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Space-Time Asymptotics of an Infinite-Dimensional Diffusion Having a Long-Range Memory

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Abstract. We develop a cluster expansion in space-time for an infinite-dimensional system of interacting diffusions where the drift term of each diffusion depends on the whole past of the trajectory; these interacting diffusions arise when considering the Langevin dynamics of a ferromagnetic system submitted to a disordered external magnetic field.

Keywords: random field Ising model, Langevin dynamics, interacting diffusion processes, space-time cluster expansions

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1. Introduction

Consider a Random Field Ising Model (RFIM) on $\mathbf{Z}^d$, i.e. a spin system on the lattice $\mathbf{Z}^d$ whose energy Hamiltonian in each finite volume Λ may be written as

$$H_\Lambda(\boldsymbol{s}) = -\sum_{\{i,j\}\in\Lambda^*} s^i s^j - \sum_{i\in\Lambda} h^i s^i, \quad \forall \boldsymbol{s} \in \{\pm 1\}^{\mathbf{Z}^d},$$

Λ^* denoting the set of all bonds in Λ (equipped e.g. with its periodic boundary conditions), and $\left(h^i\right)_{i\in\mathbf{Z}^d}$ being a fixed realisation corresponding to a family of i.i.d. symmetric random variables h^i having variance σ^2. Such site disordered spin systems have been studied in the Mathematical Physics litterature since the mid eighties. Using a rigourous renormalisation method, Bricmont and Kupiainen were able to prove in [3] that the low temperature ground states associated with a 3-dimensional Ising model weakly perturbed through a Bernoulli random field display ferromagnetic ordering, thus (partially) settling a controversy on the lower critical dimension d of such RFIMs; Aizenman and Wehr brought this controversy to its end shortly afterwards, by proving in [1] that in 2 dimensions an *arbitrarily weak* disordered external magnetic field leads to a breakdown of the first order phase transition occuring in the standard Ising model on $\mathbf{Z}^2$, so that $d_c = 2$. Such equilibrium properties were later examined in a "soft spin" setting by C. Külske, who considered a 3-dimensional RFIM where the original discrete spins $s^i = \pm 1$ are being replaced by continuous spin variables $x^i \in \mathbf{R}$. The Boltzmann factor corresponding to the inverse temperature parameter β then becomes

$$\exp\Big\{-\sum_{i\in\Lambda} U(x^i) + \beta \sum_{\{i,j\}\in\Lambda^*} x^i x^j + \beta \sum_{i\in\Lambda} h^i x^i \Big\} \tag{1.1}$$

for some "double well" single site potential $U : \mathbf{R} \to \mathbf{R}$, e.g. $U(x) = Cx^4 - 2Cx^2$, introduced in order to obtain some resemblance with the original discrete spin setting. In [12], a new renormalisation method is developped in order to prove the Bricmont−Kupiainen result in this "unbounded spin" setting.

The present paper is concerned with the dynamics corresponding to such continuous spin RFIMs. For a fixed realisation of the external field $\boldsymbol{h} = (h^i)_{i\in\mathbf{Z}^d}$ (Gaussian or Bernoulli), the Langevin dynamics associated with the Boltzmann factor (1.1) consists in the system of interacting diffusions $\big(\mathcal{S}^{\boldsymbol{h}}_\Lambda\big)$ given by

$$\big(\mathcal{S}^{\boldsymbol{h}}_\Lambda\big) \begin{cases} dx^i_t = dw^i_t - U'(x^i_t)dt + \beta \sum_{j\sim i} x^j_t dt + \beta h^i dt \\ (i \in \Lambda, t \geq 0) \end{cases}$$

$\big\{(w^i_t)_{t\geq 0}; i \in \mathbf{Z}^d\big\}$ denoting here a family of i.i.d. standard Brownian motions, and the notation $j \sim i$ indicating that i and j are nearest neighbours in Λ; as for the initial condition given to this system, one may for example consider any probability measure μ_0 on the real line having finite second moment, and set: $Law(\boldsymbol{x}|_{t=0}) = \mu_0^{\otimes\Lambda}$.

Fixing a bounded (arbitrarily large) time horizon $[0, T]$, one may then let $\Lambda \nearrow \mathbf{Z}^d$ and observe that such system of interacting diffusions obeys a strong law of large numbers. Indeed, letting $P^{\boldsymbol{h}}_{\Lambda,T}$ denote the probability law corresponding to $\big(\mathcal{S}^{\boldsymbol{h}}_\Lambda\big)$ considered during time $[0, T]$, one may establish that the law under $dP^{\boldsymbol{h}}_{\Lambda,T}(\boldsymbol{x})$ of the empirical process associated with $\boldsymbol{x} = \big\{(x^i_t)_{0\leq t\leq T}; i \in \Lambda\big\}$ converges towards a Dirac mass concentrated at some asymptotic dynamics Q_T ([2] contains a complete proof in the Gaussian case, together with quenched and annealed large deviations estimates). In the case of a "Bernoulli" random field, Q_T may be characterised as the law of the system

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i)dt + \beta \sum_{j\sim i} x_t^j dt \\ \qquad\qquad + \sigma\beta \tanh\Big\{\sigma\beta\Big(x_t^i - x_0^i + \int_0^t \Big(U'(x_u^i) - \beta\sum_{j\sim i} x_u^j\Big) du\Big)\Big\} dt, \\ Law(\boldsymbol{x}|_{t=0}) = \mu_0^{\otimes \mathbf{Z}^d} \qquad (i \in \mathbf{Z}^d,\ 0 \le t \le T). \end{cases}$$

As may be seen immediately, the diffusions in the above system have a short range spatial interaction, whereas this interaction is of a long range nature in time, due to the presence of the functional

$$\tanh\Big\{\sigma\beta\Big(x_t^i - x_0^i + \int_0^t \Big(U'(x_u^i) - \beta\sum_{j\sim i} x_u^j\Big)\, du\Big)\Big\}$$

in the drift term associated with x_t^i. Of course, letting

$$v_t^i = x_t^i - x_0^i + \int_0^t \Big(U'(x_u^i) - \beta\sum_{j\sim i} x_u^j\Big) du,$$

one may also define Q_T as the $\boldsymbol{x}$-marginal corresponding to the system

$$\begin{cases} dx_t^i = dv_t^i - U'(x_t^i)dt + \beta \sum_{j\sim i} x_t^j dt \\ dv_t^i = dw_t^i + \sigma\beta \tanh\{\sigma\beta v_t^i\} dt \\ (i \in \mathbf{Z}^d,\ 0 \le t \le T) \end{cases}$$

In the setting of a Gaussian random field, the asymptotic dynamics Q_T may be similarly characterised as the $\boldsymbol{x}$-marginal of

$$\begin{cases} dx_t^i = dv_t^i - U'(x_t^i)dt + \beta \sum_{j\sim i} x_t^j dt \\ dv_t^i = dw_t^i + \gamma_t v_t^i dt \\ (i \in \mathbf{Z}^d,\ 0 \le t \le T,\ \gamma_t = \sigma^2\beta^2/(1+\sigma^2\beta^2 t)) \end{cases}$$

So in this setting, the white noise $\{(w_t^i)_{t\ge 0}; i \in \mathbf{Z}^d\}$ driving the Langevin dynamics associated with a standard ferromagnetic spin system has to be replaced by a family $\{(v_t^i)_{t\ge 0}; i \in \mathbf{Z}^d\}$ of Ornstein – Uhlenbeck processes having a time dependent friction coefficient, $\gamma_t = \sigma^2\beta^2/(1+\sigma^2\beta^2 t)$.

Having performed such thermodynamic limit for the empirical process, one may then let $T \to +\infty$ and wonder about the large time properties of the corresponding asymptotic dynamics. At this stage, it should be noted that the customary methods relying on coercive inequalities for the associated Markov generator do not seem to be of much help here, since we are dealing with *degenerate* Markov processes on $\left(\mathbf{R}^2\right)^{\mathbf{Z}^d}$, moreover the second of these processes is also *time inhomogeneous*.

On the other hand, much work has been invested recently in the Statistical Mechanics litterature in order to implement some cluster expansion methods both for interacting diffusions systems (starting with [11], further developped in [6,17,18]) and for some one-dimensional, non-Markovian diffusions viewed as Gibbs measures on path space (see for example [19] or [15]). [18] considers a particular system of interacting diffusions representing a quantum crystal and establishes the validity of a cluster expansion in space-time for this system, in the "light mass" limit. In [17], space-time cluster expansions are being developped for certain classes of systems of interacting diffusions considered in a weak interaction regime. [15] establishes the validity of a cluster expansion method for some probability measures on path space $\mathcal{C}\left(\mathbf{R};\mathbf{R}\right)$ which are associated with some "reasonable" external potentials (corresponding e.g. to a diffusion evolving in a double well U) and with a 2 body interaction potential $\mathcal{W}$ of the type

$$\mathcal{W}\left(u,t;x_u,x_t\right) = F\left(t-u;x_u,x_t\right)$$

for some functional $F(s;x,y)$ decaying rapidly when $s \to +\infty$. Finally in [6], Dai Pra and the first author consider a general system of interacting diffusions given by

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i)dt + b_\varepsilon\left(\theta_t^i \boldsymbol{x}\right)dt \\ (i \in \mathbf{Z}^d, t \in \mathbf{R}) \end{cases}$$

θ_t^i being a space-time shift on $\mathcal{C}\left(\mathbf{R};\mathbf{R}\right)^{\mathbf{Z}^d}$:

$$\theta_t^i \boldsymbol{x} = \boldsymbol{y} = \left(y_u^j\right)_{u\le 0}^{j\in\mathbf{Z}^d}, \qquad y_u^j = x_{t+u}^{i+j},$$

and $b_\varepsilon : \mathcal{C}\left(]-\infty,0];\mathbf{R}\right)^{\mathbf{Z}^d} \to \mathbf{R}$ being simply a measurable adapted functional on $\mathcal{C}\left(]-\infty,0];\mathbf{R}\right)^{\mathbf{Z}^d}$ that is both local in space and time, and that satisfies further a uniform boundedness assumption ($\|b_\varepsilon\|_\infty \le \varepsilon$). Of course, the (eventual) lack of regularity of b_ε and its non Markovian nature show that the mere existence of a weak solution is not at all obvious for such systems. Dai Pra and Rœlly establish the existence of a weak solution (and some of its asymptotic properties) for such systems by developping a cluster expansion in space-time and considering the regime of small interactions (where ε is "small"). Such method may be carried out by giving a proper reference measure to the path space $\mathcal{C}(\mathbf{R};\mathbf{R})^{\mathbf{Z}^d}$, e.g. independent bridges based on the diffusions $\left\{(x_t^i)_{t\in\mathbf{R}}; i\in\mathbf{Z}^d\right\}$ given by

$$dx_t^i = dw_t^i - U'(x_t^i)dt, \tag{1.2}$$

and by considering an energy Hamiltonian $\left\{H^\varepsilon_{\Lambda\times I};\ \Lambda\subset\mathbf{Z}^d,\ I\subset\mathbf{R}\right\}$ corresponding to the drift term b_ε, Λ is a finite subset of $\mathbf{Z}^d$, I is an interval. For a fixed space-time window $V=\Lambda\times I$, this energy Hamiltonian is actually given by

$$H^\varepsilon_V(\boldsymbol{x}) = -\sum_{i\in\left(\Lambda\cup\partial\Lambda\right)}\left[\int_I b_\varepsilon\left(\theta^i_t\boldsymbol{x}\right)dB^i_t(\boldsymbol{x}) - \frac{1}{2}\int_I b_\varepsilon\left(\theta^i_t\boldsymbol{x}\right)^2 dt\right],$$

the notation $\partial\Lambda$ corresponding to a certain locality in space that was assumed for b_ε, and the functionals $B^i_t(\boldsymbol{x})$ being defined by

$$B^i_t(\boldsymbol{x}) = x^i_t - x^i_0 + \int_0^t U'(x^i_s)ds.$$

Considering the partition functions $\mathcal{Z}^\varepsilon_V$ associated with such reference measure and energy Hamiltonian on Ω, one is then interested in the asymptotic behaviour of $\mathcal{Z}^\varepsilon_V$ for large V. In [6], the validity of a cluster representation in space-time was established in the regime of small interactions ($\varepsilon\le\varepsilon_0$), together with an exponential estimate for the cluster coefficients appearing in this expansion. This means that the partition function $\mathcal{Z}^\varepsilon_V$ may be decomposed as

$$\mathcal{Z}^\varepsilon_V = 1 + \sum\nolimits^V_{\mathcal{R}=(\Gamma_1,\ldots,\Gamma_n)}\prod_{r=1}^s K_{\Gamma_r},$$

$\sum^V_{\mathcal{R}}$ denoting a sum over all "clusters" (compatible collection of "contours") contained in the volume V, and K_{Γ_r} being a coefficient such that

$$|K_\Gamma| \le \lambda(\varepsilon)^{|\Gamma|},$$

for some $\lambda(\varepsilon)=O(\varepsilon)$ and some positive quantity $|\Gamma|$ measuring the size of each contour Γ.

