Invariance principle for martingale-difference random fields

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Résumé

On présente d'abord un critère de convergence vers le processus de Wiener à paramètre ν-dimensionnel, pour $\nu \geq 1$. Puis on l'applique pour montrer qu'un champ aléatoire différence de martingales sur $\mathbb{Z}^\nu$ satisfait un principe d'invariance.

Abstract

A convergence criterium to the multi-parameter Wiener process is proved. Then, it is used to establish that a martingale-difference random field on the lattice satisfies an invariance principle.

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MOTS CLES : Théorème central-limite, processus de Wiener à paramètre multi-dimensionnel, champ différence de martingales, principe d'invariance
KEY WORDS : Central limit theorem, multi-parameter Wiener process, martingale-difference random field , invariance principle

1 Introduction

In this paper we are interested in functional central limit theorem, in a other words invariance principle, for martingale-difference random fields on the lattice $\mathbb{Z}^\nu$. In [4], various examples of martingale-difference random fields have been described. A particularly important class of such fields consists in Gibbsian fields with supereven potential.

A central limit theorem for martingale-difference random fields was first shown in [3], and then generalised to a 1-dimensional functional theorem in [5]. We present here a complete multi-dimensional invariance principle, which is proved owing to a convergence criterium for random fields to multi-parameter Wiener process presented in the next Section.

2 A convergence criterium to the multiparameter Wiener process

NOTATIONS

Let $\mathbf{T}^\nu$ be the ν-fold Cartesian product of the closed unit interval $[0,1]$, for $\nu \geq 1$. We consider on $\mathbf{T}^\nu$ the usual order: for $\mathbf{s},\mathbf{t} \in \mathbf{T}^\nu$, $\mathbf{s} = (s^{(1)},\ldots,s^{(\nu)})$, $\mathbf{t} = (t^{(1)},\ldots,t^{(\nu)})$, we write $\mathbf{s} < \mathbf{t}$ (or $\mathbf{s} \leq \mathbf{t}$) if $s^{(i)} < t^{(i)}$ (or $s^{(i)} \leq t^{(i)}$), $i = 1,\ldots,\nu$. For $\mathbf{t}_1,\mathbf{t}_2 \in \mathbf{T}^\nu, \mathbf{t}_1 < \mathbf{t}_2$, we will denote by $(\mathbf{t}_1,\mathbf{t}_2]$ the ν-dimensional interval $\{\mathbf{s} \in \mathbf{T}^\nu : \mathbf{t}_1 < \mathbf{s} \leq \mathbf{t}_2\}$ which is often called a block. In other words

$$(\mathbf{t}_1,\mathbf{t}_2] = \prod_{i=1}^{\nu}(t_1^{(i)},t_2^{(i)}].$$

We also denote for $\mathbf{t} \in \mathbf{T}^\nu$, $|\mathbf{t}| = \max_{1\leq i\leq\nu} |t^{(i)}|$.

C_ν is the set of all continuous functions on $\mathbf{T}^\nu$ endowed with the uniform metric.

Following the terminology of [1], we call a function $x : \mathbf{T}^\nu \to \mathbb{R}$ a step function, if x is a linear combination of functions of the form:

$$\mathbf{t} \mapsto I_{E_1\times\cdots\times E_\nu}(\mathbf{t}) \ ,$$

where each E_k is either a left-closed, right-open subinterval of $[0,1]$, or the singleton $\{1\}$ and I_E denotes the indicator of the set E. Let D_ν be the uniform closure, in the space of all bounded functions from $\mathbf{T}^\nu$ to $\mathbb{R}$, of the vector subspace of step functions. Then the functions of D_ν are a multi-dimensional version of "cad-lag" functions.

One introduces on D_ν a metric topology (which coincides with Skorohod topology if $\nu = 1$) for which the space D_ν is a complete separable metric space and the Borel σ-algebra coincides with the σ-algebra generated by the coordinate mappings (see [6], [2]).

