Renewal in Hawkes processes with self-excitation and inhibition

Manon Costa, Carl Graham, Laurence Marsalle and Viet Chi Tran

Institut de Mathématiques de Toulouse

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Outline

1. Hawkes Processes
   - Point Processes
   - Hawkes processes
   - Existence and Uniqueness

2. An auxiliary Markov process
   - Definition
   - Renewal times
   - Deviation inequality

Point Processes

- Model **successive occurrences of random events** in time.

- We consider **point measures** on $\mathbb{R}$ and set $\mathcal{N}(\mathbb{R})$ is the set of counting measures on $\mathbb{R}$.
  For a point measure N, a Borel set B and a measurable function f

$$N(f) = \int f(x)N(dx) = \sum_{x \in supp(N)} f(x),$$

$$N(B) = N(\mathbf{1}_B) = Card(supp(N) \cap B).$$

Point Processes

- Model successive occurrences of random events in time.
- We consider **point measures** on $\mathbb{R}$ and set $\mathcal{N}(\mathbb{R})$ is the set of counting measures on $\mathbb{R}$.
- Given a filtration $(\mathcal{F}_t)_{t\geq 0}$, the **conditional intensity** is the function Λ such that

$$\Lambda(t) = \lim_{h\to 0} \frac{1}{h}\mathbb{E}\Big(N([t,t+h))|\mathcal{F}_t\Big)$$

  Then

$$\mathbb{E}(N(f)) = \mathbb{E}\Big(\int f(x)\Lambda(s)ds\Big)$$

- Example : Poisson point process $\Lambda(t) = \lambda > 0$.

Definition of Hawkes Processes

- Let $\lambda > 0$ and h a measurable function from $(0, +\infty) \to \mathbb{R}$ and $\mathfrak{m}$ a probability measure on $\mathcal{N}((-\infty, 0))$.

  The point process N^h **is a Hawkes process** on $(0, \infty)$ with initial condition $N^0 \sim \mathfrak{m}$ if the conditional intensity function of $N^h_{(0,\infty)}$ is

$$\Lambda^h(t) = \left(\lambda + \int_{-\infty}^{t} h(t-u) N^h(du)\right)^+$$

- h is the **reproduction function**.

  $h > 0$: self excitation,
  $h < 0$: inhibition.

Motivations

- Earthquake occurrences (Hawkes Adamopoulos 1973, Ogata 1988)
- Financial markets (Bacry et al. 2011)
- Neuronal transmissions (Reynaud-Bourret 2013, Delattre et al. 2016, Chevalier 2017, Hadara Löcherbach 2017)

Our aims :

- Existence, Uniqueness, Coupling with $h^+ = \max(0, h)$
- Probability toolbox for statistics.

$$\frac{1}{T}\int_0^T f(N(\cdot + t)_{(-A,0]})dt$$

Existence and uniqueness

Assumptions on h :

- $L(h) = \sup\{t > 0, |h(t)| > 0\} < \infty$
- $||h^+||_1 < 1$
- $\mathbb{E}_{\mathfrak{m}}(N^0(-L(h), 0]) < \infty$.

Proposition

Under these assumptions, the Hawkes process can be constructed as the unique strong solution of

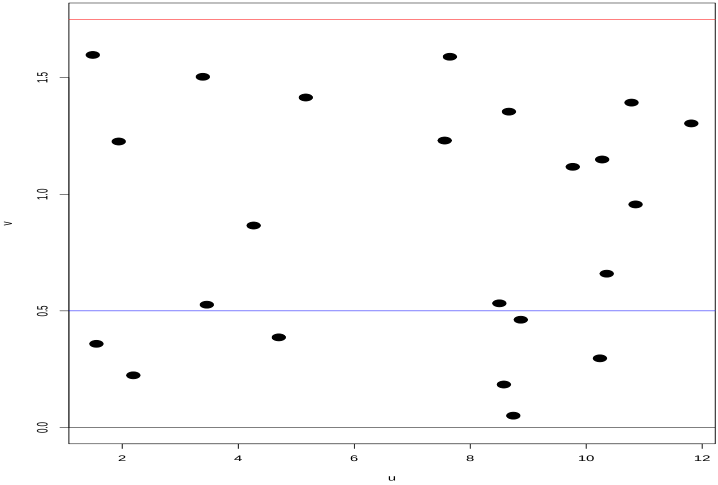
$$N^h = N^0 + \int_{\mathbb{R}_+^2} \delta_u \mathbf{1}_{\{\theta \leq \Lambda^h(u)\}} Q(du, d\theta)$$

