Various models around the Cucker-Smale model and their flocking results

Tagung des Deutsch-Französischen Doktorandenkollegs

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Joint work with P. Cattiaux and Laure Pédèches

Modelling collective behavior... Flocking

"Flocking" behavior is a particular kind of collective behavior that can be easily found in nature while observing the collective motion of a large number of individuals.

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"Flocking" behavior is a particular kind of collective behavior that can be easily found in nature while observing the collective motion of a large number of individuals.

"Flocking" property means that :

- the distance between two individuals **remains bounded**
- individuals move **in the same direction**

Modelling collective behavior... Flocking

"Flocking" behavior is a particular kind of collective behavior that can be easily found in nature while observing the collective motion of a large number of individuals.

Modelling issues :

- How local interactions at the individuals scale may lead to collective behavior ?
- Which kind of rules drive the local interactions ?

Cucker-Smale Model

Founding papers : Cucker S. et Smale S. 2007

- *Cucker F., Smale S., "On the mathematics of emergence", Japan J. Math. 2007*
- *Cucker F., Smale S., "Emergent behavior in flocks", IEEE Trans. Automat. Control, 2007*

Consider a group of N individuals, the i-th being represented by its position $x_i \in \mathbb{R}^d$ and velocity $v_i \in \mathbb{R}^d$.

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)(v_j - v_i).$$

- λ measures the strength of the interaction force between individuals.
- Fonction $t \mapsto (\psi_{ij}(t))_{ij}$ is called *communication rate* and $\psi_{ij}(t) \geq 0$ characterises the influence of individual j on individual i.
- A classical choice is $\psi_{ij}(t) = \psi(|x_i(t) - x_j(t)|)$ where ψ is usually chosen positive, decreasing, (ex $\psi(r) = \frac{1}{(1+r^2)^\beta}$, for a given β).

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Cucker-Smale model is thus an agent-centered (or microscopic) mean-field deterministic model, linear on velocities.

A velocity-attracting model

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N}\psi(|x_i - x_j|)(v_j - v_i). \tag{1}$$

Remark : Somming (1) over i, leads to :

- $\bar{v}(t) := \frac{1}{N}\sum_{i=1}^{N} v_i(t) = \bar{v}^0$ constant
- $\bar{x}(t) := \frac{1}{N}\sum_{i=1}^{N} x_i(t) = \bar{x}^0 + t\bar{v}^0$.

Consider, for $t > 0$

$$z(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}|v_i(t) - v_j(t)|^2 \left(= 2N\sum_{i=1}^{N}|v_i(t) - \bar{v}(t)|^2\right).$$

We thus have :

$$\frac{dz}{dt} = -\frac{\lambda}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\psi(|x_i - x_j|)|v_i - v_j|^2 \leq 0.$$

Can we prove the alignement of velocities along $\bar{v}$? Under which conditions ?

Flocking

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)(v_j - v_i).$$

We say that the group of individuals $\{(x_i(t), v_i(t))\}_{i=1}^{N}$ *flocks* if :

$$\forall 1 \leq i, j \leq N, \quad \sup_{t \geq 0} |x_i(t) - x_j(t)| < \infty, \quad \lim_{t \to \infty} |v_i(t) - v_j(t)| = 0$$

The flocking condition can be re-written using the center of mass and the mean velocity :

$$\forall 1 \leq i \leq N, \quad \sup_{t \geq 0} |x_i(t) - \bar{x}(t)| < \infty \qquad \lim_{t \to \infty} |v_i(t) - \bar{v}(t)| = 0$$

or equivalently :

$$\forall 1 \leq i \leq N, \quad \sup_{t \geq 0} |x_i(t) - \bar{x}(t)| < \infty \qquad \lim_{t \to \infty} z(t) = 0$$

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   - What about non-symmetric comm. rates ?
2. What about adding noise ?
   - What kind of noise ? Where ?
   - Different notions of stochastic flocking
3. Three examples in this framework
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   - Noisy environment and general communication rate
   - Flocking with positive probability