Our aim in the present paper is to establish the validity of such a space-time cluster expansion for the asymptotic dynamics Q_∞ arising in the Bernoulli RFIM (time being now extended to the whole semi-infinite interval $[0;+\infty[$). The novelty here is that we have to deal with interacting diffusions $(x^i_t)^{i\in\mathbf{Z}^d}_{t\ge0}$ which display a local interaction in space as well as a *long range* memory (in time); moreover in the present situation, the influence of x^i_u over x^i_t does not seem to decay so rapidly for $|t-u|\longrightarrow+\infty$. In the case of a Bernoulli random field, one may still establish the validity of a high temperature cluster expansion in space-time for Q_∞, together with an exponential estimation of the corresponding cluster coefficients; among other consequences, the (space and time) correlation functions associated with Q_∞ may then be shown to decay exponentially fast in the High Temperature regime (see [16], §3 in Chapter 5). On the other hand, despite various attempts, there does not seem to be any way of establishing such a space-time cluster expansion for the asymptotic dynamics Q_∞ arising in the Gaussian setting.

The next section is dedicated to a brief derivation of the large deviations estimates and strong law of large numbers leading to the consideration of Q_∞ in the Bernoulli setting. Then in section 3 we show that Q_∞ may also be presented as a Gibbs measure in space-time, and establish correspondingly the validity of a space-time cluster expansion in the high temperature regime, together with exponential estimates for the cluster coefficients. This implies in particular that the non-Markovian interacting diffusions system under consideration displays exponential ergodicity in the high temperature regime.

2. Spatial large deviations and the asymptotic dynamics Q_∞

2.1. Gibbsian nature of the annealed dynamics

Recall that for a fixed (Bernoulli) realisation of the random field $\boldsymbol{h} = \left(h^i, i \in \mathbf{Z}^d\right)$, $P^{\boldsymbol{h}}_{\Lambda,T}$ denotes the law of the interacting diffusions system $\left(\mathcal{S}^{\boldsymbol{h}}_{\Lambda,T}\right)$ given through the stochastic differentials

$$dx^i_t = dw^i_t - U'(x^i_t)dt + \beta \sum_{j \sim i} x^j_t dt + \beta h^i dt,$$

where $i, j \in \Lambda$ and $j \sim i$ means that i and j are nearest neighbours, whereas time t varies in a bounded interval $[0, T]$. (For simplicity, we shall always assume that a finite box Λ is being equipped with its periodic boundary conditions.) $P^{\boldsymbol{h}}_{\Lambda,T}$ is thus a probability measure on path space $\mathcal{C}([0,T];\mathbf{R})^\Lambda$, and we then define the averaged probability measure $P_{\Lambda,T}$ via the identity

$$P_{\Lambda,T}(A) = \mathsf{E}_{\boldsymbol{h}}\left(P^{\boldsymbol{h}}_{\Lambda,T}(A)\right)$$

holding for any Borel set $A \subset \mathcal{C}([0,T];\mathbf{R})^\Lambda$ (here and in the sequel, $\mathsf{E}_{\boldsymbol{h}}$ denotes an average taken with respect to the realisations of the random external magnetic field $\boldsymbol{h}$). It turns out that the averaged probability measure $P_{\Lambda,T}$ may also be viewed as the weak solution associated with a new stochastic differential system.

Proposition 2.1. *For fixed $\Lambda \subset\subset \mathbf{Z}^d$ and $T > 0$,*

$$P_{\Lambda,T} = \mathsf{E}_{\boldsymbol{h}}\left[P^{\boldsymbol{h}}_{\Lambda,T}\right]$$

may be characterised as the law of the interacting diffusions system $(\mathcal{S}_{\Lambda,T})$ *given by*

$$\begin{cases} dx_t^i = dw_t^i - U'(x_t^i)dt + \beta \sum\limits_{j\sim i} x_t^j dt \\ \qquad + \sigma\beta \tanh\left\{\sigma\beta\left(x_t^i - x_0^i + \int\limits_0^t \left(U'(x_s^i) - \beta \sum\limits_{j\sim i} x_s^j\right) ds\right)\right\} dt, \\ Law(\boldsymbol{x}|_{t=0}) = \mu_0^{\otimes\Lambda} \qquad (i \in \Lambda,\ 0 \le t \le T). \end{cases}$$

Proof. Let p_T denote the probability law on $\mathcal{C}\big([0,T];\mathbf{R}\big)$ corresponding to the diffusion

$$dx_t = dw_t - U'(x_t)dt \tag{2.1}$$

having initial condition μ_0. We also consider the restriction of p_T to the σ-algebra $\mathcal{F}_t$ associated with the time interval $[0,t] \subset [0,T]$ and denote it by p_t, whilst $P_{\Lambda,t}$ similarly denotes the restriction of $P_{\Lambda,T}$ to the σ-algebra $\mathcal{F}_t^\Lambda$ in $\mathcal{C}\big([0,T];\mathbf{R}\big)^\Lambda$. Then according to the Fubini and Girsanov theorems:

$$P_{\Lambda,T} \ll p_T^{\otimes\Lambda},$$

and

$$M_t^\Lambda = \frac{dP_{\Lambda,t}}{dp_t^{\otimes\Lambda}}$$

is a positive $p_T^{\otimes\Lambda}$-martingale with mean 1 such that

$$\begin{aligned} M_t^\Lambda(\boldsymbol{x}) &= \mathsf{E}_{\boldsymbol{h}}\left[\exp\left\{\beta \sum_{i\in\Lambda} \int\limits_0^t \Big(\sum_{j\sim i} x_s^j + h^i\Big) dw_s^i - \frac{\beta^2}{2} \sum_{i\in\Lambda} \int\limits_0^t \Big(\sum_{j\sim i} x_s^j + h^i\Big)^2 ds\right\}\right] \\ &= \exp\left(\beta \sum_{i\in\Lambda} \int\limits_0^t \Big(\sum_{j\sim i} x_s^j\Big) dw_s^i - \frac{\beta^2}{2} \sum_{i\in\Lambda} \int\limits_0^t \Big(\sum_{j\sim i} x_s^j\Big)^2 ds\right) \\ &\quad \times \mathsf{E}_{\boldsymbol{h}}\left[\exp\left\{\beta\big(\boldsymbol{h}_\Lambda; A_{\beta,t}(\boldsymbol{x})\big) - \frac{\beta^2 t}{2}(\boldsymbol{h}_\Lambda; \boldsymbol{h}_\Lambda)\right\}\right], \end{aligned}$$

$A_{\beta,t}(\boldsymbol{x})$ being the Λ-dimensional vector defined by

$$A_{\beta,t}^i(\boldsymbol{x}) = w_t^i(\boldsymbol{x}) - \beta \int\limits_0^t \Big(\sum_{j\sim i} x_s^j\Big) ds = x_t^i - x_0^i + \int\limits_0^t \Big(U'(x_s^i) - \beta \sum_{j\sim i} x_s^j\Big) ds.$$

Observing that

$$\mathsf{E}_{\boldsymbol{h}}\Big[\exp\Big\{\beta(\boldsymbol{h}_\Lambda; A_{\beta,t}(\boldsymbol{x})) - \frac{\beta^2 t}{2}(\boldsymbol{h}_\Lambda;\boldsymbol{h}_\Lambda)\Big\}\Big]$$
$$= \Big(\prod_{i\in\Lambda}\cosh(\sigma\beta A^i_{\beta,t}(\boldsymbol{x}))\Big)\exp\Big\{-\frac{\sigma^2\beta^2 t}{2}|\Lambda|\Big\},$$

and using Ito's formula, we then obtain:

$$\begin{aligned}
\log M^\Lambda_t(\boldsymbol{x}) &=_{mart} \sum_{i\in\Lambda}\log\cosh\big(\sigma\beta A^i_{\beta,t}(\boldsymbol{x})\big) + \beta\sum_{i\in\Lambda}\int_0^t\Big(\sum_{j\sim i}x^j_s\Big)dw^i_s \\
&=_{mart} \sum_{i\in\Lambda}\int_0^t \sigma\beta\tanh\big(\sigma\beta A^i_{\beta,s}(\boldsymbol{x})\big)dA^i_{\beta,s}(\boldsymbol{x}) + \beta\sum_{i\in\Lambda}\int_0^t\Big(\sum_{j\sim i}x^j_s\Big)dw^i_s \\
&=_{mart} \sum_{i\in\Lambda}\int_0^t \sigma\beta\tanh\big(\sigma\beta A^i_{\beta,s}(\boldsymbol{x})\big)dw^i_s + \beta\sum_{i\in\Lambda}\int_0^t\Big(\sum_{j\sim i}x^j_s\Big)dw^i_s,
\end{aligned}$$

the sign $=_{mart}$ meaning here that the two $p_T^{\otimes\Lambda}$-semimartingales under consideration (on the left-hand side and on the right-hand side of the equality) have the same martingale part. At this point Girsanov's theorem may be applied a second time, which yields the announced characterisation of $P_{\Lambda,T}$. □

Naturally, one may also introduce the auxiliary variables

$$v^i_t(\boldsymbol{x}) = x^i_t - x^i_0 + \int_0^t\Big(U'(x^i_s) - \beta\sum_{j\sim i}x^j_s\Big)ds$$

and view $P_{\Lambda,T}$ as the $\boldsymbol{x}$-marginal of a Markov system of interacting diffusions taking values in $\big(\mathbf{R}^2\big)^\Lambda$, and the classical results of Shiga and Shimizu [20] may then be applied to establish that such Markov system of interacting diffusions also has a unique strong solution when extending the spatial index set to $\mathbf{Z}^d$ and letting t vary in $[0,+\infty[$. But let us first extend the spatial index set to $\mathbf{Z}^d$ whilst keeping $[0,T]$ as our time horizon, and call Q_T the $\boldsymbol{x}$-marginal of the unique strong solution associated with

$$(\mathcal{S}_T)\begin{cases} dx^i_t = dv^i_t - U'(x^i_t)dt + \beta\sum_{j\sim i}x^j_t dt, \\ dv^i_t = dw^i_t + \sigma\beta\tanh\big\{\sigma\beta v^i_t\big\}dt, \\ v^i_0 \equiv 0, \quad law\big(\,\boldsymbol{x}|_{t=0}\big) = \mu_0^{\otimes\mathbf{Z}^d} \quad (i\in\mathbf{Z}^d,\ 0\le t\le T). \end{cases}$$

Q_T is a probability measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$, and just as in [2] we may view Q_T as a Gibbs measure corresponding to a certain translation invariant family of interaction functionals on this infinite-dimensional path space. Indeed, for fixed $\Lambda \subset\subset \mathbf{Z}^d$ the Radon−Nykodím derivative $M_T^\Lambda = dP_{\Lambda,T}/dp_T^{\otimes\Lambda}$ has a Girsanov exponent which may be decomposed as

$$\begin{aligned}
\log M_T^\Lambda(\boldsymbol{x}) &= \sum_{i\in\Lambda}\int_0^T \Big(\sigma\beta\tanh(\sigma\beta A^i_{\beta,s}(\boldsymbol{x})) + \beta\sum_{j\sim i} x^j_s\Big)\,dw^i_s \\
&\quad - \frac{1}{2}\sum_{i\in\Lambda}\int_0^T \Big(\sigma\beta\tanh(\sigma\beta A^i_{\beta,s}(\boldsymbol{x})) + \beta\sum_{j\sim i} x^j_s\Big)^2 ds \\
&= \sum_{i\in\Lambda}\Big\{\int_0^T \sigma\beta\tanh(\sigma\beta A^i_{\beta,s}(\boldsymbol{x}))dA^i_{\beta,s}(\boldsymbol{x})) \\
&\quad - \frac{\sigma^2\beta^2}{2}\int_0^T \tanh^2(\sigma\beta A^i_{\beta,s}(\boldsymbol{x}))\,ds - \frac{\beta^2}{2}\int_0^T\Big(\sum_{j\sim i}x^j_s\Big)^2 ds \\
&\quad + \beta\int_0^T\Big(\sum_{j\sim i}x^j_s\Big)dx^i_s + \beta\int_0^T\Big(\sum_{j\sim i}x^j_s\Big)U'(x^i_s)ds\Big\} \\
&= \sum_{i\in\Lambda}\Big\{\log\cosh\big(\sigma\beta A^i_{\beta,T}(\boldsymbol{x})\big) - \frac{\sigma^2\beta^2}{2}\int_0^T\Big(\tanh^2\big(\sigma\beta A^i_{\beta,s}(\boldsymbol{x})\big) \\
&\quad + \frac{1}{1+\sigma^2\beta^2 A^i_{\beta,s}(\boldsymbol{x})^2}\Big)ds + \beta\int_0^T\Big(\sum_{j\sim i}x^j_s\Big)U'(x^i_s)\,ds \\
&\quad - \frac{\beta^2}{2}\int_0^T\Big(\sum_{j\sim i}x^j_s\Big)^2 ds\Big\} + \beta\sum_{j\sim i}\big[x^i_T x^j_T - x^i_0 x^j_0\big],
\end{aligned}$$

which suggests that the projections of Q_T onto a space $\mathcal{C}\big([0,T];\mathbf{R}\big)^\Lambda$ of finite-volume configurations satisfy the DLR equations associated with a translation invariant family $\Psi = \big(\psi_A\big)_{A\subset\subset \mathbf{Z}^d}$ of interaction functionals defined on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$. More precisely, letting