$$\Lambda^h(t) = \left(\lambda + \int_{-\infty}^{t} h(t-u) N^h(du)\right)_+$$

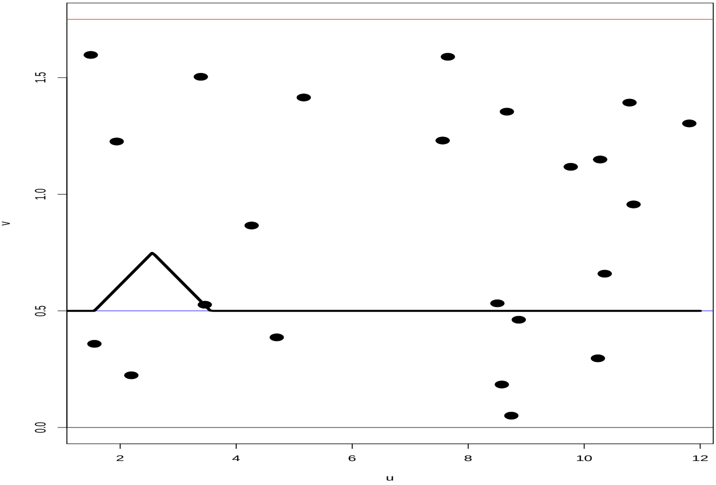
where Q is a Poisson point measure on $\mathbb{R}_+^2$ with Lebesgue intensity

Picard iteration method (Brémaud Massoulié 1996).