We define the modulus of continuity of an element $x \in D_\nu$ by

$$w_x(\delta) = w(x,\delta) = \sup\{|x(\mathbf{t}) - x(\mathbf{s})| : \mathbf{t},\mathbf{s} \in \mathbf{T}^\nu, |\mathbf{s}-\mathbf{t}| < \delta\} \ , \delta > 0.$$

If $\{X(\mathbf{t}),\mathbf{t} \in \mathbf{T}^\nu\}$ is a stochastic process then the increment $X(B)$ of X

around a block $B = (\mathbf{s}, \mathbf{t}] \subset \mathbf{T}^\nu$ is defined by

$$X(B) = \sum_{\substack{\alpha_i=0,1 \\ i=1,\dots,\nu}} (-1)^{\nu-\sum_{i=1}^{\nu}\alpha_i} X\Big(s^{(1)}+\alpha_1\big(t^{(1)}-s^{(1)}\big),\dots,s^{(\nu)}+\alpha_\nu\big(t^{(\nu)}-s^{(\nu)}\big)\Big).$$

Let $\hat{B} = (\hat{s}, \hat{t}], \hat{s}, \hat{t} \in \mathbf{T}^{\nu-1}$ be a fixed block in $\mathbf{T}^{\nu-1}$. If $(s,t] \subset [0,1]$, then evidently $(s,t] \times \hat{B}$ is a block in $\mathbf{T}^\nu$.
For $h > 0$ we will denote by $\Delta_{t,t+h}$ the block $(t, t+h] \times \hat{B}$.

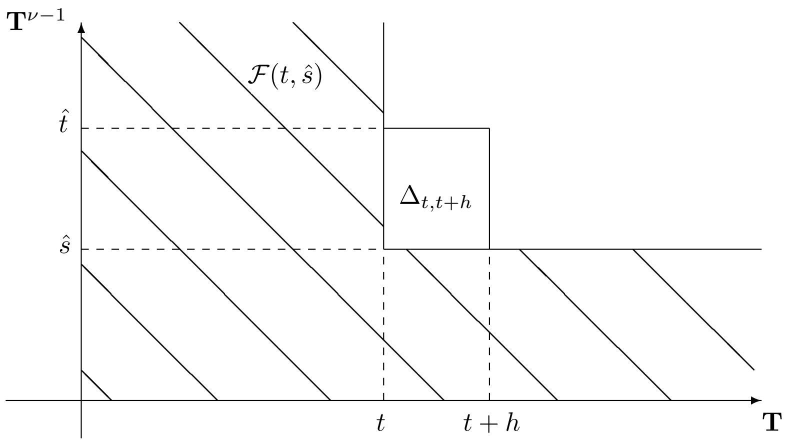

Figure 1: The weak past σ-algebra

We recall that a stochastic process $\{W(\mathbf{t}), \mathbf{t} \in \mathbf{T}^\nu\}$ is called a ν-parameter Wiener process if

1) $P(W \in C_\nu) = 1$, $P(W(\mathbf{t}) = 0) = 1$ for each $\mathbf{t} \in \mathbf{T}^\nu_o$, where $\mathbf{T}^\nu_o = \{\mathbf{t} \in \mathbf{T}^\nu \colon \exists 1 \le j \le \nu$ such that $t^{(j)} = 0\}$ is the "lower boundary" of $\mathbf{T}^\nu$.

2) If $B_1, \dots, B_k$ are pairwise disjoint blocks in $\mathbf{T}^\nu$, then the increments $W(B_1), \dots, W(B_k)$ are independent normal random variables with means zero and variances $|B_1|, \dots, |B_k|$, where $|B|$ denotes the ν-dimensional volume of a block B from $\mathbf{T}^\nu$.

For $\mathbf{t} \in \mathbf{T}^\nu$ we define the "weak past" of $\mathbf{t}$ by

$$\mathbf{T}^\nu_-(\mathbf{t}) = \{\mathbf{s} \in \mathbf{T}^\nu \colon \exists 1 \le j \le \nu \text{ such that } s^{(j)} \le t^{(j)}\}$$

and put

$$\mathcal{F}(\mathbf{t}) = \sigma\big\{X(\mathbf{u}), \mathbf{u} \in \mathbf{T}^\nu_-(\mathbf{t})\big\}.$$

Now we can formulate conditions, which will characterize a random element X of D_ν as a ν-parameter Wiener process.

Condition 1. For $t \in [0,1), \hat{B} = (\hat{s}, \hat{t}] \subset \mathbf{T}^{\nu-1}$

a) $\lim_{h \downarrow 0} \frac{1}{h} E\big\{\big|E(X(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s}))\big|\big\} = 0,$

b) $\lim_{h\downarrow 0}\frac{1}{h}E\{|E(X^2(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s}))-h|\hat{B}||\}=0$.