$$\Delta^i = \big\{j\in\mathbf{Z}^d : |j-i| = 0 \text{ or } 1\big\},$$

one may successively define the translation invariant functionals ψ_A for each finite subset $A \subset \mathbf{Z}^d$ through

$$\begin{aligned}
\psi_{\{i\}}(\boldsymbol{x}) &= \beta^2 d \int_0^T (x_s^i)^2 ds, \\
\psi_{\{i,j\}}(\boldsymbol{x}) &= -\beta \Big\{ \big[x_T^i x_T^j - x_0^i x_0^j\big] + \int_0^T \big(U'(x_s^i) x_s^j + U'(x_s^j) x_s^i\big) ds \Big\}, \quad |j-i| = 1, \\
\psi_{\{i,j\}}(\boldsymbol{x}) &= 2\beta^2 \int_0^T x_s^i x_s^j ds \quad \text{when } |j-i| = \sqrt{2}, \qquad (2.2) \\
\psi_{\{i,j\}}(\boldsymbol{x}) &= \beta^2 \int_0^T x_s^i x_s^j ds \quad \text{when } |j-i| = 2, \\
\psi_{\Delta^i}(\boldsymbol{x}) &= -\log\cosh\big(\sigma\beta A_{\beta,T}^i(\boldsymbol{x})\big) + \frac{\sigma^2\beta^2}{2} \int_0^T \Big(\tanh^2\big(\sigma\beta A_{\beta,s}^i(\boldsymbol{x})\big) \\
&\qquad + \frac{1}{1 + \sigma^2\beta^2 A_{\beta,s}^i(\boldsymbol{x})^2} \Big) ds,
\end{aligned}$$

letting further $\psi_A \equiv 0$ whenever A is not of the preceding type, and the Radon-Nykodím derivative M_T^Λ may then be expressed as the exponential of the sum

$$-\sum_{A \subset \Lambda} \psi_A(\boldsymbol{x}),$$

Λ still being equipped with its periodic boundary conditions.

One may check that the infinite volume dynamics Q_T satisfies the DLR equations relative to the interaction $\big(\psi_A\big)_{A \subset\subset \mathbf{Z}^d}$, and it then remains to show there are no other Gibbs measures corresponding to $\big(\psi_A\big)_{A \subset\subset \mathbf{Z}^d}$ and to the reference measure $p_T^{\otimes \mathbf{Z}^d}$.

Proposition 2.2. *Let Q be a probability measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$, and assume that Q is a Gibbs measure corresponding to the interaction Ψ and to the reference measure $p_T^{\otimes \mathbf{Z}^d}$. Then Q is the $\boldsymbol{x}$-marginal of an infinite dimensional diffusion $\big(\boldsymbol{x}_t, \boldsymbol{v}_t\big)_{0 \le t \le T}$ solving $\big(\mathcal{S}_T\big)$, consequently: $Q = Q_T$.*

Proof. The identification of any Gibbs measure as a weak solution for an infinite dimensional system of interacting diffusions follows from an integration by parts formula that was developped and used in this context by Cattiaux, Rœlly and

Zessin, see in particular Théorème 2.11 in [5]. As for the fact that Q then has to coincide with Q_T, it follows from unicity in Shiga and Shimizu's classical results (see Theorem 4.1 in [20]); note that we have equipped our interacting diffusions system with a product initial condition, therefore there can be no such phenomena as a phase transition occuring at time $t = 0$. □

Since Q_T is a Gibbs measure corresponding to a translation invariant interaction Ψ, it should be expected to satisfy some spatial large deviations estimates for $\Lambda \nearrow \mathbf{Z}^d$; there are indeed several reference papers establishing large deviations estimates for the empirical process of a spin system evolving under a Gibbs measure on the configuration space $X^{\mathbf{Z}^d}$ (X being a Polish space), see e.g. [4] or [10]. The next subsection is devoted to a precise statement of such a spatial Large Deviations Principle (LDP) for the Gibbs measure Q_T.

2.2. Large deviations of the empirical process

For each cubic box $\Lambda \subset\subset \mathbf{Z}^d$ and for each configuration $\boldsymbol{x} \in \mathcal{C}\big([0,T];\mathbf{R}\big)^{\Lambda}$, one may define a probability measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$, the empirical process associated with $\boldsymbol{x}$, in the following way:

$$\hat{\pi}_{\boldsymbol{x}}^{(\Lambda)} = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\delta_{\left(per.\boldsymbol{x}\right)^{(i)}},$$

where $\boldsymbol{y} = per.\boldsymbol{x} \in \mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$ is a Λ-periodic configuration on $\mathbf{Z}^d$ whose restriction to Λ coincides with $\boldsymbol{x}$, and where $\big(\boldsymbol{y}^{(i)}\big)^j = y^{j+i}$, for all $j \in \mathbf{Z}^d$.

The empirical process $\hat{\pi}_{\boldsymbol{x}}^{(\Lambda)}$ thus defines a shift invariant probability measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$, whose 1-site marginal coincides with the empirical measure associated with $\boldsymbol{x}$. Now in the case where $\boldsymbol{x}$ is distributed according to a product measure $p^{\otimes\Lambda}$ on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\Lambda}$, the law of the empirical process obeys a Large Deviation Principle (LDP) on the scale $|\Lambda|$ and according to a good rate function

$$\mathcal{H} : \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big) \longrightarrow [0;+\infty]$$

known as the specific entropy relative to $p^{\otimes\mathbf{Z}^d}$ and defined on the set $\mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big)$ consisting of all shift invariant probability measures on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}$ as the following limit:

$$\mathcal{H}(\pi) = \lim_{\Lambda\nearrow\mathbf{Z}^d}\frac{1}{|\Lambda|}H\big(\pi_\Lambda \mid p^{\otimes\Lambda}\big),$$

π_Λ denoting here the Λ-marginal of π, and $H\big(\cdot \mid p^{\otimes\Lambda}\big) : \mathcal{M}\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\Lambda}\big) \to [0;+\infty]$ being the relative entropy corresponding to $p^{\otimes\Lambda}$.

As a convenient generalisation of the preceding LDP, one may then consider the case where $\boldsymbol{x}$ is being distributed according to the projection onto $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\Lambda}$ (say with periodic boundary conditions) of a Gibbs measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathrm{Z}^d}$ corresponding to the reference measure $p^{\otimes \mathrm{Z}^d}$ and to a translation invariant interaction $\Psi = \big(\psi_A\big)_{A\subset\subset \mathrm{Z}^d}$. As was shown in several papers (see for example [4] or [10]), the law of the empirical process then also obeys a LDP, at least when the interaction Ψ satisfies an additional boundedness assumption such as

$$\sum_{A\ni O} \|\psi_A\|_\infty < +\infty, \tag{2.3}$$

and the new rate functional $\mathcal{I}^\Psi : \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathrm{Z}^d}\big) \to [0;+\infty]$ may then be defined by

$$\mathcal{I}^\Psi(\pi) = \mathcal{H}(\pi) - \int\limits_{\mathcal{C}\left([0,T];\mathrm{R}\right)^{\mathrm{Z}^d}} \mathcal{U}^\Psi(\boldsymbol{x})d\pi(\boldsymbol{x}),$$

where $\mathcal{U}^\Psi(\boldsymbol{x}) = \sum_{A\ni O} \psi_A(\boldsymbol{x})/|A|$.

We are in the situation where $p = p_T$ and where Ψ coincides with the interaction (2.2) defined in the preceding subsection; in such a situation, (2.3) does not hold true: our translation invariant interaction Ψ has a finite range, but the individual functionals ψ_A fail to be uniformly bounded. Nevertheless, one may still prove in such a context that the law of the empirical process satisfies a LDP on the scale $|\Lambda|$ and according to the good rate functional $\mathcal{I}^\Psi$. Of course some verifications are needed in order to make sure that such LDP still holds true, and some further verifications are needed in order to prove that the following variational principle is indeed valid for Ψ.

Proposition 2.3. *Let $Q \in \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathrm{Z}^d}\big)$. Then Q is a minimiser associated with the good rate functional $\mathcal{I}^\Psi$ if and only if Q is a Gibbs measure on $\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathrm{Z}^d}$ corresponding to the interaction Ψ and to the reference measure $p_T^{\otimes \mathrm{Z}^d}$.*

Sketch of proof. The validity of such a variational principle is established in [9, Chapter 15] for Gibbs measures satisfying the summability condition (2.3). As for the present situation, one may decompose the proof into the following three steps:

1°). *Any $Q \in \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathrm{Z}^d}\big)$ for which the integral*

$$\int\limits_{\mathcal{C}\left([0,T];\mathrm{R}\right)^{\mathrm{Z}^d}} \mathcal{U}^\Psi(\boldsymbol{x})dQ(\boldsymbol{x})$$

is finite satisfies

$$\mathcal{H}(Q) \geq \int_{\mathcal{C}\left([0,T];\mathrm{R}\right)^{\mathrm{Z}^d}} \mathcal{U}^{\Psi}(\boldsymbol{x})\, dQ(\boldsymbol{x}).$$

Indeed, using the L^1 version of the multidimensional ergodic theorem enables us to view the integral in the right-hand side above as

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \int \mathcal{U}^{\Psi}(\boldsymbol{x}^{(i)})\, dQ(\boldsymbol{x}),$$

and the limit above may then be seen to coincide with

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \int \mathcal{U}^{\Psi}((per.\, \boldsymbol{x}_\Lambda)^{(i)})\, dQ_\Lambda(\boldsymbol{x}_\Lambda),$$

where Q_Λ stands for the Λ-marginal of Q. Hence

$$\begin{aligned}\mathcal{H}(Q) &- \int \mathcal{U}^{\Psi}(\boldsymbol{x}) dQ(\boldsymbol{x}) \\ &= \lim_{\Lambda} \frac{1}{|\Lambda|} \int dQ_\Lambda(\boldsymbol{x}_\Lambda) \Big\{ \sum_{i\in\Lambda} \mathcal{U}^{\Psi}((per.\, \boldsymbol{x}_\Lambda)^{(i)}) - \ln\Big(\frac{dQ_\Lambda}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big)\Big\},\end{aligned}$$

and Jensen's inequality applied to ln then yields

$$\begin{aligned}&\int dQ_\Lambda \Big\{ \sum_{i\in\Lambda} \mathcal{U}^{\Psi}((per.\, \boldsymbol{x}_\Lambda)^{(i)}) - \ln\Big(\frac{dQ_\Lambda}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big)\Big\} \\ &\qquad \leq \ln\Big\{ \int dQ_\Lambda \frac{\exp\big(\sum_{i\in\Lambda} \mathcal{U}^{\Psi}((per.\, \boldsymbol{x}_\Lambda)^{(i)})\big)}{dQ_\Lambda/dp_T^{\otimes\Lambda}(\boldsymbol{x}_\Lambda)} \Big\} = 0\end{aligned}$$

for each fixed Λ.

2°). *Q_T is such that*

$$\mathcal{H}(Q_T) = \int \mathcal{U}^{\Psi}(\boldsymbol{x}) dQ_T(\boldsymbol{x}).$$

In order to prove this equality, one uses the fact that Q_T is a Gibbs measure corresponding to Ψ, so that for each finite box $\Lambda \subset \mathbf{Z}^d$:

$$dQ_T(\boldsymbol{x}_\Lambda) = \int dQ_T(\boldsymbol{x}_{\Lambda^c}) \frac{\exp\big\{-\beta \sum_{\Gamma\cap\Lambda\neq\emptyset} \psi_\Gamma(\boldsymbol{x}_\Lambda \vee \boldsymbol{x}_{\Lambda^c})\big\}}{\mathcal{Z}_\Lambda^{\Psi}(\boldsymbol{x}_{\Lambda^c})},$$

where

$$\mathcal{Z}_\Lambda^{\Psi}(\boldsymbol{x}_{\Lambda^c}) = \int dp_T^{\otimes\Lambda}(\boldsymbol{x}_\Lambda) \exp\Big\{-\beta \sum_{\Gamma\cap\Lambda\neq\emptyset} \psi_\Gamma(\boldsymbol{x}_\Lambda \vee \boldsymbol{x}_{\Lambda^c})\Big\},$$

$\boldsymbol{x}_\Lambda \vee \boldsymbol{x}_{\Lambda^c}$ denoting a combination of configurations $\boldsymbol{x}_\Lambda$ (in the volume Λ) and $\boldsymbol{x}_{\Lambda^c}$ (in Λ^c).