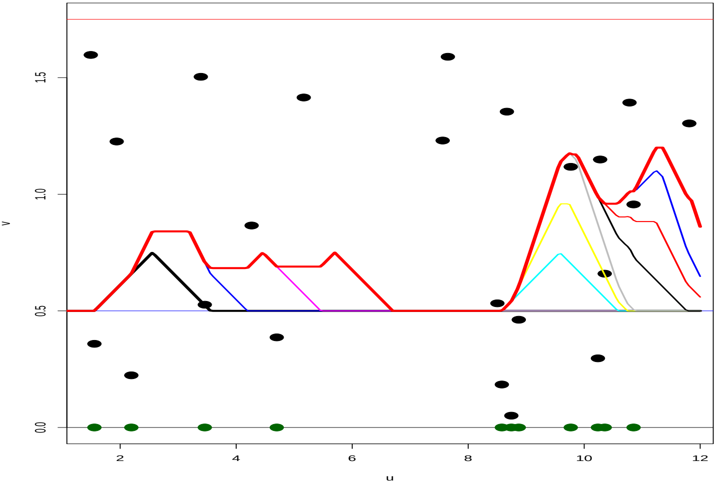
Construction - $h > 0$



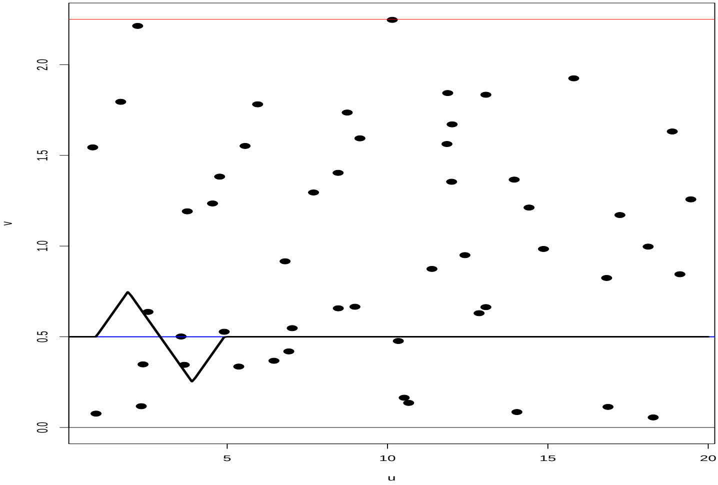
Construction - $h > 0$



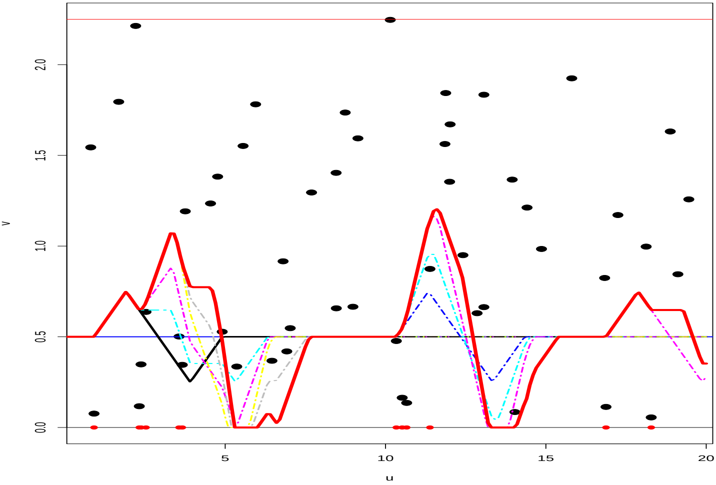
Construction - $h > 0$



Construction



Construction (Thinning)



Coupling argument

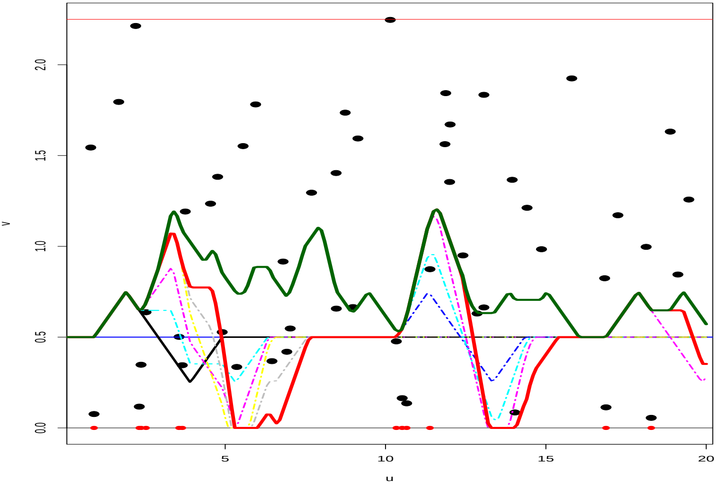
- Let $h^+ = \max(h, 0)$.

We can construct N^h and N^{h^+} using the same Poisson Point measure Q, then

$$N^h(B) \leq N^{h^+}(B), \quad \forall B \in \mathcal{B}\mathbb{R}_+.$$

This allows to control N^h with N^{h^+} which is easier to study.

Construction



Cluster representation $h \geq 0$

In the case where $h \geq 0$, the Hawkes process can be constructed as

- Ancestors immigrate at rate λ.
- An atom in s is an individual with lifetime $L(h)$
  It reproduces at rate $h(\cdot - s)$:
  - The number of its offspring is $Poisson(||h^+||_1)$
  - The births of the children are drawn independently on $(s, s + L(h))$ with density $h(\cdot)/||h^+||_1$

The Hawkes process is then a superposition of independent subcritical Galton Watson trees

[Hawkes, Oakes 1974]

- When h can take negative values, it would lead to a pruned version of this construction.

An auxiliary Markov process

- Let $A > L(h)$ be a window size of interest.
- We are interested in the long time behavior of

$$\frac{1}{T}\int_0^T f(N(\cdot + t))dt$$

where f is a function of $\mathcal{N}((-A,0])$ locally bounded.

