mean orientation $\Omega \in \mathbb{S}$ satisfy the following first order PDE system

$$\begin{cases} \partial_t \varrho + \nabla_x \cdot (\varrho c \Omega) = 0 \\ \varrho\big(\partial_t \Omega + \tilde{c}(\Omega \cdot \nabla_x)\Omega\big) + \lambda(\mathbb{I} - \Omega \otimes \Omega)\nabla_x \varrho = 0, \end{cases} \tag{6}$$

for an appropriate coefficient $\tilde{c}\big(\kappa(\varrho)\big)$ and a parameter $\lambda(\varrho)$.

A generalization

A generalization

We consider the same model as before, but now we assume that the flock is comprised of two different populations, let's say A and B, that differ in their dynamics for the diffusion coefficient.
More precisely, the dynamics of our model is given by the coupled system

$$\begin{cases} dX_t^i = V_t^i dt, \ dY_t^i = W_t^i dt \\ dV_t^i = \sqrt{2d}(\mathbb{I} - V_t^i \otimes V_t^i) \circ dB_t^i + (\mathbb{I} - V_t^i \otimes V_t^i) J_t^i(X_t^i) dt \\ dW_t^i = \sqrt{2b}(\mathbb{I} - W_t^i \otimes W_t^i) \circ dB_t^i + (\mathbb{I} - W_t^i \otimes W_t^i) J_t^i(Y_t^i) dt, \end{cases} \tag{7}$$

where, for $Z_t^i = X_t^i$ or Y_t^i, the function J is defined as

$$J_t^i(Z_t^i) = \frac{1}{N_A} \sum_{j=1}^{N_A} K(|Z_t^i - X_t^j|) V_t^j + \frac{1}{N_B} \sum_{j=1}^{N_B} K(|Z_t^i - Y_t^j|) W_t^j.$$

A generalization

Following what we have done in the previous case, it is easy to obtain

$$\begin{cases} \partial_t f_t + v \cdot \nabla_x f_t = -\nabla_v \cdot \left((\mathbb{I} - v \otimes v) \bar{J}_{f+g} f_t \right) + d\ \Delta_v f_t \\ \partial_t g_t + v \cdot \nabla_x g_t = -\nabla_v \cdot \left((\mathbb{I} - v \otimes v) \bar{J}_{f+g} g_t \right) + b\ \Delta_v g_t. \end{cases} \tag{8}$$

We expect the equilibria to reflect, in some way, the difference in the diffusion coefficient of the two populations. Again, we define the collision operator

$$Q(f) = -\nabla_v \cdot \left((\mathbb{I} - v \otimes v) J_{f+g} f \right) + d\ \Delta_v f,$$

and are interested in the functions f such that $Q(f) = 0$. In order to do so, we introduce a modified Von Mises-Fischer distribution

$$M^d_{\kappa\Omega}(v) = \frac{e^{\frac{\kappa v \cdot \Omega}{d}}}{\int_{\mathbb{S}} e^{\frac{\kappa w \cdot \Omega}{d}} dw}.$$

It is easy to show that, if f and g are two functions such that Q(f) and Q(g) are zero, then they are of the form

$$f = \varrho_f\, C_1 \exp\left(\frac{\kappa v \cdot \Omega}{d}\right), \quad g = \varrho_g\, C_2 \exp\left(\frac{\kappa v \cdot \Omega}{b}\right). \tag{9}$$

and the new compatibility condition reads

$$1 = d\varrho_f \frac{c(\kappa/d)}{\kappa/d} + b\varrho_g \frac{c(\kappa/b)}{\kappa/b}. \tag{10}$$

Phase transition

Since $\frac{c(\kappa)}{\kappa} \to \frac{1}{n}$ as $\kappa \to 0$, we can summarize our results as

Compatibility condition

- $\to$ If $d\varrho_f + b\varrho_g \leq n$, then $\kappa = 0$ is the unique solution of (10). The only equilibria are the isotropic ones, $f = \varrho_f$ and $g = \varrho_g$.
- $\to$ If $d\varrho_f + b\varrho_g > n$, then (10) has 2 roots: $\kappa = 0$ and $\kappa(\varrho) > 0$. The equilibria for $\kappa = 0$ are $f = \varrho_f$ and $g = \varrho_g$; the ones associated to $\kappa(\varrho)$ consist of the Von Mises-Fischer distributions $\varrho_f M^d_{\kappa(\varrho)\Omega}$ and $\varrho_g M^b_{\kappa(\varrho)\Omega}$, for $\Omega \in \mathbb{S}$ arbitrary.

Definition (Convergence rates)

Let X be a Banach space with norm $\|\cdot\|$ and let $f : \mathbb{R}_+ \to X$.

- We say that f converges exponentially fast to a function f_∞ with **global rate** r if there exists a constant $C = C(\|f_0\|)$, such that
$$\|f(t) - f_\infty\| \leq Ce^{-rt}$$
for all $t \geq 0$.
- We say that the convergence is of **asymptotic rate** r if the above holds for a constant $C = C(f_0)$ depending on f_0 and not only on $\|f_0\|$.
- We say that the convergence is of **asymptotic algebraic rate** α if there exists a constant $C = C(f_0)$ such that
$$\|f(t) - f_\infty\| \leq C/t^\alpha.$$

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