Bounded from below $\psi : \forall r \in \mathbb{R},\ |\psi(r)| \geq \ell$

CS'07, bounded from below ψ

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi(|x_i(t) - x_j(t)|)(v_j - v_i).$$

Suppose that ψ is bounded from below by $\ell > 0$, then :

$$\frac{dz}{dt} = -\frac{\lambda}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \psi(|x_i - x_j|)|v_i - v_j|^2 \leq -\frac{\lambda\ell}{N} z(t)$$

thus : $z(t) \leq z^0 e^{-\frac{\lambda\ell}{N}t} \underset{t\to\infty}{\longrightarrow} 0$ and, for all $t \geq 0$ and all $1 \leq i, j \leq N$:

$$|x_i(t) - x_j(t)| \leq |x_i^0 - x_j^0| + \int_0^t \sqrt{z(s)}ds \leq |x_i^0 - x_j^0| + \frac{2\sqrt{z^0}N}{\lambda\ell} \text{ bounded.}$$

NB : If z decreases fast enough, then, the group necessarily swarms.

Case $\psi(r) = \frac{1}{(1+r^2)^\beta}$, *a priori* not bounded from below

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \frac{1}{(1+|x_i(t)-x_j(t)|^2)^\beta}(v_j - v_i).$$

Let

$$\psi_\ell(u) = \inf_{0\le r\le u} \psi(r) \text{ and } T_R = \inf\left\{t,\ \max_{1\le i,j\le N} |x_i(t)-x_j(t)| \ge R\right\},$$

then, for all $t \le T_R$ and ,$1 \le i,j \le N$:

$$|x_i(t)-x_j(t)| \le |x_i^0 - x_j^0| + \frac{\sqrt{z(0)}N}{\lambda\psi_\ell(R^2)}.$$

There thus exists initial data that lead to flocking, whatever β is.

A few flocking results... case $\psi(r) = \frac{1}{(1+r^2)^\beta}$

[Cucker Smale '07], [Ha, Tadmor'08], [Ha Liu '09]

Let $(x_i(t), v_i(t))_{1\leq i\leq N}$ be the solution to (1) associated with initial data $(x_i^0, v_i^0)_{1\leq i\leq N}$,

"Unconditional flocking" : case $\beta \in [0, 1/2]$

There exist $x_m > 0$ and $x_M > 0$ such that, for all $t \geq 0$, $1 \leq i \leq N$:

$$x_m \leq |x_i(t) - \bar{x}(t)| \leq x_M, \quad \text{et } |v_i(t) - \bar{v}| \leq |v_i^0 - \bar{v}| e^{-\psi_M t}$$

"Conditional flocking" : case $\beta > 1/2$

If moreover $(x_i^0, v_i^0)_{1\leq i\leq N}$ satisfy

$$(1+2N\|x^0 - \bar{x}^0\|)^{\frac{1-2\beta}{2}} > \frac{3N(2N)^{3/2}}{\lambda}\|v^0 - \bar{v}^0\| \left[\left(\frac{1}{2\beta}\right)^{\frac{1}{2\beta-1}} - \left(\frac{1}{2\beta}\right)^{\frac{1-2\beta}{2}}\right],$$

Then conclusion still holds.

Non-Symmetric case : Which difference ?

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = \frac{\lambda}{N}\sum_{j=1}^{N} \psi_{ij}(X(t))(v_j - v_i)$$

The communication rate is said to be *non symetric* when

$$\psi_{ij}(X(t)) \neq \psi_{ji}(X(t)).$$

Cucker-Smale '07 : $\psi_{ij}(X(t)) = \frac{1}{(1+|x_i(t)-x_j(t)|^2)^\beta}$ symetric

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Principal difference : Summing (1) over i used to lead to :

- $\frac{d\bar{v}}{dt} = 0$
- the equation was *dissipative* :

$$\frac{d}{dt}\left(\sum_{i,j}|v_i(t) - v_j(t)|^2\right) = -\frac{\lambda}{N}\sum_{i,j}\psi_{ij}(X(t))\,|v_i(t) - v_j(t)|^2 \leq 0$$

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What kind of noise ? Where ?

Goal : Add noise into the fully deterministic Cucker-Smale interaction.
Questions ?