Condition 2.

$$\sup_{\mathbf{t}\in\mathbf{T}^\nu} E\{X^2(\mathbf{t})\}<+\infty\ .$$

Condition 3. For $0\le t<1$

$$\lim_{\alpha\to\infty}\limsup_{h\downarrow 0}\frac{1}{h}\int_{X^2(\Delta_{t,t+h})\ge\alpha h} X^2(\Delta_{t,t+h})dP=0$$

The following Theorems 1 and 2 are multidimensional extensions of Theorems 19.3 and 19.4 of [2] respectively.

Theorem 1 *Let X be a random element of D_ν with $P(X\in C_\nu)=1$ and $P(X(\mathbf{t})=0)=1$ for each $\mathbf{t}\in\mathbf{T}^\nu_o$. If X satisfies conditions 1-3 then X is a ν-parameter Wiener process.*

Proof : Let $B, B_1,\dots,B_k$, be the following family of disjoint blocks in $\mathbf{T}^\nu$: $B=(s,t]\times\hat{B}, B_j=(s_j,t_j]\times\hat{B}_j$, where $\hat{B}=(\hat{s},\hat{t}], \hat{B}_j=(\hat{s}_j,\hat{t}_j]\subset\mathbf{T}^{\nu-1}, j=1,\dots,k$. Without loss of generality (by reordering the blocks) we can assume that $\hat{B}_j\subset\mathbf{T}^\nu_-((s,\hat{s})), j=1,\dots,k$. We suppose that $t<1$.

Let $\lambda_1,\dots,\lambda_k$ be real numbers and let

$$Z=\lambda_1X(B_1)+\cdots+\lambda_kX(B_k)$$

Consider the characteristic functional defined for $\lambda\in\mathbb{R}, s\le t<1$ by

(1) $\quad \psi(t,\lambda)=E\{\exp[iZ+i\lambda X((s,t]\times\hat{B})]\}$.

We want to show that ψ satisfies the following differential equation :

(2) $\quad \dfrac{\partial}{\partial t}\psi(t,\lambda)=-\dfrac{1}{2}\lambda^2|\hat{B}|\psi(t,\lambda)$.

It is clear that for $h>0$, $t+h\le 1$,

$$X\big((s,t+h]\times\hat{B}\big)=X\big((s,t]\times\hat{B}\big)+X\big((t,t+h]\times\hat{B}\big)\ .$$

We have that

$$\begin{aligned}&\frac{1}{h}\big[\psi(t+h,\lambda)-\psi(t,\lambda)\big]\\&=\frac{1}{h}E\left\{\exp\big[iZ+i\lambda X(B)\big]\ \big[\exp(i\lambda X(\Delta_{t,t+h}))-1\big]\right\}\\&=\frac{1}{h}E\left\{\exp\big[iZ+i\lambda X(B)\big]\cdot\big[i\lambda X(\Delta_{t,t+h})-\frac{\lambda^2}{2}X^2(\Delta_{t,t+h})+r(\lambda X(\Delta_{t,t+h}))\big]\right\}\end{aligned}$$

where r is the remaining term in the expansion of the exponential function. This implies that

$$\begin{aligned}&\frac{1}{h}\big[\psi(t+h,\lambda)-\psi(t,\lambda)\big]+\frac{1}{2}\lambda^2|\hat{B}|\psi(t,\lambda)\\&=E\left\{\exp(iZ+i\lambda X(B))\big[\frac{i\lambda}{h}X(\Delta_{t,t+h})+\frac{\lambda^2}{2}\big(|\hat{B}|-\frac{1}{h}X^2(\Delta_{t,t+h})\big)+\right.\\&\left.+\frac{1}{h}r(\lambda X(\Delta_{t,t+h}))\big]\right\}\\&=\Psi_1+\Psi_2+\Psi_3\end{aligned}$$

where

$$\begin{aligned}
\Psi_1 &= \frac{i\lambda}{h} E\left\{\exp[iZ + i\lambda X(B)] X(\Delta_{t,t+h})\right\}, \\
\Psi_2 &= \frac{\lambda^2}{2h} E\left\{\exp[iZ + i\lambda X(B)] \cdot [h|\hat{B}| - X^2(\Delta_{t,t+h})]\right\}, \\
\Psi_3 &= \frac{1}{h} E\left\{\exp[iZ + i\lambda X(B)] \cdot r(\lambda X(\Delta_{t,t+h}))\right\},
\end{aligned}$$