We define

$$X_t = (S_t N^h)_{(-A,0]} = N^h_{(t-A,t]}(\cdots + t)$$

Definition

$$X_t = (S_t N^h)_{(-A,0]} = N^h_{(t-A,t]}(\cdots + t)$$

X_t is the measure N^h restricted on $(t-A,t]$ shifted back in $(-A,0]$

Proposition

The process $(X_t, t \geq 0)$ is a strong Markov process with initial condition $X_0 = N^0_{(-A,0]}$ in $\mathbb{D}(\mathbb{R}_+, \mathcal{N}((-A,0]))$.

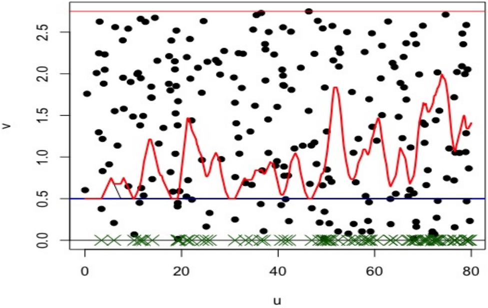
- The properties of $\frac{1}{T}\int_0^T f(X_t)dt$ will be studied using **renewal properties**

Renewal times

$$X_t = (S_t N^h)_{(-A,0]} = N^h_{(t-A,t]}(\cdots + t)$$

We consider

$$\tau = \inf\{t > 0, X_{t-} \neq \emptyset, X_t = \emptyset\}$$
$$= \inf\{t > 0, N[t-A,t) \neq 0, N(t-A,t] = 0\}$$



Renewal times

$$\tau_0 = \inf\{t \geq 0 : X_t = \emptyset\} \qquad \text{(First entrance time of } \emptyset\text{)}$$

$$\tau_k = \inf\{t > \tau_{k-1} : X_{t-} \neq \emptyset, X_t = \emptyset\}\,. \qquad \text{(Successive return times at } \emptyset\text{)}$$

Theorem

- The τ_k for $k \geq 0$ are finite stopping times, a.s.
- The delay $(X_t)_{t\in[0,\tau_0)}$ is independent of the cycles $(X_{\tau_{k-1}+t})_{t\in[0,\tau_k-\tau_{k-1})}$ for $k \geq 1$.
- These cycles are i.i.d. and distributed as $(X_t)_{t\in[0,\tau)}$ under $\mathbb{P}_\emptyset$. In particular their durations $(\tau_k - \tau_{k-1})_{k\geq 1}$ are distributed as τ under $\mathbb{P}_\emptyset$, so that $\lim_{k\to+\infty} \tau_k = +\infty$, $\mathbb{P}_{\mathfrak{m}}$-a.s.

Exponential moments of τ

From the coupling

$$\mathbb{P}(\tau \leq \tau^+) = 1$$

- **It is sufficient to control τ^+.**

We use the **cluster representation** of Hawkes processes for $h \geq 0$ and consider a **queueing issue**.

- Ancestors/clients migrate/arrive at rate λ
- Each client has a service length $H + A$ where
  - H is the length of the Galton Watson tree
  - A is the window size.

$M/G/\infty$ queue (infinite number of servers)

Queueing issue

Proposition ((Reynaud-Bouret, Roy, 2007))

- The Galton Watson tree length satisfies

$$\forall x \geq 0, \quad \mathbb{P}(H_1 > x) \leq Ce^{\gamma x},$$

with $\gamma = (||h^+||_1 - \log(||h^+||_1) - 1)/L(h)$ and $C = \exp(1 - ||h^+||_1)$.

Proposition

The renewal times τ^+ satisfies $\forall \alpha < \min(\lambda, \gamma)$

$$\mathbb{E}_\emptyset(e^{\alpha\tau}) < \infty.$$

(proof based on a formula by Takacs for the Laplace transform $\mathbb{E}(e^{-sB})$ of the busy period, then work for showing that the abscissa of convergence is $\sigma_c \leq -\gamma$)

Ergodic Theorem

Theorem

$$\frac{1}{T}\int_0^T f(X_s)ds \quad \xrightarrow[T\to\infty]{} \quad \pi_A(f) = \frac{1}{\mathbb{E}_\emptyset(\tau)}\mathbb{E}_\emptyset\left(\int_0^\tau f(X_s)ds\right) \quad a.s.$$