- Introduce a stochastic term into the kinetic mean-field dynamic (diffusion term ? which form ?)
- Define a stochastic counterpart for the flocking property
- Asymptotic time behavior of the stochastic Cucker-Smale-inspired models ?

What kind of noise ? Where ?

Personal freedom...

- Each individual has its own alea
- Modelled by a diffusion term of form $\sigma_i(t)dW_i(t)$
  - with W_i independent d-dimensional Brownian motions
  - with $\sigma_i(t)$ only depending on $(x_i(t), v_i(t))$
- [Cucker, Mordecki'08], [Ha, Lee, Levy'09] : $\sigma_i = \sqrt{D} I_d$, [Pédèches'16]

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t) \left(v_j(t) - v_i(t)\right) dt + \sigma(x_i(t), v_i(t)) \; dW_i(t)$$

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Noisy environnement...

- Common noise for all the individuals, intensity might depend on position/velocity of each individual
- Modelled by a diffusion term of form $\sigma_i(t)dW(t)$, with W d-dimensional Brownian motion
- [Ahn, Ha'10]

$$dv_i(t) = \frac{\lambda}{N}\sum_{j=1}^{N}\psi_{ij}(t)\,(v_j(t) - v_i(t))\,dt + \sigma(x_i(t), v_i(t))\;dW(t)$$

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Noisy environnement...

- Modelled by a diffusion term of form $\sigma_i(t)dW(t)$, with W d-dimensional Brownian motion

Noisy perception in the interaction...

- Imperfect perception of the distance with the others
- Modeled by a diffusion term of form $\sum_{j=1}^{N} \sigma_{i,j}(t)(v_j(t) - v_i(t))dW_{i,j}(t)$, with $W_{i,j}$ d-dimensional independent Brownian motions
- [Ton, Link, Yagi'14], [Erban, Haskovec, Sun '15], [Sun-Lin '15]

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} (v_j(t) - v_i(t)) \left[\psi_{ij}(t)\ dt + \sigma_{ij}(t)\ dW_{ij}(t)\right]$$

What kind of noise ? Where ?

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Which differences ?

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)\,(v_j(t) - v_i(t))\,dt + \sigma(x_i(t), v_i(t))dW_i(t) \qquad (2)$$

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)\,(v_j(t) - v_i(t))\,dt + \sigma(x_i(t), v_i(t))dW(t) \qquad (3)$$

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} (v_j(t) - v_i(t))\,[\psi_{ij}(t)dt + \sigma_{ij}(t)dW_{ij}(t)] \qquad (4)$$

Equilibrium : Remember that the CS model (1) admits $v_i(t) = \bar{v}^0$ as an equilibrium of the velocities ($\psi_{ij} = \psi_{ji}$)

- In the case (4) it's still true,
- In the general case of models (2) and (3), there is no immediate equilibrium.

Case of a noisy environnement...

In the case (3), if $\sigma_i(t) = D(v_i(t) - v_e)$ where $v_e \in \mathbb{R}^d$ is given, then $v_i = v_e$ is an equilibrium (see [Ahn, Ha '10])

$$dv_i(t) = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t) \left(v_j(t) - v_i(t)\right) dt + \sigma(x_i(t), v_i(t)) dW(t)$$

Case σ constant : the dynamics can be split into two parts :

- the dynamics of the mean velocity : $\bar{v}(t)$ is driven by a purely stochastic process : $d\bar{v}(t) = \sigma dW(t)$
- the distance to the mean velocity : $\hat{v}_i(t) = v_i(t) - \bar{v}(t)$ satisfies the initial deterministic Cucker-Smale problem :

$$\frac{d\hat{v}_i}{dt} = \frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t)(\hat{v}_j(t) - \hat{v}_i(t))$$

And now... Laure...