Let us estimate Ψ_1. We have

$$\begin{aligned}
|\Psi_1| &\le \frac{|\lambda|}{h} \left| E\left\{\exp(iZ + i\lambda X(B)) E(X(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s})\right\}\right| \\
&\le \frac{|\lambda|}{h} E\left\{\left|E[X(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s})]\right|\right\}
\end{aligned}$$

Hence by Condition 1 a) Ψ_1 tends to 0 as $h \downarrow 0$.

Concerning Ψ_2 we can write

$$\begin{aligned}
|\Psi_2| &\le \frac{\lambda^2}{2h} E\left\{\left|E[h|\hat{B}| - X^2(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s})]\right|\right\} \\
&= \frac{\lambda^2}{2h} E\left\{\left|h|\hat{B}| - E[X^2(\Delta_{t,t+h})/\mathcal{F}(t,\hat{s})]\right|\right\}
\end{aligned}$$

which tends to 0 as $h \downarrow 0$, by Condition 1 b).

To estimate Ψ_3 we note that

$$|r(v)| \le v^3 \quad \text{and} \quad |r(v)| \le v^2.$$

Therefore

$$\begin{aligned}
|\Psi_3| &\le \frac{1}{h} E\left\{|r(\lambda X(\Delta_{t,t+h}))|\right\} \\
&\le \frac{1}{h} \int_{X^2(\Delta_{t,t+h})<\alpha h} |\lambda|^3 |X(\Delta_{t,t+h})|^3 dP + \frac{\lambda^2}{h} \int_{X^2(\Delta_{t,t+h})\ge\alpha h} X^2(\Delta_{t,t+h}) dP \\
&\le |\lambda|^3 \alpha^{3/2} h^{1/2} + \frac{\lambda^2}{h} \int_{X^2(\Delta_{t,t+h})\ge\alpha h} X^2(\Delta_{t,t+h}) dP.
\end{aligned}$$

By Condition 3 we conclude that Ψ_3 tends to 0 as $h \downarrow 0$.

Thus we have proved that ψ satisfies the differential equation (2) in the domain : $\lambda \in \mathbb{R}$, $s \le t < 1$. This implies that, in this domain,

$$\psi(t,\lambda) = \exp\left[-\frac{1}{2}\lambda^2 |\hat{B}|(t-s)\right] \psi(s,\lambda).$$

Since

$$\psi(s,\lambda) = E\{\exp(iZ)\}$$

it follows that

$$\psi(t,\lambda) = \exp\left(-\frac{1}{2}\lambda^2 |B|\right) E\{\exp(iZ)\}.$$

or equivalently,

$$\begin{aligned}
& E\left\{\exp[i\lambda_1 X(B_1) + \cdots + i\lambda_k X(B_k) + i\lambda X((s,t) \times \hat{B})]\right\} \\
(3) \qquad &= E\left\{\exp[i\lambda_1 X(B_1) + \cdots + i\lambda_k X(B_k)]\right\} \exp[-\frac{1}{2}\lambda^2 |\hat{B}|(t-s)]
\end{aligned}$$

It follows from Condition 2 and $P(X \in C_\nu) = 1$ that (3) remains true also for $t = 1$.

Now by taking $k = 1$ and $B_1 = \emptyset$ we find that for any block $B \subset \mathbf{T}^\nu$, $X(B)$ is a normal random variable with mean zero and variance $|B|$.
Taking $k = 1$ and B_1, B arbitrary but disjoint, we find that

$$E\left\{\exp[i\lambda_1 X(B_1) + i\lambda X(B)]\right\} = \exp(-\frac{1}{2}\lambda_1|B_1|)\cdot\exp(-\frac{1}{2}\lambda|B|),$$

which means that $X(B_1), X(B)$ are independent normal random variables with means zero and variances $|B_1|$ and $|B|$ respectively.