Moreover

$$\mathbb{P}_\mathfrak{m}\left((S_t N^h)|_{[0,+\infty)} \in \cdot\right) \xrightarrow[t\to\infty]{\text{total variation}} \mathbb{P}_{\pi_A}\left(N^h|_{[0,+\infty)} \in \cdot\right).$$

Sketch of the proof

- The idea is to decompose the paths of X_t into independent excursions outside $\emptyset$.

$$\int_0^T f(X_s)ds = \int_0^{\tau_0} f(X_s)ds + \sum_{k=1}^{K_T} I_k(f) + \int_{\tau_{K_T}}^T f(X_s)ds,$$

where $I_k(f) = \int_{\tau_{k-1}}^{\tau_k} f(X_s)ds$ and $K_T = \max\{k \geq 0, \tau_k \leq T\}$.

- The strong law of large numbers implies that

$$\frac{1}{K_T}\sum_{k=1}^{K_T} I_k(f) \quad \underset{T\to\infty}{\longrightarrow} \quad \mathbb{E}_\emptyset\Big(\int_0^\tau f(X_s)ds\Big) = \mathbb{E}_\emptyset(\tau)\pi_A(f)$$

Central limit Theorem

Theorem

Assume

$$\sigma^2(f) \triangleq \frac{1}{\mathbb{E}_\emptyset(\tau)} \mathbb{E}_\emptyset \left(\left(\int_0^\tau \left(f((S_t N^h)|_{(-A,0]}) - \pi_A f \right) dt \right)^2 \right) < \infty$$

Then

$$\sqrt{T} \left(\frac{1}{T} \int_0^T f((S_t N^h)|_{(-A,0]})\, dt - \pi_A f \right) \xrightarrow[T\to\infty]{\text{in law}} \mathcal{N}(0, \sigma^2(f)) .$$

Deviation inequalities

Theorem

Let $\alpha > 0$ such that $\mathbb{E}_\emptyset(e^{\alpha\tau}) < \infty$.
We set

$$v = \frac{2(b-a)^2}{\alpha^2}\left\lfloor \frac{T}{\mathbb{E}_\emptyset(\tau)} \right\rfloor \mathbb{E}_\emptyset(e^{\alpha\tau}) e^{\alpha \mathbb{E}_\emptyset(\tau)}, \quad \text{and} \quad c = \frac{|b-a|}{\alpha}.$$

Then for all $\varepsilon > 0$

$$\mathbb{P}_\emptyset\left(\left| \frac{1}{T}\int_0^T f((S_t N^h)|_{(-A,0]}) - \pi_A f \right| \geq \varepsilon \right) \leq 4 \exp\left(- \frac{\left(T\varepsilon - |b-a|\mathbb{E}_\emptyset(\tau)\right)^2}{4\left(2v + c(T\varepsilon - |b-a|\mathbb{E}_\emptyset(\tau))\right)} \right),$$

Deviation inequalities

Theorem

Let $\alpha > 0$ such that $\mathbb{E}_\emptyset(e^{\alpha\tau}) < \infty$.
We set

$$v = \frac{2(b-a)^2}{\alpha^2}\left\lfloor \frac{T}{\mathbb{E}_\emptyset(\tau)} \right\rfloor \mathbb{E}_\emptyset(e^{\alpha\tau}) e^{\alpha \mathbb{E}_\emptyset(\tau)}, \quad \text{and} \quad c = \frac{|b-a|}{\alpha}.$$

For all $1 \geq \eta > 0$

$$P_\emptyset\left(\left| \frac{1}{T}\int_0^T f((S_t N^h)|_{(-A,0]})dt - \pi_A f \right| \geq \varepsilon_\eta \right) \leq \eta\,, \tag{1}$$

where

$$\varepsilon_\eta = \frac{1}{T}\left(|b-a|\mathbb{E}_\emptyset(\tau) - 2c\log\left(\frac{\eta}{4}\right) + \sqrt{4c^2\log^2\left(\frac{\eta}{4}\right) - 8v\log\left(\frac{\eta}{4}\right)} \right).$$

Thank you for your attention!