Using again the L^1 multidimensional ergodic theorem enables us to view both $\mathcal{H}(Q)$ and $\int \mathcal{U}^\Psi dQ$ as the following limit:

$$\mathcal{H}(Q_T) - \int_\Omega \mathcal{U}^\Psi(\boldsymbol{x}) dQ_T(\boldsymbol{x}) = \lim_\Lambda \frac{1}{|\Lambda|} \int dQ_T(\boldsymbol{x}_{\Lambda^c}) - \ln\big(\mathcal{Z}_\Lambda^\Psi(\boldsymbol{x}_{\Lambda^c})\big),$$

and using Jensen's inequality applied to $(-\ln)$ and to the probability measure

$$\exp\Big\{ -\beta \sum_{\Gamma\cap\Lambda\neq\emptyset} \psi_\Gamma(\boldsymbol{x}_\Lambda^{(\Lambda)})\Big\} \cdot dp_T^{\otimes\Lambda}(\boldsymbol{x}_\Lambda)$$

finishes the proof of the fact that

$$\mathcal{H}(Q_T) - \int \mathcal{U}^\Psi(\boldsymbol{x}) dQ_T(\boldsymbol{x}) \leq 0.$$

3°). *Any* $Q \in \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbb{Z}^d}\big)$ *satisfying*

$$\int_\Omega \mathcal{U}^\Psi dQ = \mathcal{H}(Q) < +\infty$$

is also such that

$$\lim_\Lambda \frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_\Lambda}{dQ_{T,\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) = 0$$

(*where* $Q_{T,\Lambda}$ *and* Q_Λ *denote the* Λ*-marginals of* Q_T *and* Q *respectively*). Indeed, for each finite box Λ one has

$$\begin{aligned}
&\frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_\Lambda}{dQ_{T,\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) \\
&\quad = \frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_\Lambda}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) - \frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_{T,\Lambda}}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) \\
&\quad = \frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_\Lambda}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) - \frac{1}{|\Lambda|} \int dQ_\Lambda(\boldsymbol{x}_\Lambda) \\
&\qquad \times \ln\Big\{ \int dQ_{T,\Lambda^c}(\boldsymbol{x}_{\Lambda^c}) \frac{\exp\{-\beta \sum_{\Gamma\cap\Lambda\neq\emptyset} \psi_\Gamma(\boldsymbol{x}_\Lambda \vee \boldsymbol{x}_{\Lambda^c})\}}{\mathcal{Z}_\Lambda^\Psi(\boldsymbol{x}_{\Lambda^c})} \Big\} \\
&\quad \leq \frac{1}{|\Lambda|} \int \ln\Big(\frac{dQ_\Lambda}{dp_T^{\otimes\Lambda}}(\boldsymbol{x}_\Lambda)\Big) dQ_\Lambda(\boldsymbol{x}_\Lambda) - \frac{1}{|\Lambda|} \int dQ_\Lambda(\boldsymbol{x}_\Lambda) \int dQ_{T,\Lambda^c}(\boldsymbol{x}_{\Lambda^c}) \\
&\qquad \times \Big\{ \Big(-\beta \sum_{\Gamma\cap\Lambda\neq\emptyset} \psi_\Gamma(\boldsymbol{x}_\Lambda \vee \boldsymbol{x}_{\Lambda^c}) \Big) - \ln\big(\mathcal{Z}_\Lambda^\Psi(\boldsymbol{x}_{\Lambda^c})\big) \Big\},
\end{aligned}$$

and in the right-hand side of the preceding inequality the first term converges to $\mathcal{H}(Q)$, while the second term has a limit that may easily be seen to coincide with $\int_\Omega \mathcal{U}^\Psi dQ$, using once again the multidimensional ergodic theorem.

In view of 1°), 2°) and 3°), any $Q \in \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big)$ minimising the good rate functional $\mathcal{I}^\Psi$ has to coincide with Q_T. □

Using the unicity of Q_T as a minimiser for the good rate function $\mathcal{I}^\Psi$, we may now state the following Strong Law of Large Numbers (SLLN) as a corollary to the annealed large deviation estimates already available.

Corollary 2.1. P-*a.s.* $(\boldsymbol{h})$*, the law of the empirical process* $p_T^{\otimes \mathbf{Z}^d}$ *under* $dP^{\boldsymbol{h}}_{\Lambda,T}(\boldsymbol{x})$ *converges to a Dirac mass concentrated at* Q_T *as* $\Lambda \nearrow \mathbf{Z}^d$*,* $\mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big)$ *being equipped with the topology of weak convergence.*

Proof. Consider a metric D on $\mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big)$ compatible with the topology of weak convergence, and for fixed $\varepsilon > 0$, let

$$\mathcal{A}_\varepsilon = \big\{Q \in \mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big) \mid D\big(Q;Q_T\big) \geq \varepsilon\big\}.$$

$\mathcal{I}^\Psi$ defines a good rate function on $\mathcal{M}_s\big(\mathcal{C}\big([0,T];\mathbf{R}\big)^{\mathbf{Z}^d}\big)$, so that it certainly attains its minimum m_ε on $\mathcal{A}_\varepsilon$, and m_ε is positive since $Q_T \notin \mathcal{A}_\varepsilon$. Applying the large deviations upper bound to $\mathcal{A}_\varepsilon$ enables one to choose a finite cubic box $\Lambda_0 \subset\subset \mathbf{Z}^d$ such that

$$P_\Lambda\big\{\boldsymbol{x} \mid \hat{\pi}^{(\Lambda)}_{\boldsymbol{x}} \in \mathcal{A}_\varepsilon\big\} \leq \exp\Big\{-\frac{m}{2}|\Lambda|\Big\} \tag{2.4}$$

whenever $\Lambda \supset \Lambda_0$.

On the other hand, according to Chebyshev's inequality, for any $\delta > 0$,

$$\mathsf{P}\,\big\{\boldsymbol{h} \mid P^{\boldsymbol{h}}_\Lambda\big\{\boldsymbol{x} \mid \hat{\pi}^{(\Lambda)}_{\boldsymbol{x}} \in \mathcal{A}_\varepsilon\big\} > \delta\big\} \leq \frac{P_\Lambda\big\{\boldsymbol{x} \mid \hat{\pi}^{(\Lambda)}_{\boldsymbol{x}} \in \mathcal{A}_\varepsilon\big\}}{\delta}.$$

As a consequence of (2.3), one has

$$\sum_\Lambda \frac{P_\Lambda\big\{\boldsymbol{x} \mid \hat{\pi}^{(\Lambda)}_{\boldsymbol{x}} \in \mathcal{A}_\varepsilon\big\}}{\delta} < +\infty,$$

and making use of the Borel−Cantelli lemma then finishes the proof. □

The preceding SLLN for the empirical process $\hat{\pi}^{(\Lambda)}_{\boldsymbol{x}}$ justifies our interest in the dynamics Q_T, which may now be viewed as an asymptotic dynamics obtained by letting $\Lambda \nearrow \mathbf{Z}^d$ and performing spatial averages. As a second step, one may then extend the time horizon to $[0;+\infty[$ and wonder about the (space and time) decorrelation properties of the infinite-dimensional dynamics Q_∞, given as the $\boldsymbol{x}$-marginal of the stochastic differential system

$$(\mathcal{S}_\infty)\begin{cases} dx_t^i = dv_t^i - U'(x_t^i)dt + \beta \sum_{j\sim i} x_t^j dt \\ dv_t^i = dw_t^i + \sigma\beta \tanh\{\sigma\beta v_t^i\}dt \\ Law(\boldsymbol{x}|_{t=0}) = \mu_0^{\otimes \mathbf{Z}^d} \qquad (i \in \mathbf{Z}^d, t \geq 0) \end{cases}$$

We are going to consider Q_∞ in the high temperature regime (where β is "small") and prove, among other facts, that the corresponding spin system decorrelates exponentially fast in space and time, using a cluster expansion on the path space $\Omega = \mathcal{M}_s\big(\mathcal{C}\big([0,+\infty[;\mathbf{R}\big)^{\mathbf{Z}^d}\big)$. To this end, we first need to present Q_∞ as a Gibbs measure on Ω *in a space-time sense*, the reference measure being now made of independent *bridges* based on the diffusion (2.1).

3. Cluster expansion in space-time

3.1. Presentation of Q_∞ as a space-time Gibbs measure

For each finite $\Lambda \subset \mathbf{Z}^d$, let

$$\Lambda^+ = \big\{i \in \mathbf{Z}^d \mid (i \in \Lambda) \text{ or } (i \sim j \text{ for some } j \in \Lambda)\big\}$$

and $\partial\Lambda = \big(\Lambda^+\big)^+ \setminus \Lambda$. For any open interval $I =]a_1;a_2[\subset \mathbf{R}_+$, define the enlargement of I by $I^+ = [0;a_2]$, and let $\mathcal{V}$ denote the set consisting of all space-time windows V of the form $V = \Lambda \times I$, where Λ is a finite subset in $\mathbf{Z}^d$ and I an open interval in $\mathbf{R}_+$. Following [6], we define a *forward* (resp. *backward*) σ-field $\mathcal{F}_V$ (resp. $\hat{\mathcal{F}}_V$) on path space $\Omega = \mathcal{C}\big(\mathbf{R}_+;\mathbf{R}\big)^{\mathbf{Z}^d}$ by

$$\mathcal{F}_V = \boldsymbol{\sigma}\big\{\omega_t^i;\ i \in \Lambda^{++}, t \in I^+\big\},$$
$$\hat{\mathcal{F}}_V = \boldsymbol{\sigma}\big\{\omega_t^i; (i;t) \notin V\big\},$$

and the *boundary* σ-field $\partial\mathcal{F}_V$ is then given by

$$\partial\mathcal{F}_V = \mathcal{F}_V \cap \hat{\mathcal{F}}_V.$$

We now need to equip Ω with a *reference specification* $\big(\Pi_V^0\big)_{V\in\mathcal{V}}$, i.e. a Markov kernel on

$$\big(\Omega; \{\mathcal{F}_V\}_{V\in\mathcal{V}}; \{\hat{\mathcal{F}}_V\}_{V\in\mathcal{V}}\big)$$

corresponding to a "free" situation (where the diffusions $\big\{(x_t^i)_{t\geq 0}; i \in \mathbf{Z}^d\big\}$ do not interact with each other). In the present setting, Ω may be conveniently equipped with the reference specification $\big(\Pi_V^0\big)_{V\in\mathcal{V}}$ defined by

$$\forall V \in \mathcal{V},\ \forall A \in \mathcal{F}_V, \quad \Pi_V^0(A) = P\big(A \mid \hat{\mathcal{F}}_V\big),$$

where $P = p^{\otimes \mathbf{Z}^d}$ and p is the stationary weak solution of the S.D.E. (2.1) with initial condition $d\nu_0(x)$ proportional to $\exp\{-2U(x)\}dx$. In order to come to the Gibbsian specification corresponding to the asymptotic dynamics Q_∞, we then let

$$\Lambda_0 = \{ i \in \mathbf{Z}^d \mid i = O \text{ or } i \sim O \}$$

and define the *potential* $\Phi = (\phi_V)_{V \in \mathcal{V}}$ on Ω by $\phi_{\Lambda \times I} \equiv 0$ whenever Λ is not a translate of Λ_0 and

$$\begin{aligned}\phi_{(i+\Lambda_0)\times I}(\boldsymbol{x}) = &-\int_I \Big(\beta \sum_{j\sim i} x_t^j + \sigma\beta \tanh \sigma\beta \tilde{B}_t^i(\boldsymbol{x})\Big) dB_t^i(\boldsymbol{x}) \\ &+ \frac{1}{2}\int_I \Big(\beta \sum_{j\sim i} x_t^j + \sigma\beta \tanh \sigma\beta \tilde{B}_t^i(\boldsymbol{x})\Big)^2 dt,\end{aligned}$$

with

$$B_t^i(\boldsymbol{x}) = x_t^i - x_0^i + \int_0^t U'(x_u^i) du$$

and

$$\tilde{B}_t^i(\boldsymbol{x}) = x_t^i - x_0^i + \int_0^t \Big(U'(x_u^i) - \beta \sum_{j\sim i} x_u^j\Big) du = B_t^i(\boldsymbol{x}) - \beta \int_0^t \Big(\sum_{j\sim i} x_u^j\Big) du,$$

otherwise. At this stage, one should remark that $\phi_{(i+\Lambda_0)\times I} \in L^2(P)$, so that $\phi_{(i+\Lambda_0)\times I}(\boldsymbol{x})$ is finite P-a.s. $(\boldsymbol{x})$, say, on $\Omega' \subset \Omega$. We then define the Hamiltonian $H = (H_V)_{V \in \mathcal{V}}$ on Ω' by

$$H_V(\boldsymbol{x}) = \sum_{\Lambda' \cap \Lambda \neq \emptyset} \phi_{\Lambda' \times I}(\boldsymbol{x}) = -\sum_{i \in \Lambda^+} \Big[\int_I b_t^i(\boldsymbol{x}) dB_t^i(\boldsymbol{x}) - \frac{1}{2} \int_I b_t^i(\boldsymbol{x})^2 dt \Big],$$

where

$$b_t^i(\boldsymbol{x}) = \beta \sum_{j\sim i} x_t^j + \sigma\beta \tanh \sigma\beta \tilde{B}_t^i(\boldsymbol{x}).$$

Observe that Φ and H are both spatially translation invariant, and that H_V is $\mathcal{F}_V$-measurable. We finally let $(\Pi_V^H)_{V \in \mathcal{V}}$ denote the new specification given by

$$\Pi_V^H(\boldsymbol{\omega}; d\boldsymbol{\omega}') = (\mathcal{Z}_V^H(\boldsymbol{\omega}))^{-1} \mathbf{1}_{\Omega'}(\boldsymbol{\omega}') \exp\{-H_V(\boldsymbol{\omega}')\} \Pi_V^0(\boldsymbol{\omega}; d\boldsymbol{\omega}')$$

if $0 < \mathcal{Z}_V^H(\boldsymbol{\omega}) < +\infty$ and zero otherwise, $\mathcal{Z}_V^H(\boldsymbol{\omega}) = \int_{\Omega'} \exp(-H_V(\boldsymbol{\omega}'))\Pi_V^0(\boldsymbol{\omega}; d\boldsymbol{\omega}')$ being the ($\partial\mathcal{F}_V$-measurable) normalisation factor corresponding to the space-time window V and to the boundary condition $\boldsymbol{\omega}$. So for each $\boldsymbol{\omega} \in \Omega$, $\Pi_V^H(\boldsymbol{\omega}; d\boldsymbol{\omega}')$

is a probability measure on Ω whose support is included in Ω'; one then says that a probability measure Q on Ω is a space-time Gibbs state corresponding to the specification $\Pi_V^H(\boldsymbol{\omega}; d\boldsymbol{\omega'})$ whenever the identity

$$Q(A \mid \hat{\mathcal{F}}_V) = \Pi_V^H(A), \quad Q\text{-a.s.}, \tag{3.1}$$

holds true for each $V \in \mathcal{V}$ and all $A \in \mathcal{F}_V$.