In the same way we can get that $X(B_1), \ldots, X(B_k), X(B)$ are pairwise independent normal random variables with zero means and variances $|B_1|, \ldots, |B_k|$ and $|B|$ respectively. But this implies the independence of $X(B_1)\ldots, X(B_k)$ and $X(B)$ in the usual sense.

This completes the proof of Theorem 1.

To formulate an asymptotic generalisation of Theorem 1 we need three new conditions which are weaker versions of Conditions 1-3.

Let $\{X_n, n \geq 1\}$ be a sequence of random processes of D_ν.

Condition 1' For $t \in [0,1), \hat{B} = (\hat{s}, \hat{t}] \subset \mathbf{T}^{\nu-1}$,

a)

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\frac{1}{h}E\left\{|E(X_n(\Delta_{t,t+h})/\mathcal{F}_n(t,\hat{s}))|\right\} = 0$$

b)

$$\lim_{h\downarrow 0}\limsup_{n\to\infty}\frac{1}{h}E\left\{|E(X_n^2(\Delta_{t,t+h})/\mathcal{F}_n(t,\hat{s})) - h|\hat{B}||\right\} = 0$$

Here $\mathcal{F}_n(t,\hat{s}) = \sigma\{X_n(\mathbf{u}), \mathbf{u} \in \mathbf{T}^\nu_-((t,\hat{s}))\}$.

Condition 2'

$$\sup_{\mathbf{t}\in\mathbf{T}^\nu}\limsup_{n\to\infty}E\{X_n^2(\mathbf{t})\} < +\infty\,.$$

Condition 3' For $0 \leq t < 1$

$$\lim_{\alpha\to\infty}\limsup_{h\downarrow 0}\limsup_{n\to\infty}\frac{1}{h}\int_{X_n^2(\Delta_{t,t+h})\geq\alpha h} X_n^2(\Delta_{t,t+h})dP = 0$$

Theorem 2 *Let $\{X_n(\mathbf{t}), \mathbf{t} \in \mathbf{T}^\nu\}$ be a sequence of random processes in D_ν, uniformly integrable for each $\mathbf{t} \in \mathbf{T}^\nu$. Suppose that, for each $\mathbf{t} \in \mathbf{T}^\nu_o$, the sequence $X_n(\mathbf{t})$ tends in probability to 0 as $n \to \infty$ and that, for any positive ε and η, there exists $\delta > 0$ such that for all sufficiently large n*

(4) $\quad P(w(X_n,\delta) \geq \varepsilon) \leq \eta.$

If $\{X_n\}$ satisfies Conditions 1'-3' then X_n converges in law to W, where W is the ν-parameter Wiener process on $\mathbf{T}^\nu$.

Proof : The tightness of the sequence $\{X_n\}$ is proven in [8], Theorem 2 or [6], Theorem 5.6. (as generalisation of Billingsley's criteria for 1-parameter processes).

Let us denote by X a weak limit of a convergent subsequence of $\{X_n\}$; then $P(X \in C_\nu) = 1$ and $P(X(\mathbf{t}) = 0) = 1$ for each $\mathbf{t} \in \mathbf{T}_o^\nu$. Since $\{X_n\}$ satisfy Conditions 1'-3' it implies that X satisfies Conditions 1-3 and also satisfies the hypotheses of Theorem 1, which completes the proof.

3 An invariance principle for martingale-difference fields

Before we present the limit theorem, let us recall some notions on the class of fields we consider.

On the ν-dimensional integer lattice $\mathbb{Z}^\nu$, we consider a real-valued random field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$. The corresponding probability space is $(\Omega, \mathcal{F}, P)$, where $\Omega = \mathbb{R}^{\mathbb{Z}^\nu}$, $\mathcal{F}$ is the σ-algebra generated by cylinder sets and P is the distribution of $\xi(\mathbf{t})$.