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How to define stochastic flocking ? [Cattiaux, D., P. '17]

Recall that **deterministic flocking** is defined as, for all $i \in \{1, ..., N\}$,

$$\lim_{t\to\infty} |v_i(t) - \bar{v}(t)| = 0 \quad \text{and} \quad \sup_{0\leqslant t<\infty} |x_i(t) - \bar{x}(t)| < \infty \ ;$$

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The most natural forms of **random flocking** :

- **almost-sure flocking** : the definition above holds **almost surely** :

$$\lim_{t\to\infty} |v_i(t) - \bar{v}(t)| = 0 \ \ a.s. \quad \text{and} \quad \sup_{0 \leqslant t < \infty} |x_i(t) - \bar{x}(t)| < \infty \ \ a.s. \, ;$$

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- $\mathbb{L}^{p,q}$**-flocking** : **convergence in $\mathbb{L}^p$ of the velocities** towards the center of mass, **boundedness of the positions around their center of mass in $\mathbb{L}^q$** :

$$\lim_{t\to\infty} \mathbb{E}\left[|v_i(t) - \bar{v}(t)|^p\right] = 0 \quad \text{and} \quad \sup_{0\leqslant t<\infty} \mathbb{E}\left[|x_i(t) - \bar{x}(t)|^q\right] < \infty.$$

If $q = 1$, we simply say that there is $\mathbb{L}^p$-flocking.

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There are others : **mean flocking** and **weak flocking**, but these two are the most demanding.

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Noisy environment and constant communication rate

Take the **same d-dimensional random noise** W impacting all particles : for $i \in \{1, ..., N\}$, $k \in \{1, ..., d\}$,

$$dv_i^k(t) = -\lambda\psi\left(v_i^k(t) - \bar{v}^k(t)\right)dt + D\left(v_i^k(t) - v_e^k\right)dW^k(t)$$

with

- $\psi > 0$ constant communication rate ;
- $W = (W^1, \ldots, W^d)$, W^k a Brownian motion ;
- $D > 0$ and $v_e \in \mathbb{R}^d$.

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General strategy :

- Step 1 : study of the evolution of $\bar{v}(t)$;
- Step 2 : study of the distance to the mean : $\hat{v}_i(t) = v_i(t) - \bar{v}(t)$.

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Step 1 : macroscopic scale $d\bar{v}^k(t) = D(\bar{v}^k(t) - v_e^k)dW^k(t)$ and thus :

$$\bar{v}^k(t) = v_e^k + (\bar{v}^k(0) - v_e^k)e^{DW_t^k - \frac{D^2 t}{2}} \underset{t\to+\infty}{\longrightarrow} v_e^k \ \text{ p.s.}$$

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Take the **same d-dimensional random noise** W impacting all particles : for $i \in \{1, ..., N\}$, $k \in \{1, ..., d\}$,

$$dv_i^k(t) = -\lambda\psi\left(v_i^k(t) - \bar{v}^k(t)\right)dt + D\left(v_i^k(t) - v_e^k\right)dW^k(t)$$

Step 2 : microscopic scale $d\hat{v}_i^k(t) = -\lambda\psi\hat{v}_i^k(t)dt - D\hat{v}_i^k dW^k(t)$, hence

$$\hat{v}_i^k(t) = \hat{v}_i^k(0)e^{DW_t^k - (\frac{D^2}{2} + \lambda\psi)t} \underset{t\to+\infty}{\longrightarrow} 0 \quad p.s.$$

Summary : $\forall i \in \{1, \ldots, N\}$, $\forall k \in \{1, \ldots, d\}$,

$$v_i^k(t) = \bar{v}^k(t) + \hat{v}_i^k(t) \underset{t\to+\infty}{\longrightarrow} v_e^k \text{ p. s.}$$

$\Rightarrow$ Unconditional almost sure flocking.

What about the other kinds of stochastic flocking ?