The asymptotic dynamics Q_∞ may now be presented as a limit corresponding to the finite-dimensional dynamics $\tilde{Q}_n$ given by

$$\tilde{Q}_n\big(d\boldsymbol{x}_n\big) = \exp\big\{ - H_{V_n}(\boldsymbol{x})\big\} \underset{i \in \Lambda_n^{++}}{\otimes} p\,|_{\mathcal{F}_{I_n}}(dx^i),$$

$V_n = \Lambda_n \times I_n$ being here a sequence of bounded space-time windows increasing up to $\mathbf{Z}^d \times \mathbf{R}_+$ (so that $\Lambda_n \nearrow \mathbf{Z}^d$, $I_n =]0; T_n[$, $T_n \to +\infty$), and $\boldsymbol{x}_n$ denoting the restriction of a configuration $\boldsymbol{x}$ to $\Lambda_n^{++} \times I_n^+$. To be more precise, one may extend each of the probability measures $\tilde{Q}_n$ to Ω by letting

$$Q_n(d\boldsymbol{x}) = \exp\big(- H_{V_n}(\boldsymbol{x})\big) P(d\boldsymbol{x}),$$

and observe that Q_n converges weakly to Q_∞. On the other hand, each of the probability measures Q_n is actually a mixture of the local specifications $\Pi_{V_n}^H$, which shows that the weak limit Q_∞ is a Gibbs measure corresponding to $\Pi^H = \big(\Pi_V^H\big)_{V \in \mathcal{V}}$ (see Lemma 2 and Proposition 1 in [17]). Moreover we may now derive a cluster expansion in space-time for some finite-volume approximation Q_n, with n arbitrarily large, and look for some exponential bounds for the corresponding contour coefficients: as long as these bounds depend only on the small parameter β, such cluster expansion will also be valid for Q_∞ itself. The main consequence of interest to us is that one may then establish that the interacting diffusions $\big\{(x_t^i)_{t\geq 0}; i \in \mathbf{Z}^d\big\}$ driven by Q_∞ decorrelate exponentially fast in space and time.

Theorem 3.1. *There exists $\beta_0 > 0$ such that for each $0 < \beta \leq \beta_0$, one may find positive constants c, C for which*

$$\left| \int_\Omega F(\boldsymbol{x})G(\boldsymbol{x})dQ_\infty(\boldsymbol{x}) - \int_\Omega F dQ_\infty \cdot \int_\Omega G dQ_\infty \right| \leq C \exp\{-cD(V_1; V_2)\}$$

whenever $F, G : \Omega \to \mathbf{R}$ are measurable with respect to $\mathcal{F}_{V_1}$, $\mathcal{F}_{V_2}$ respectively and such that $\|F\|_\infty$, $\|G\|_\infty \leq 1$, $D(V_1; V_2)$ standing for a measure of the distance separating the bounded space-time windows $V_1 = \Lambda_1 \times I_1$ and $V_2 = \Lambda_2 \times I_2$.

Considering bounded functionals $F(\boldsymbol{x})$ which depend on $\boldsymbol{x}$ only through $\boldsymbol{x}_t^\Lambda = \big\{(x_t^i); i \in \Lambda\big\}$ yields the following exponential ergodicity statement:

Corollary 3.1. *For each $0 < \beta < \beta_0$, there exist positive constants c, C such that*

$$\Big| \int F\big(\boldsymbol{x}_t^\Lambda\big) F\big(\boldsymbol{x}_{t+T}^\Lambda\big) dQ_\infty - \int F\big(\boldsymbol{x}_t^\Lambda\big) dQ_\infty \cdot \int F\big(\boldsymbol{x}_{t+T}^\Lambda\big) dQ_\infty \Big| \leq C \cdot e^{-cT}$$

whenever $F : \Omega \to [0;1]$ is a bounded measurable functional of $\boldsymbol{x}$ depending on $\boldsymbol{x}$ only through $\boldsymbol{x}_t^\Lambda = \big\{(x_t^i); i \in \Lambda\big\}$, for some t and some finite box $\Lambda \subset \mathbf{Z}^d$ (c and C do not depend on Λ).

In the next subsection, we establish the validity of a cluster expansion in space-time for Q_∞ considered in the high temperature regime, and give exponential estimates for the corresponding contour coefficients in Proposition 3.2. The above theorem may then be seen to follow from the validity of such exponential estimates, as was established in a general setting in [16] (§3 in Chapter 3).

3.2. Construction of a cluster expansion for Q_∞

For simplicity, we first consider the Markovian case where $\sigma^2 = 0$ and derive a space-time cluster expansion for some finite-volume approximation Q_n following the method developped by Dai Pra and the second author in [6]. In our case some extra care has to be taken in the exponential estimation of the contour coefficients, since the interaction term $\beta\big(\sum_{j\sim i} x_t^j\big) dt$ appearing in the drift of our stochastic differential is not a uniformly bounded one. We then consider the non-Markovian setting where $\sigma^2 > 0$; in this case the notion of contour has to be modified, but in the end one may again derive a satisfactory cluster expansion in space-time, where the validity of some exponential estimates for the contour coefficients may be seen to follow from the uniform boundedness of the original random field variables $h^i \in (\pm\sigma)$.

3.2.1. Markovian case ($\sigma^2 = 0$)

We recall that ν_0 denotes the probability measure on $\mathbf{R}$ yielding a reversible equilibrium measure for the diffusion (2.1) and equip the stochastic differential system $\mathcal{S}_\infty$ with the initial condition $\nu_0^{\otimes \mathbf{Z}^d}$. For fixed $y, z \in \mathbf{R}$, we further let $\mathcal{W}_I^{y,z}(d\omega)$ denote a stochastic bridge associated with the diffusion (2.1) considered on the interval I. Fixing $a > 0$, we then let $I_n = [0; na]$, whereas $\big(\Lambda_n\big)_{n\geq 1}$ is defined recursively through the relations

$$\Lambda_0 = \big\{ i \in \mathbf{Z}^d \mid |i| = 0 \text{ or } |i| = 1 \big\}, \quad \Lambda_{n+1} = \Lambda_n^+, \ \forall n \in \mathbf{N}.$$

In this Markovian context, the partition function $\mathcal{Z}_n$ associated with Q_n,

$$\mathcal{Z}_n = \int\limits_\Omega \exp\{-H_{V_n}(\boldsymbol{x})\} P(d\boldsymbol{x}),$$

may be decomposed as the following integral over $\boldsymbol{y_n} \in \mathbf{R}^{(n+1)|\Lambda_{n+2}|}$:

$$\mathcal{Z}_n = \int\limits_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}} Z_n(\boldsymbol{y_n}) \prod_{k\in\Lambda_{n+2}, 0\le j\le n-1} q_a(y_{j+1}^k; y_j^k) \bigotimes_{k\in\Lambda_{n+2}, 0\le j\le n} \nu_0(dy_j^k), \tag{3.2}$$

$q_a(y_{j+1}^k; y_j^k) d\nu_0(y_j^k)$ denoting a transition probability density on any time interval of length a for the diffusion (2.1) considered in its stationary regime, and $Z_n(\boldsymbol{y_n})$ being defined by

$$\begin{aligned} Z_n(\boldsymbol{y_n}) &= \int\limits_\Omega \exp\big\{-H_{V_n}(\boldsymbol{x})\big\} \bigotimes_{k\in\Lambda_{n+2}, 0\le j\le n-1} \mathcal{W}_{I_j}^{y_j^k, y_{j+1}^k}(dx^k) \\ &= \prod_{0\le j\le n-1} \int\limits_\Omega \exp\big\{-H_{\Lambda_n\times I_j}(\boldsymbol{x})\big\} \bigotimes_{k\in\Lambda_{n+2}} \mathcal{W}_{I_j}^{y_j^k, y_{j+1}^k}(dx^k) \qquad (3.3) \\ &= \prod_{0\le j\le n-1} \int\limits_\Omega \prod_{k\in\Lambda_{n+1}} \exp\big\{-\phi_{(k+\Lambda_0)\times I_j}(\boldsymbol{x})\big\} \bigotimes_{k\in\Lambda_{n+2}} \mathcal{W}_{I_j}^{y_j^k, y_{j+1}^k}(dx^k). \end{aligned}$$

Analysing the spatial product $\prod_{k\in\Lambda_{n+1}} \exp\big\{-\phi_{(k+\Lambda_0)\times I_j}(\boldsymbol{x})\big\}$ first, and letting $\phi_{k;j} = \phi_{(k+\Lambda_0)\times I_j}$, one obtains

$$\begin{aligned} \prod_{k\in\Lambda_{n+1}} \exp\{-\phi_{k;j}(\boldsymbol{x})\} &= \prod_{k\in\Lambda_{n+1}} \{1 + (\exp\{-\phi_{k;j}(\boldsymbol{x})\} - 1)\} \\ &= 1 + \sum_L \sum_{k\in L} (\exp\{-\phi_{k;j}(\boldsymbol{x})\} - 1) \\ &= 1 + \sum_{s\ge 1} \sum_{\gamma_1^j,\ldots,\gamma_s^j} \prod_{m=1}^{s} \prod_{k\in\gamma_m^j} (\exp\{-\phi_{k;j}(\boldsymbol{x})\} - 1), \end{aligned}$$

where $\sum_L$ denotes a sum over all nonempty subsets of Λ_{n+1}, and where $\sum_{\gamma_1^j,\ldots,\gamma_s^j}$ stands for a summation over all maximal "Λ_0-connected" components of $L\times I_j$, so that $L\times I_j = \big(\gamma_1^j \times I_j\big) \cup \ldots \cup \big(\gamma_s^j \times I_j\big)$, the latter decomposition being the finest one for which $(\gamma_r^j + \Lambda_0) \cap (\gamma_{r'}^j + \Lambda_0) = \emptyset$, for all $1 \le r \ne r' \le s$. Integrating back (and still using the Markov property) we have:

$$\begin{aligned} Z_n(\boldsymbol{y_n}) = \prod_{j=0}^{n-1} \int\limits_\Omega \Big\{ 1 + \sum_{s\ge 1} \sum_{\gamma_1^j,\ldots,\gamma_s^j} \prod_{m=1}^{s} \prod_{k\in\gamma_m^j} \big(\exp\{-\phi_{k;j}(\boldsymbol{x})\} - 1\big) \Big\} \\ \bigotimes_{k\in\Lambda_{n+2}} \mathcal{W}_{I_j}^{y_j^k, y_{j+1}^k}(dx^k). \end{aligned}$$

The time product $\prod_{j=0}^{n-1} q_a(y_{j+1}^k; y_j^k)$ may also be expanded as

$$1+\sum_{\tau}\sum_{I_j\in\tau}\big(q_a(y_{j+1}^k;y_j^k)-1\big)=1+\sum_{p\geq 1}\sum_{\tau_1^k,\dots,\tau_p^k}\prod_{u=1}^{p}\prod_{I_j\in\tau_u^k}\big(q_a(y_{j+1}^k;y_j^k)-1\big),$$

where $\sum_\tau$ runs over all non-ordered collections of intervals of the type $I_j = \big[ja;(j+1)a\big]$, $0\leq j\leq n-1$, and where the summation $\sum_{\tau_1^k,\dots,\tau_s^k}$ is taken over all pairwise non-intersecting collections of consecutive time intervals $\tau_u^k=\{(k;I_j), (k;I_{j+1}),\dots,(k;I_{j+r})\}$.