Let $\mathcal{I}$ be the σ-algebra of invariant subsets of Ω :

$$\mathcal{I} = \{A \in \mathcal{F} \colon \tau_{\mathbf{u}}(A) = A \text{ for each } \mathbf{u} \in \mathbb{Z}^\nu\}$$

where $\{\tau_{\mathbf{u}}, \mathbf{u} \in \mathbb{Z}^\nu\}$ is the group of translations, acting on Ω by

$$(\tau_{\mathbf{u}} X)(t) = X(\mathbf{t} - \mathbf{u}), \mathbf{t} \in \mathbb{Z}^\nu.$$

Definition 1 *A random field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$ is called translation invariant (homogeneous) if $P(\tau_{\mathbf{u}}(A)) = P(A)$ for each $A \in \mathcal{F}$ and $\mathbf{u} \in \mathbb{Z}^\nu$.*

Definition 2 *A translation invariant random field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$ is called ergodic if P is trivial on the σ-algebra of invariant subsets, i.e. $P(A) = 0$ or $P(A) = 1$ for each $A \in \mathcal{I}$.*

For $\mathbf{u} = (u^{(1)}, \ldots, u^{(\nu)}) \in \mathbb{Z}^\nu$ let

$$\mathbb{Z}_-^\nu(\mathbf{u}) = \{\mathbf{t} \in \mathbb{Z}^\nu \colon \exists j, 1 \le j \le \nu \text{ such that } t^{(j)} \le u^{(j)}\}$$

and let $\mathbb{Z}_+^\nu(\mathbf{u}) = \mathbb{Z}^\nu \setminus \mathbb{Z}_-^\nu(\mathbf{u})$.

For a random field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$ we put

(5) $\quad \mathcal{P}(\mathbf{u}) = \sigma\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}_-^\nu(\mathbf{u})\}$

Definition 3 *We call a random field $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$ a martingale-difference if for each $\mathbf{t} \in \mathbb{Z}^\nu$*

(6) $\quad E(\xi(\mathbf{t})/\mathcal{P}(\mathbf{t} - \mathbf{1})) = 0 \, a.s.$

where $\mathbf{t} - \mathbf{1} = (t^{(1)} - 1, \ldots, t^{(\nu)} - 1)$.

Note that our definition of martingale-difference random field is weaker than the definition given in [3], where the filtration $\mathcal{P}(\mathbf{t}-\mathbf{1})$ (past of $\mathbf{t}-\mathbf{1}$) is replaced by the filtration generated by all sites of $\mathbb{Z}^\nu$ different of $\mathbf{t}$.

The following Theorem 3 is the main result of the present paper. It is a multidimensional extension of Theorem 23.1 of [2].

Theorem 3 *Let $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$ be a translation invariant, ergodic, martingale-difference random field with finite second moment $0 < \sigma^2 = E\{\xi^2(0)\} < \infty$. Let*

$$(7) \quad X_n(\mathbf{t}) = \frac{1}{\sigma n^{\nu/2}} \sum_{\substack{\mathbf{u}\in\mathbb{Z}^\nu \\ 0<\mathbf{u}\leq[n\mathbf{t}]}} \xi(\mathbf{u}), \mathbf{t} \in \mathbf{T}^\nu ,$$

where $[n\mathbf{t}] = ([nt^{(1)}], \ldots, [nt^{(\nu)}])$ and $[\cdot]$ denotes the integer part of a number.

Then

$$X_n \stackrel{\mathcal{D}}{\to} W ,$$

where W is the ν-parameter Wiener process on $\mathbf{T}^\nu$.

Proof : To prove the theorem it is enough to show that the sequence $\{X_n(\mathbf{t})\}$ of D_ν-valued random elements defined by (7) satisfies the hypotheses of Theorem 2.

From (6) we get that

$$(8) \quad E(\xi(\mathbf{s})/\mathcal{P}(\mathbf{t})) = 0 \, a.s.$$

for any $\mathbf{s} \in \mathbb{Z}^\nu_+(\mathbf{t})$.

If $B = (s,t] \times \hat{B}$ is a block in $\mathbf{T}^\nu$, $\hat{B} = (\hat{s}, \hat{t}] \subset \mathbf{T}^{\nu-1}$, then by $[n\hat{B}]$ we denote the block $([n\hat{s}], [n\hat{t}]]$ and by $[nB]$ the block $([ns],[nt]] \times [n\hat{B}]$. Note that $[nB] \subset \mathbb{Z}^\nu$.