Noisy environment and constant communication rate

$$\bar{v}^k(t) = v_e^k + (\bar{v}^k(0) - v_e^k)e^{DW_t^k - \frac{D^2 t}{2}}$$
$$\hat{v}_i^k(t) = \hat{v}_i^k(0)e^{DW_t^k - (\frac{D^2}{2} + \lambda\psi)t}$$

$\mathbb{L}^1$**-flocking :**

$$\mathbb{E}\left(|\hat{v}_i^k(t)|\right) = |\hat{v}_i^k(0)|e^{-\lambda\psi t}\mathbb{E}\left(e^{DW_t^k - \frac{D^2 t}{2}}\right) = |\hat{v}_i^k(0)|e^{-\lambda\psi t} \underset{t\to+\infty}{\longrightarrow} 0.$$

Positions : $\bar{x}_i^k(t) = \bar{x}_i^k(0) + \int_0^t \bar{v}_i^k(s)ds$.
Hence :

$$\sup_{t\geq 0} \mathbb{E}\left(|\hat{x}_i^k(t)|\right) \leq |\hat{x}_i^k(0)| + \int_0^{+\infty} \mathbb{E}\left(|\hat{v}_i^k(s)|\right) ds$$
$$\leq |\hat{x}_i^k(0)| + \int_0^{+\infty} |\hat{v}_i^k(0)|e^{-\lambda\psi s}ds < \infty$$

$\Rightarrow$ Unconditional $\mathbb{L}^1$-flocking.

Noisy environment and constant communication rate

$$\bar{v}^k(t) = v_e^k + (\bar{v}^k(0) - v_e^k)e^{DW_t^k - \frac{D^2 t}{2}}$$

$$\hat{v}_i^k(t) = \hat{v}_i^k(0)e^{DW_t^k - (\frac{D^2}{2} + \lambda\psi)t}$$

$\mathbb{L}^2$-**flocking :**

Positions : as before,

$$\sup_{t\geq 0} \mathbb{E}\left(|\hat{x}_i^k(t)|\right) \leq |\hat{x}_i^k(0)| + \int_0^{+\infty} |\hat{v}_i^k(0)|e^{-\lambda\psi s} ds < \infty$$

Velocities :

$$\mathbb{E}\left(|\hat{v}_i^k(t)|^2\right) = |\hat{v}_i^k(0)|^2 e^{(D^2 - 2\lambda\psi)t}\, \mathbb{E}\left(e^{2DW_t^k - 2D^2 t}\right) = |\hat{v}_i^k(0)|^2 e^{(D^2 - 2\lambda\psi)t}.$$

$\mathbb{L}^2$-flocking $\Leftrightarrow D^2 < 2\lambda\psi$

A case with almost sure flocking, but no $\mathbb{L}^2$-flocking

Two realizations of $t \mapsto |\hat{v}(t)| = |v(t) - \bar{v}(t)|$, with an initial configuration for which there is **no $\mathbb{L}^2$-flocking**, with

- $d = 2$;
- $N = 9$;
- $\lambda = 10$;
- $D = 7$;

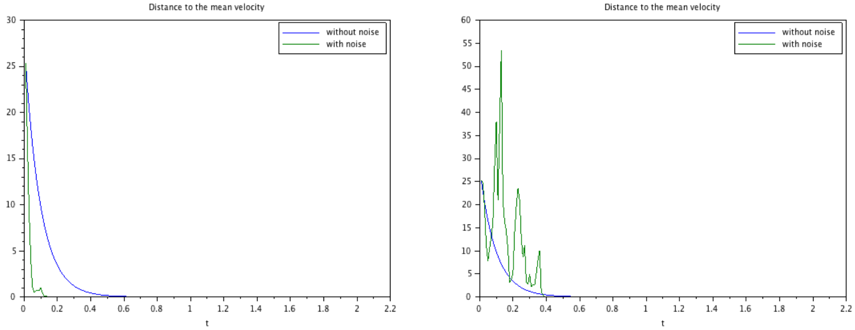

FIGURE: Evolution of $t \mapsto |\hat{v}(t)| = |v(t) - \bar{v}(t)|$.

1. A few Cucker-Smale models, What about flocking ?
   - Cucker-Smale Model 2007, Flocking
   - Choice of a symmetric communication rate
   - What about non-symmetric comm. rates ?
2. What about adding noise ?
   - What kind of noise ? Where ?
   - Different notions of stochastic flocking
3. Three examples in this framework
   - Noisy environment and constant communication rate
   - Noisy environment and general communication rate
   - Flocking with positive probability

Noisy environment and general communication rate

Take the **same d-dimensional random noise** $W(t)$ impacting all particles : for $i \in \{1, ..., N\}$, $k \in \{1, ..., d\}$

$$dv_i^k = -\frac{\lambda}{N} \sum_{j=1}^{N} \psi_{ij}(t) \left(v_i^k - v_j^k\right) dt + D \left(v_i^k - v_e^k\right) dW(t),$$

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with

- $\psi_{ij} = \psi_{ij}(v(.), x(.))$ locally Lipschitz, non-negative and symmetric (for instance the **Cucker-Smale rate**) ;
- $D > 0$ and $v_e \in \mathbb{R}^d$.