Inserting both of these expansions in the expression (3.2) obtained for $\mathcal{Z}_n$, one obtains

$$\begin{aligned}\mathcal{Z}_n=&\int_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}}\prod_{j=0}^{n-1}\int_{\Omega}\Big(1+\sum_{s\geq 1}\sum_{\gamma_1^j,\dots,\gamma_s^j}\prod_{m=1}^{s}\prod_{k\in\gamma_m^j}\big(\exp\{-\phi_{k;j}(\boldsymbol{x})\}-1\big)\Big)\\&\bigotimes_{k\in\Lambda_{n+2}}\mathcal{W}_{I_j}^{y_j^k,\,y_{j+1}^k}(dx^k)\\&\times\prod_{k\in\Lambda_{n+2}}\Big(1+\sum_{p\geq 1}\sum_{\tau_1^k,\dots,\tau_p^k}\prod_{u=1}^{p}\prod_{I_j\in\tau_u^k}\big(q_a(y_{j+1}^k;y_j^k)-1\big)\Big)\\&\bigotimes_{k\in\Lambda_{n+2},0\leq j\leq n}\nu_0(dy_j^k)\end{aligned}\tag{3.4}$$

so that

$$\mathcal{Z}_n=1+\sum_{v\geq 1}\sum_{\Gamma^1,\dots,\Gamma^v}\prod_{l=1}^{v}K_{\Gamma^l},\tag{3.5}$$

where

$$\Gamma^l=\big\{\gamma_1^{j_1},\dots,\gamma_s^{j_s};\tau_1^{k_1},\dots,\tau_p^{k_p}\big\}$$

is a nonempty collection of contours γ and temporal series τ satisfying

$$\big(\gamma_m^{j_m}+\Lambda_0\big)\cap\big(\gamma_{m'}^{j_{m'}}+\Lambda_0\big)=\emptyset,\quad \tau_u^{k_u}\cap\tau_{u'}^{k_{u'}}=\emptyset,\quad \forall m\neq m', u\neq u'.$$

The coefficient K_Γ attached to each aggregate Γ is given by

$$\begin{aligned}K_\Gamma=&\int_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}}\prod_{m=1}^{s}\int_{\Omega}\prod_{k\in\gamma_m^j}\big(\exp\{-\phi_{k;j}(\boldsymbol{x})\}-1\big)\bigotimes_{k\in\Lambda_{n+2}}\mathcal{W}_{I_j}^{y_j^k,\,y_{j+1}^k}(dx^k)\\&\times\prod_{u=1}^{p}\prod_{I_j\in\tau_u^k}\big(q_a(y_{j+1}^{k_u};y_j^{k_u})-1\big)\bigotimes_{(k;j)\in[\overline{\Gamma}]}\nu_0(dy_j^k),\end{aligned}\tag{3.6}$$

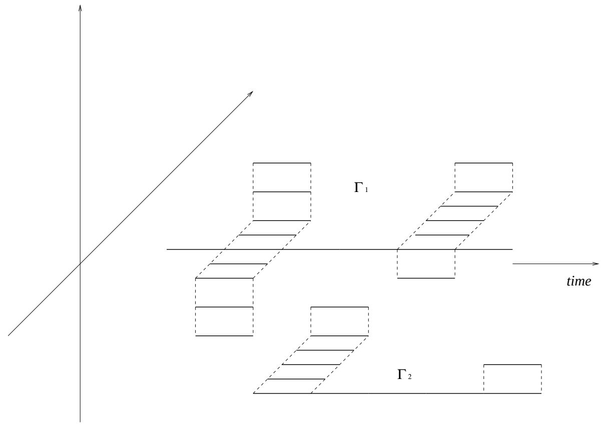

Figure 1. Two non-intersecting aggregates Γ_1 and Γ_2.

so that

$$\mathcal{Z}_n = 1 + \sum_{v \geq 1} \sum_{\Gamma^1, \dots, \Gamma^v} \prod_{l=1}^{v} K_{\Gamma^l}, \tag{3.7}$$

the sum $\sum_{\Gamma^1,\dots,\Gamma^v}$ running over arbitrary finite collections of 2 by 2 non intersecting aggregates, and $[\,\overline{\Gamma}\,]$ denoting the set of all vertices $(k;j)$ appearing in Γ. Letting further $\overline{\Gamma}$ denote the set consisting of all temporal edges appearing in Γ, one may then establish the validity of an exponential upper bound of the type

$$|K_\Gamma| \leq \lambda(\beta)^{|\overline{\Gamma}|},$$

for some $\lambda(\beta) = O(\beta)$.

Indeed, using a generalised Hölder inequality (stated and proved in [18], Lemma 5.5), one may first show that

$$\begin{aligned} |K_\Gamma| \leq & \prod_{m=1}^{s} \prod_{k \in \gamma_m^{j_m}} \left(\int F_{j_m}^k(\boldsymbol{y_n})^{\rho_1} \underset{l \in (k+\Lambda_0)}{\otimes} \nu_0(dy_{j_m}^l)\nu_0(dy_{j_m+1}^l) \right)^{1/\rho_1} \\ & \times \prod_{u=1}^{p} \prod_{I_j \in \tau_u^{k_u}} \left(\int \left| q_a(y_{j+1}^{k_u}; y_j^{k_u}) - 1 \right|^{\rho_2} \nu_0(dy_j^{k_u})\, \nu_0(dy_{j+1}^{k_u}) \right)^{1/\rho_2}, \end{aligned}$$

the function $F_j^k(\boldsymbol{y_n})$ being defined on $\mathbf{R}^{(n+1)|\Lambda_{n+2}|}$ by

$$F_j^k(\boldsymbol{y_n}) = \left(\int_\Omega |\exp\{-\phi_{k;j}(\boldsymbol{x})\} - 1|^{\rho_1} \underset{l \in (k+\Lambda_0)}{\otimes} \mathcal{W}_{I_j}^{y_j^k,\, y_{j+1}^k}(dx^k) \right)^{1/\rho_1}$$

and the exponents ρ_1, ρ_2 satisfying $2(1+2d)/\rho_1 + 2/\rho_2 \le 1$. This enables one to control the spatial interactions and the time interactions separately; more precisely, one may then prove the existence of upper bounds $M_1 = M_1(a,\beta)$, $M_2 = M_2(a,\beta) > 0$ depending both on the time scale a and on the inverse temperature parameter β, and for which for any $k \in \mathbf{Z}^d$, for any $j \in \mathbf{N}$

$$\Big(\int F_j^k(\boldsymbol{y_n})^{\rho_1} \underset{l\in(k+\Lambda_0)}{\otimes} \nu_0(dy_j^l)\nu_0(dy_{j+1}^l) \Big)^{1/\rho_1} \le M_1, \tag{3.8}$$

$$\Big(\int \big| q_a(y_{j+1}^k; y_j^k) - 1 \big|^{\rho_2} \nu_0(dy_j^k)\, \nu_0(dy_{j+1}^k) \Big)^{1/\rho_2} \le M_2, \tag{3.9}$$

whereas

$$\lim_{\beta\searrow 0} M_1(a(\beta),\beta) = \lim_{\beta\searrow 0} M_2(a(\beta),\beta) = 0 \tag{3.10}$$

when the time scale a is chosen properly as a function of the inverse temperature parameter β.

More precisely, using ultracontractivity of the reference diffusion (2.1) enables one to establish the existence of positive constants a_0 and C for which

$$\Big(\int |q_a(y;x) - 1|\, \nu_0(dy)\, \nu_0(dx) \Big)^{1/4} \le C a^{-1/2}$$

as soon as $a \ge a_0$ (see the end of the proof of Proposition 5 in [6]); as we shall see, the choice of a time scale $a(\beta) = \beta^{-1/2}$ turns out to be a convenient one (cf. proof of Proposition 3.1), and in this case one obtains an upper bound M_2 of the type $C \cdot \beta^{-1/4}$ in (3.9). As for (3.8), it may be seen to follow from the basic estimate derived below.

Lemma 3.1. *Fix $U(x) = Cx^4 - 2Cx^2$ for some $C > 0$, and recall that $I_j = [ja; (j+1)a] \subset \mathbf{R}_+$. There exists a constant $K > 0$ depending only on C and on the dimension d of the lattice for which*

$$\int_\Omega \exp\Big\{ \alpha \int_{I_j} \Big(\sum_{l\sim k} x_t^l \Big)^2 dt \Big\} P(d\boldsymbol{x}) \le K \exp\{K\alpha^2 a^2\},$$

for any $\alpha > 0$, $j \in \mathbf{N}$, $k \in \mathbf{Z}^d$.

Proof. Let us first observe that

$$\begin{aligned} \int_\Omega \exp\Big\{ \alpha \int_{I_j} \Big(\sum_{l\sim k} x_t^l \Big)^2 dt \Big\} P(d\boldsymbol{x}) &\le \int_\Omega \exp\Big\{ 2\alpha d \sum_{l\sim k} \int_{I_j} (x_t^l)^2 dt \Big\} P(d\boldsymbol{x}) \\ &= \Big(\int_\Omega \exp\Big\{ 2\alpha d \int_{I_j} (x_t^k)^2 dt \Big\} P(d\boldsymbol{x}) \Big)^{2d}, \end{aligned}$$

so that
$$\int_\Omega \exp\Big\{\alpha \int_{I_j} \Big(\sum_{l\sim k} x_t^l\Big)^2 dt\Big\} P(d\boldsymbol{x}) \le \big(f(2\alpha d)\big)^{2d}$$
for f defined through
$$f(z) = \int \exp\Big\{z \int_I \omega_t^2 dt\Big\} p(d\omega),$$
p being the probability distribution associated with the reference stationary diffusion (2.1), and $I \subset \mathbf{R}_+$ being any interval of length a. Observing that
$$f(z) = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \int \Big(\int_I \omega_t^2 dt\Big)^n p(d\omega),$$
we then obtain
$$\begin{aligned} |f(z)| &\le \sum_{n=0}^{+\infty} \frac{|z|^n}{n!} \int \Big(\int_I \omega_t^2 dt\Big)^n p(d\omega) \\ &\le \sum_{n=0}^{+\infty} \frac{|z|^n}{n!} a^{n-1} \int \int_I \omega_t^{2n} dt\, p(d\omega) \\ &= \sum_{n=0}^{+\infty} \frac{(a|z|)^n}{a\cdot n!} \Big(\int \omega_0^{2n} p(d\omega)\Big) \times a \\ &= \sum_{n=0}^{+\infty} \frac{(a|z|)^n}{n!} \int_{\mathrm{R}} x^{2n} \exp\{-2U(x)\} dx, \end{aligned}$$
having used Hölder's inequality for the first inequality and then the stationarity of our reference diffusion process (2.1). Hence,
$$\begin{aligned} |f(z)| &\le \int_{\mathrm{R}} \sum_n \frac{(ax^2|z|)^n}{n!} \exp\{-2U(x)\} dx \\ &= \int_{\mathrm{R}} \exp\{ax^2|z| - 2U(x)\} dx \\ &= \exp\Big\{2C\Big(1 + \frac{a|z|}{4C}\Big)^2\Big\} \int_{\mathrm{R}} \exp\Big\{-2C\Big(x^2 - \frac{a|z|+4C}{4C}\Big)^2\Big\} dx. \end{aligned}$$
Setting $A = (a|z| + 4C)/(4C)$ and taking into account the fact that the two-parameter integral
$$I(A,C) = \int_{\mathrm{R}} \exp\{-2C(x^2 - A)^2\} dx$$

satisfies

$$\sup_{A \geq 1} I(A, C) = K_1(C) < +\infty,$$

we then have:

$$\begin{aligned}
&\int\limits_{\Omega} \exp\Big\{\alpha \int\limits_{I_j} \Big(\sum_{l \sim k} x_t^l\Big)^2 dt\Big\} P(d\boldsymbol{x}) \\
&\quad \leq |f(2\alpha d)|^{2d} \leq K_1(C)\Big(\exp\Big\{2C\Big(1 + \frac{a\alpha d}{2C}\Big)^2\Big\}\Big)^{2d} \\
&\quad = K_1(C)\exp\Big\{4Cd\Big(1 + \frac{a\alpha d}{2C}\Big)^2\Big\},
\end{aligned}$$

which finishes the proof. □

The proof of inequality (3.9) may now be seen to follow from the estimate given in the preceding lemma, and we give the full details of such a derivation in the non-Markovian case where $\sigma^2 > 0$ (see Proposition 3.1 and its proof).

3.2.2. Non-Markovian case ($\sigma^2 > 0$)

In order to obtain a satisfactory cluster expansion in the non-Markovian setting where $\sigma^2 > 0$, we shall now take into account the fact that the SLLN characterising Q_∞ as an asymptotic dynamics is of a self-averaging nature.

Lemma 3.2. *For any space-time window $V \in \mathcal{V}$ and for $\boldsymbol{x} \in \Omega'$, the (space-time) Boltzmann weight*

$$\exp\{-H_V(\boldsymbol{x})\} = \exp\Big\{\sum_{i \in \Lambda^+}\Big[\int\limits_I b_t^i(\boldsymbol{x})dB_t^i(\boldsymbol{x}) - \frac{1}{2}\int\limits_I b_t^i(\boldsymbol{x})^2 dt\Big]\Big\}$$

may also be presented as

$$\exp\{-H_V(\boldsymbol{x})\} = \mathsf{E}_{\boldsymbol{h}}\Big[\exp\Big\{\sum_{i \in \Lambda^+}\Big[\int\limits_I c_t^i(\boldsymbol{x};\boldsymbol{h})dB_t^i(\boldsymbol{x}) - \frac{1}{2}\int\limits_I c_t^i(\boldsymbol{x};\boldsymbol{h})^2 dt\Big]\Big\}\Big],$$

where $c_t^i(\boldsymbol{x};\boldsymbol{h})$ is given by

$$c_t^i(\boldsymbol{x};\boldsymbol{h}) = \beta\Big(\sum_{j \sim i} x_t^j + h^i\Big).$$

Proof. The proof is similar to that of Proposition 2.1. Indeed, considering first the particular case where $V = \{i\} \times [0, t]$, one may introduce

$$M_t^i(\boldsymbol{x}) = \exp\Big\{\int\limits_0^t b_u^i(\boldsymbol{x})dB_u^i(\boldsymbol{x}) - \frac{1}{2}\int\limits_0^t b_u^i(\boldsymbol{x})^2 du\Big\}$$

and observe that $\big(M_t^i(\boldsymbol{x})\big)_{t\geq 0}$ is a positive martingale with mean 1 under $dP(\boldsymbol{x})$ such that

$$\log M_t^i(\boldsymbol{x}) =_{mart} \int_0^t b_u^i(\boldsymbol{x}) dB_u^i(\boldsymbol{x})$$

(using here again the notation introduced in the proof of Proposition 2.1). Remembering the expressions given for the functionals $b_t^i(\boldsymbol{x})$, $B_t^i(\boldsymbol{x})$ and $\tilde{B}_t^i(\boldsymbol{x})$, we then have

$$\begin{aligned}
\log M_t^i(\boldsymbol{x}) &=_{mart} \int_0^t \Big(\beta \sum_{j\sim i} x_u^j\Big) dB_u^i(\boldsymbol{x}) + \sigma\beta \int_0^t \tanh\big(\sigma\beta \tilde{B}_u^i(\boldsymbol{x})\big) dB_u^i(\boldsymbol{x}) \\
&=_{mart} \int_0^t \Big(\beta \sum_{j\sim i} x_u^j\Big) dB_u^i(\boldsymbol{x}) + \sigma\beta \int_0^t \tanh\big(\sigma\beta \tilde{B}_u^i(\boldsymbol{x})\big) d\tilde{B}_u^i(\boldsymbol{x}) \\
&=_{mart} \int_0^t \Big(\beta \sum_{j\sim i} x_u^j\Big) dB_u^i(\boldsymbol{x}) + \log\cosh\big(\sigma\beta \tilde{B}_t^i(\boldsymbol{x})\big),
\end{aligned}$$

the last equality following from Itô's formula. Now the second summand in the latter term may also be presented as

$$\log \mathsf{E}_{\boldsymbol{h}} \left[\exp\{\beta h^i \tilde{B}_t^i(\boldsymbol{x})\}\right],$$

which establishes Lemma 3.2 in the particular case where $V = \{i\} \times [0,t]$. The general case may be proved along the same lines. □

This representation of the Boltzmann weights turns out to be very convenient for our purposes, at least in the Bernoulli setting; one may indeed replace the identity (3.2) obtained for $\mathcal{Z}_n$ by an expected value

$$\mathcal{Z}_n = \mathsf{E}_{\boldsymbol{h}} \Bigg[\int_{\mathrm{R}^{n|\Lambda_{n+2}|}} Z_n^{\boldsymbol{h}}(\boldsymbol{y_n}) \prod_{\substack{k\in\Lambda_{n+2},\\ 0\leq j\leq n-1}} q_a(y_{j+1}^k; y_j^k) \bigotimes_{\substack{k\in\Lambda_{n+2},\\ 0\leq j\leq n}} \nu_0(dy_j^k)\Bigg] \tag{3.11}$$

where, correspondingly to (3.3), $Z_n^{\boldsymbol{h}}(\boldsymbol{y_n})$ is now given by

$$Z_n^{\boldsymbol{h}}(\boldsymbol{y_n}) = \prod_{0\leq j\leq n-1} \int_\Omega \prod_{k\in\Lambda_{n+1}} \exp\big\{-\phi^{\boldsymbol{h}}_{(k+\Lambda_0)\times I_j}(\boldsymbol{x})\big\} \bigotimes_{k\in\Lambda_{n+2}} \mathcal{W}_{I_j}^{y_j^k, y_{j+1}^k}(dx_\cdot^k), \tag{3.12}$$

and where

$$\phi^{\boldsymbol{h}}_{(k+\Lambda_0)\times I_j}(\boldsymbol{x}) = -\int_I \Big(\beta\sum_{l\sim k} x_t^l + \beta h^k\Big) dB_t^k(\boldsymbol{x}) + \frac{1}{2}\int_I \Big(\beta \sum_{l\sim k} x_t^l + \beta h^k\Big)^2 dt.$$

Following step by step the development given precedingly in the Markovian case, one then obtains

$$\mathcal{Z}_n = 1 + \sum_{v\geq 1}\sum_{\Gamma^1,\ldots,\Gamma^v} \mathsf{E}_{\boldsymbol{h}}\Big[\prod_{l=1}^{v} K^{\boldsymbol{h}}_{\Gamma^l}\Big], \tag{3.13}$$

the coefficient $K^{\boldsymbol{h}}_{\Gamma^l}$ being now given by

$$K^{\boldsymbol{h}}_{\Gamma} = \int_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}} \prod_{m=1}^{s}\int_{\Omega}\prod_{k\in\gamma^j_m}\big(\exp\{-\phi^{\boldsymbol{h}}_{k;j}(\boldsymbol{x})\}-1\big) \underset{k\in\Lambda_{n+2}}{\otimes} \mathcal{W}^{y^k_j,\,y^k_{j+1}}_{I_j}(dx^k_{\cdot})$$
$$\times\prod_{u=1}^{p}\prod_{I_j\in\tau^k_u}\big(q_a(y^{k_u}_{j+1};y^{k_u}_j)-1\big) \underset{(k;j)\in[\overline{\Gamma}]}{\otimes} \nu_0(dy^k_j), \tag{3.14}$$

for each aggregate $\Gamma=\big\{\gamma^{j_1}_1,\ldots,\gamma^{j_s}_s;\tau^{k_1}_1,\ldots,\tau^{k_p}_p\big\}$.

In order to view the average $\mathsf{E}_{\boldsymbol{h}}\big[\prod_{l=1}^{v}K^{\boldsymbol{h}}_{\Gamma^l}\big]$ as a product running over some new aggregates, one should then partition the collection $\Gamma^1,\ldots,\Gamma^v$ into a convenient collection of (two by two disjoint) subsets

$$\Theta_1=\big\{\Gamma^{e_1,1},\ldots,\Gamma^{e_{n_1},1}\big\},\ldots,\Theta_{\tilde v}=\big\{\Gamma^{e_1,\tilde v},\ldots,\Gamma^{e_{n_{\tilde v}},\tilde v}\big\}\qquad(\tilde v\leq v).$$

To be more precise, one may define the "spatial support" associated with

$$\Gamma=\big\{\gamma^{j_1}_1,\ldots,\gamma^{j_s}_s;\tau^{k_1}_1,\ldots,\tau^{k_p}_p\big\}$$

as

$$\mathrm{supp}_{\mathrm{Z}^d}(\Gamma)=\{k\in\mathbf{Z}^d:\ \exists 1\leq m\leq s,(k;j_m)\in\gamma^{j_m}_m\}$$

and then decompose $\{\Gamma^1,\ldots,\Gamma^v\}$ into a union $\{\Gamma^1,\ldots,\Gamma^v\}=\cup_{l=1}^{\tilde v}\Theta_l$, each of the classes $\Theta_l=\{\Gamma^{e_1,l},\ldots,\Gamma^{e_n,l}\}$ being maximal among all subsets of $\{\Gamma^1,\ldots,\Gamma^v\}$ satisfying

$$\forall 1\leq n'\leq n,\ \exists\, n''\neq n',\quad \mathrm{supp}_{\mathrm{Z}^d}\big(\Gamma^{e_{n'}}\big)\cap\mathrm{supp}_{\mathrm{Z}^d}\big(\Gamma^{e_{n''}}\big)\neq\emptyset.$$

One thus obtains:

$$\mathsf{E}_{\boldsymbol{h}}\Big[\prod_{l=1}^{v}\prod_{m=1}^{s}\prod_{k\in\gamma^{j_m}_m}\big(\exp\{-\phi^{\boldsymbol{h}}_{k,j_m}(\boldsymbol{x})\}-1\big)\Big]$$
$$=\prod_{l=1}^{\tilde v}\mathsf{E}_{\boldsymbol{h}}\Big[\prod_{n'=1}^{n}\prod_{m=1}^{s}\prod_{k\in\mathrm{supp}(\gamma^{j_m}_m)}\big(\exp\{-\phi^{\boldsymbol{h}}_{k,j_m}(\boldsymbol{x})\}-1\big)\Big],$$

so that

$$\mathcal{Z}_n = 1+\sum_{\tilde v\geq 1}\sum_{\Theta^1,\ldots,\Theta^{\tilde v}}\prod_{l=1}^{\tilde v}\tilde K_{\Theta^l}, \tag{3.15}$$

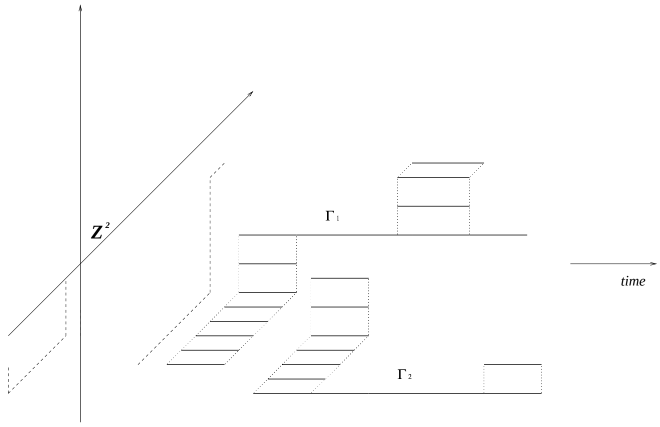

Figure 2. Two aggregates, Γ_1 and Γ_2, having disjoint supports.

the new cluster coefficients $\tilde{K}_\Theta$ being now given by

$$\begin{aligned}\tilde{K}_\Theta = & \int_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}} \prod_{(k;I_j)\in\overline{\Theta}} \big(q_a(y^k_{j+1};y^k_j)-1\big) \underset{(k;j)\in[\overline{\Theta}]}{\otimes} \nu_0(dy^k_j) \\ & \times \Big\{ \int_\Omega \mathsf{E}_{\boldsymbol{h}} \Big[\prod_{(k;j)\in[\overline{\Theta}]} \big(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\big)\Big] \underset{(k;I_j)\in\overline{\Theta}}{\otimes} \mathcal{W}^{y^k_j,\, y^k_{j+1}}_{I_j}(dx^k_\cdot) \Big\} \\ = \mathsf{E}_{\boldsymbol{h}} \Big[& \int_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}} \prod_{(k;I_j)\in\overline{\Theta}} \big(q_a(y^k_{j+1};y^k_j)-1\big) \underset{(k;j)\in[\overline{\Theta}]}{\otimes} \nu_0(dy^k_j) \\ & \times \Big\{ \int_\Omega \prod_{(k;j)\in[\overline{\Theta}]} \big(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\big) \underset{(k;I_j)\in\overline{\Theta}}{\otimes} \mathcal{W}^{y^k_j,\, y^k_{j+1}}_{I_j}(dx^k_\cdot) \Big\}\Big]. \end{aligned} \tag{3.16}$$

For a fixed realisation of $\boldsymbol{h}$, one may then apply the generalised Hölder inequality stated as Lemma 5.5 in [18], first to the integral

$$\int_\Omega \prod_{(k;j)\in[\overline{\Theta}]} \big(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\big) \underset{(k;I_j)\in\overline{\Theta}}{\otimes} \mathcal{W}^{y^k_j,\, y^k_{j+1}}_{I_j}(dx^k_\cdot),$$

whose absolute value is bounded from above by the product

$$\prod_{n'=1}^{n} \prod_{m=1}^{s} \prod_{k\in\mathrm{supp}(\gamma^{jm}_m)} \Big(\int_\Omega \big(\exp\{-\phi^{\boldsymbol{h}}_{k,j_m}(\boldsymbol{x})\}-1\big)^\rho \underset{l\in(k+\Lambda_0)}{\otimes} \mathcal{W}^{y^l_{j_m},\, y^l_{j_m+1}}_{I_{j_m}}(dx^l_\cdot)\Big)^{1/\rho}$$

$$= \prod_{n'=1}^{n} \prod_{m=1}^{s} \prod_{k \in \mathrm{supp}(\gamma_m^{j_m})} F_{k,\, j_m}^{\boldsymbol{h}} \Big(\big\{ y_j^l \big\}_{j=j_m, j_m+1}^{l \in k+\Lambda_0} \Big)$$

for $\rho = 4(2d+1)$ (since each of the edges $\big(k; I_j\big)$ cannot appear in more than $2d+1$ of the contours γ or τ pertaining to the collection Θ); one may secondly apply this generalised Hölder inequality to

$$\int\limits_{\mathrm{R}^{(n+1)|\Lambda_{n+2}|}} \prod_{n'=1}^{n} \Big\{ \prod_{u=1}^{p} \big(q_a(y_{j+1}^{k_u}; y_j^{k_u}) - 1 \big) \\ \times \prod_{m=1}^{s} \prod_{k \in \mathrm{supp}(\gamma_m^{j_m})} F_{k,\, j_m}^{\boldsymbol{h}} \Big(\big\{ y_j^l \big\}_{j=j_m, j_m+1}^{l \in k+\Lambda_0} \Big) \Big\} \bigotimes_{(k;j) \in [\overline{\Theta}]} \nu_0(dy_j^k),$$

whose absolute value is bounded from above by

$$\prod_{n'=1}^{n} \Big\{ \prod_{m=1}^{s} \prod_{k \in \mathrm{supp}(\gamma_m^{j_m})} \Big(\int\limits_{\mathrm{R}^2} |q_a(y_{j+1}^{k_u}; y_j^{k_u}) - 1|^4 d\nu_0(y_j^{k_u})\, d\nu_0(y_{j+1}^{k_u}) \Big)^{1/4} \Big\} \\ \times \Big\{ \prod_{m=1}^{s} \prod_{k \in \mathrm{supp}(\gamma_m^{j_m})} \Big(\int F_{k,\, j_m}^{\boldsymbol{h}} \Big(\big\{ y_j^l \big\}_{j=j_m, j_m+1}^{l \in (k+\Lambda_0)} \Big)^{\rho} \\ \bigotimes_{l \in (k+\Lambda_0)} d\nu_0(y_{j_m}^l)\, d\nu_0(y_{j_m+1}^l) \Big)^{1/\rho} \Big\}$$

(since each of the space-time vertices $(k;j)$ appears at most $2(1+2d)$ times as an extremity of an edge pertaining to Θ).