Theorem [Cattiaux-D.-P. '17]

The system **flocks almost surely**. However, if $2\lambda \sup_{i,j,x,v} \psi_{i,j}(v,x) \leqslant D^2$, there is **no $\mathbb{L}^2$-flocking**.

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Remarks :

- Struggle between λ and D.
- In the deterministic case, **unconditional flocking** for the Cucker-Smale communication rate **only if** $\gamma \leqslant 1/2$... **multiplicative noise** “improves" that.

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Flocking with a positive probability

Take the **same d-dimensional random noise** $W(t)$ for all particles,

$$dv_i = -\frac{\lambda}{N}\sum_{j=1}^{N}\psi_{ij}(t)\,(v_i - v_j)\,dt + \sigma(v_i)\,dW(t)\,,$$

with

- $\psi_{i,j} = \widetilde{\psi}(|x_i - x_j|)$ locally Lipschitz, non-negative and non-increasing (for instance the **Cucker-Smale rate**) ;
- σ globally K-Lipschitz continuous.

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Theorem [Cattiaux-D.-P. '17]

Under some assumptions on $x(0)$, $v(0)$, K and $\widetilde{\psi}$, there is **flocking with a positive probability**, that is there exists $p \in (0,1]$ such that

$$\mathbb{P}\Big(\forall i \in \{1, \dots, N\},\ \lim_{t\to\infty} |v_i(t) - \bar{v}(t)| = 0 \quad \text{and} \sup_{0\leqslant t<\infty} |x_i(t) - \bar{x}(t)| < \infty\Big) \geqslant p$$

A case with partial flocking

Two realizations of $t \mapsto |\hat{v}(t)| = |v(t) - \bar{v}(t)|$, with an initial configuration – **that does not satisfy the hypotheses of the theorem** – for which there is **flocking in the deterministic case**, with

- $d = 2\,;$
- $N = 9\,;$
- $\lambda = 10\,;$
- σ diagonal, $\sigma^{k,k}(v) = 1 + \sin\left(v^k\right)$
- $\psi(x,y) = \frac{1}{1+|x-y|^2}$.

$$dv_i^k(t) = -\frac{10}{9}\sum_{j=1}^{9}\frac{v_i^k - v_j^k}{1+|x_i - x_j|^2}\,dt + \left(1 + \sin\left(v_i^k\right)\right)dW^k(t)\,, \quad k \in \{1,2\}$$

A case with partial flocking

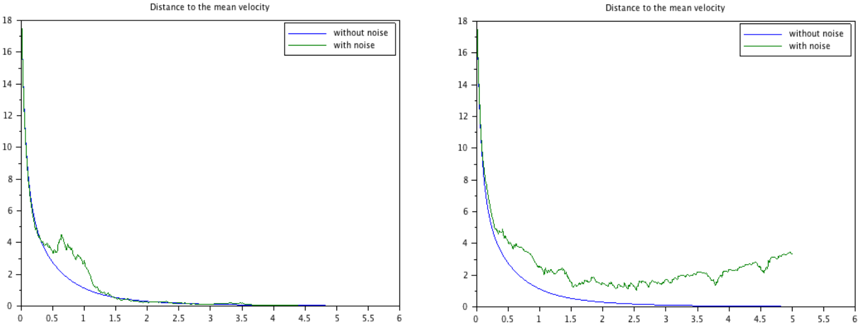

FIGURE: Evolution of $t \mapsto |\hat{v}(t)| = |v(t) - \bar{v}(t)|$.

The **probability of flocking** looks to be **strictly between** 0 **and** 1...

Thank you !