Controlling the term

$$\Big(\int\limits_{\mathrm{R}^2} \big| q_a(y_{j+1}^{k_u}; y_j^{k_u}) - 1 \big|^4 d\nu_0(y_j^{k_u})\, d\nu_0(y_{j+1}^{k_u}) \Big)^{1/4} \tag{3.17}$$

requires of course no new ingredient, and we are left with the control of the expected value

$$\mathsf{E}_{\boldsymbol{h}} \Big[\prod_{n'=1}^{n} \prod_{m=1}^{s} \prod_{k \in \mathrm{supp}(\gamma_m^{j_m})} \Big(\int F_{k,\, j_m}^{\boldsymbol{h}} \Big(\big\{ y_j^l \big\}_{j=j_m, j_m+1}^{l \in (k+\Lambda_0)} \Big)^{\rho} \\ \bigotimes_{l \in (k+\Lambda_0)} d\nu_0(y_{j_m}^l) d\nu_0(y_{j_m+1}^l) \Big)^{1/\rho} \Big],$$

where

$$F_{k,j_m}^{\boldsymbol{h}} \Big(\big\{ y_j^l \big\}_{j=j_m,\, j_m+1}^{l \in (k+\Lambda_0)} \Big)^{\rho} = \int\limits_{\Omega} \big(\exp\{-\phi_{k,j_m}^{\boldsymbol{h}}(\boldsymbol{x})\} - 1 \big)^{\rho} \bigotimes_{l \in (k+\Lambda_0)} \mathcal{W}_{I_{j_m}}^{y_{j_m}^l,\, y_{j_m+1}^l}(dx^l).$$

Using the uniform boundedness of the variables h^i, one may actually give a satisfactory upper bound that is valid almost surely in $\boldsymbol{h}$ (having assumed β is small enough).

Proposition 3.1. *There exists a positive constant β_0 for which the following holds true: whenever $0 < \beta \leq \beta_0$, one may choose a time scale $a = a(\beta) > 0$ so that for $\rho = 4(2d+1)$*

$$\left(\int_\Omega F^{\boldsymbol{h}}_{k,\, j_m} \left(\{y^l_j\}^{l \in k+\Lambda_0}_{j=j_m, j_m+1} \right)^\rho \underset{l \in k+\Lambda_0}{\otimes} d\nu_0(y^l_{j_m}) d\nu_0(y^l_{j_m+1}) \right)^{1/\rho} = O(\beta)$$

uniformly in k, j_m and $\boldsymbol{h}$.

Proof. Let us recall that $q_t(y_2; y_1)$ denotes the transition density of the reference diffusion (2.1),

$$dx_t = dw_t - U'(x_t)\, dt,$$

with respect to its invariant reversible measure ν_0; q_t may be defined through the equalities

$$p\big\{x_t \in dy_2 \mid x_{t=0} = y_1\big\} = q_t(y_2; y_1)\nu_0(dy_2),$$

and since the single-site potential U has been defined as $U(x) = Cx^4 - 2Cx^2$, we know that the diffusion (2.1) is ultracontractive (cf. [13]). Hence, $q_t(y_2; y_1)$ converges to 1 uniformly in $y_1, y_2 \in \mathbf{R}$, and a fortiori

$$\forall A > 1,\, \exists\, a_0 \in \mathbf{R}_+,\ \forall a > a_0,\, \forall y_1, y_2 \in \mathbf{R}, \quad q_a(y_2; y_1) \geq \frac{1}{A}.$$

Choosing a time-scale $a > 0$ that is large enough, we may thus replace the integral

$$\left(\int \left(\int_\Omega \left(\exp\{-\phi^{\boldsymbol{h}}_{k,j_m}(\boldsymbol{x})\} - 1 \right)^\rho \underset{l \in (k+\Lambda_0)}{\otimes} \mathcal{W}^{y^l_{j_m},\, y^l_{j_m+1}}_{I_{j_m}}(dx^l_\cdot) \right) \underset{l \in (k+\Lambda_0)}{\otimes} d\nu_0(y^l_{j_m}) d\nu_0(y^l_{j_m+1}) \right)^{1/\rho}$$

by

$$\left(\int \left(\int_\Omega \left(\exp\{-\phi^{\boldsymbol{h}}_{k,j_m}(\boldsymbol{x})\} - 1 \right)^\rho \underset{l \in (k+\Lambda_0)}{\otimes} \mathcal{W}^{y^l_{j_m},\, y^l_{j_m+1}}_{I_{j_m}}(dx^l_\cdot) \right) \underset{l \in (k+\Lambda_0)}{\otimes} \frac{d\nu_0(y^l_{j_m}) d\nu_0(y^l_{j_m+1})}{q_a(y^l_{j_m}, y^l_{j_m+1})} \right)^{1/\rho},$$

thereby loosing only a constant factor $A^{(2d+1)/\rho}$. But the latter integral coincides with

$$\Bigl(\int_\Omega \bigl(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\bigr)^\rho P(d\boldsymbol{x})\Bigr)^{1/\rho},$$

and we then have:

$$\begin{aligned}
&\int_\Omega \bigl(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\bigr)^\rho P(d\boldsymbol{x}) \\
&\quad= \int_\Omega \Bigl(\int_0^1 \phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\exp\{-\tau\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}d\tau\Bigr)^\rho P(d\boldsymbol{x}) \\
&\quad= \int_{[0,1]^\rho}\int_\Omega (\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x}))^\rho \exp\{-(\tau_1+\ldots+\tau_\rho)\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}P(d\boldsymbol{x})\,d\tau_1\ldots d\tau_\rho \\
&\quad= \int_{[0,1]^\rho} \frac{d^\rho}{dz^\rho}\mathcal{T}^{\boldsymbol{h}}(z)\Big|_{z=\tau_1+\ldots+\tau_\rho} d\tau_1\ldots d\tau_\rho,
\end{aligned}$$

where for any $z\in\mathbf{C}$,

$$\mathcal{T}^{\boldsymbol{h}}(z)=\int_\Omega \exp\{-z\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}P(d\boldsymbol{x}).$$

According to Cauchy's formula:

$$\Bigl|\frac{d^\rho}{dz^\rho}\mathcal{T}^{\boldsymbol{h}}(z)\Bigr| \le \frac{\rho!}{r^\rho}\sup_{|\zeta-z|=r}|\mathcal{T}^{\boldsymbol{h}}(\zeta)|,$$

whenever $\mathcal{T}^{\boldsymbol{h}}$ is holomorphic on an open domain containing $B(z;r)$, and we also know that for $\zeta=\xi_1+\iota\xi_2,\ \xi_1,\xi_2\in\mathbf{R}$,

$$|\mathcal{T}^{\boldsymbol{h}}(\zeta)| \le \int_\Omega \bigl|\exp\{-\zeta\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}\bigr|P(d\boldsymbol{x}) = \int_\Omega \exp\{-\xi_1\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}P(d\boldsymbol{x}).$$

Factorising $\exp\{-\xi_1\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}$ into

$$\exp\Bigl(\xi_1\int_{I_{j_m}} c^k_t(\boldsymbol{x};\boldsymbol{h})dB^k_t(\boldsymbol{x}) - \xi_1^2\int_{I_{j_m}} c^k_t(\boldsymbol{x};\boldsymbol{h})^2dt\Bigr)\cdot\exp\Bigl(\Bigl(\xi_1^2-\frac{\xi_1}{2}\Bigr)\int_{I_{j_m}} c^k_t(\boldsymbol{x};\boldsymbol{h})^2dt\Bigr)$$

and using the Cauchy–Schwarz inequality together with the P-martingale property of the square of the first factor then yields:

$$\int_\Omega \bigl(\exp\{-\phi^{\boldsymbol{h}}_{k,j}(\boldsymbol{x})\}-1\bigr)^\rho P(d\boldsymbol{x}) \le \frac{\rho!}{r^\rho}\Bigl(\int_\Omega \exp\Bigl\{(2\xi^2-\xi)\int_{I_{j_m}} c^k_t(\boldsymbol{x};\boldsymbol{h})^2dt\Bigr\}P(d\boldsymbol{x})\Bigr)^{1/2},$$

$\xi > 0$ being chosen so that $(\xi^2 - \xi/2)$ is larger than any of the $(\xi_1^2 - \xi_1/2)$'s appearing when using an auxiliary parameter ζ such that $|\zeta - (\tau_1 + \ldots + \tau_\rho)| = r$.

At this stage the integrand

$$\exp\Big\{(2\xi^2 - \xi)\int_{I_{jm}} c_t^k(\boldsymbol{x};\boldsymbol{h})^2 dt\Big\}$$

may be estimated from above by

$$\exp\{(4\xi^2 - 2\xi)a\sigma^2\beta^2\}\exp\Big\{(4\xi^2 - 2\xi)\beta^2\int_{I_{jm}}\Big(\sum_{l\sim k} x_t^l\Big)^2 dt\Big\},$$

and replacing further $(4\xi^2 - 2\xi)$ by $4(\rho + r)^2$, it thus remains to control

$$\frac{\rho!}{r^\rho}\exp\{4(\rho + r)^2 a\sigma^2\beta^2\}\Big(\int_\Omega \exp\Big\{4(\rho + r)^2\beta^2\int_{I_{jm}}\Big(\sum_{l\sim k} x_t^l\Big)^2 dt\Big\}dP(\boldsymbol{x})\Big)^{1/2}.$$

Taking $a(\beta) = \beta^{-1/2}$ and using Lemma 3.1 enables us to replace the above term by

$$f_\beta(r) = \frac{\rho!}{r^\rho}\exp\{4(\rho + r)^2\sigma^2\beta^{3/2}\}\cdot K\cdot\exp\{16K\beta(\rho + r)^4\},$$

and $\min_{r>0} f_\beta(r)$ may be seen to decrease to 0 as $\beta \searrow 0$ by setting e.g. $r_\beta = \beta^{-1/\rho} = \beta^{-1/4(2d+1)}$, for which one obtains the existence of $\tilde{K}$ such that $f_\beta(r_\beta) \le \tilde{K}\beta$ for all β small enough. This finishes the proof. □

We are now in a position to give exponential estimates for the cluster coefficients $\tilde{K}_\Theta$ appearing in the decomposition (3.15).

Proposition 3.2. *There exists an $\beta_0 > 0$ for which the following holds true: whenever $0 < \beta \le \beta_0$, one may choose a time scale $a = a(\beta) > 0$ so that each of the cluster coefficients $\tilde{K}_\Theta$ appearing in the decomposition* (3.15) *of the partition function $\mathcal{Z}_n$ satisfies*

$$|\tilde{K}_\Theta| \le C\cdot\big(\lambda(\beta)\big)^{|\Theta|},$$

$|\Theta|$ denoting the number of temporal edges $(k; I_j)$ appearing in the cluster Θ, and λ being such that

$$\lim_{\beta\searrow 0}\lambda(\beta) = 0.$$

Proof. This is a simple consequence of inequality (3.9) and of Proposition 3.1, yielding an exponential control of the contributions associated with “time contours” and “spatial contours” respectively. □

As a consequence of such exponential control of $\tilde{K}_\Theta$, we may further assert that the asymptotic dynamics Q_∞ is exponentially ergodic in a space-time sense (Theorem 3.1). The link between such exponential control of the cluster coefficients $\tilde{K}_\Theta$ and an exponential decay of correlations under the Gibbs measure Q_∞ may be found in [16] (see Lemma 1 in §2 of Chapter 3).

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