It is easy to see that

$$X_n(B) = \frac{1}{\sigma n^{\nu/2}} \sum_{\mathbf{u}\in[nB]} \xi(\mathbf{u})$$

Therefore by (8)

$$E(X_n(\Delta_{t,t+h})/\mathcal{F}_n(t,\hat{s})) = 0 \ a.s. ,$$

where $\mathcal{F}_n(t,\hat{s}) = \sigma\{\xi(\mathbf{u}), \mathbf{u} \in \mathbb{Z}^\nu_-(t,\hat{s}), 0 < \mathbf{u} \leq \mathbf{n}\} \subset \mathcal{P}(([nt],[n\hat{s}]))$ (see (5)).

Using again (8) we find that

$$E\big(X_n^2(\Delta_{t,t+h})/\mathcal{F}_n(t,\hat{s})\big) = \sum_{\mathbf{u}\in[n\Delta_{t,t+h}]} E\big(\xi^2(\mathbf{u})/\mathcal{F}_n(t,\hat{s})\big) .$$

Hence

$$\frac{1}{h} E\left\{\left|E\big(X_n^2(\Delta_{t,t+h})/\mathcal{F}_n(t,\hat{s})\big) - h|\hat{B}|\right|\right\}$$
$$= \frac{|\hat{B}|}{\sigma^2} E\left\{\left|E\Big[\frac{1}{n^\nu h|\hat{B}|} \sum_{\mathbf{u}\in[n\Delta_{t,t+h}]} \xi^2(\mathbf{u}) - \sigma^2/\mathcal{F}_n(t,\hat{s})\Big]\right|\right\}$$

The last term tends to zero as $n \to \infty$ by ergodicity. (Indeed since $|[n\Delta_{t,t+h}]|$ is equivalent to $n^\nu h|\hat{B}|$, we have, by the mean ergodic theorem, that $\frac{1}{n^\nu h|\hat{B}|}\sum_{\mathbf{u}\in[n\Delta_{t,t+h}]} \xi^2(\mathbf{u}) \to \sigma^2$ when n tends to $+\infty$).

Thus Condition 1' is fulfilled.

Condition 2' follows from the fact that

$$E\{X_n^2(\mathbf{t})\} = \frac{1}{\sigma^2 n^\nu} \sum_{0<\mathbf{u}\leq[n\mathbf{t}]} E\{\xi^2(\mathbf{u})\}$$

tends to 1, as $n \to \infty$.

Now we will show that to complete the proof of Theorem 3 it is sufficient to prove that

$$(9) \quad \lim_{\alpha\to\infty} \sup_n E_\alpha\big(\frac{1}{n^\nu} \max_{|\mathbf{k}|\le n} S^2(\mathbf{k})\big) = 0\,,$$

where

$$S(\mathbf{k}) = \sum_{\mathbf{t}\le\mathbf{k}} \xi(\mathbf{t}), \mathbf{k} \in \mathbb{Z}_+^\nu(0)\,,$$

$$E_\alpha(Y) = \int_{\{Y\ge\alpha\}} Y dP\,.$$

Suppose that (9) holds.

According to a simple multidimensional extension of Theorem 8.4 from [2], in order to verify the tightness condition (4) of Theorem 2, it is sufficient to show that for any $\varepsilon > 0$, there exist $\lambda > 1$ and n_o such that

$$(10) \quad P\big(\max_{|\mathbf{k}|\le n} |S(\mathbf{k})| \ge \lambda n^{\nu/2}\big) \le \frac{\varepsilon}{\lambda^2}\,, n \ge n_o.$$

But

$$P\big(\frac{1}{n^\nu} \max_{|\mathbf{k}|\le n} |S^2(\mathbf{k})| \ge \lambda^2\big) \le \frac{1}{\lambda^2} E_{\lambda^2}\big(\frac{1}{n^\nu} \max_{|\mathbf{k}|\le n} |S^2(\mathbf{k})|\big)$$

which together with (9) implies (10).

To get the uniform integrability of $\{X_n^2(\mathbf{t})\}$ for each $\mathbf{t} \in \mathbf{T}^\nu$, we note that

$$E_\alpha\{X_n^2(\mathbf{t})\} \le E_\alpha\big(\frac{1}{\sigma n^\nu} \max_{|\mathbf{k}|\le n} S^2(\mathbf{k})\big)\,.$$

Using the translation invariance of $\{\xi(\mathbf{t}), \mathbf{t} \in \mathbb{Z}^\nu\}$, we can rewrite Condition 3' into the form :

$$\lim_{\alpha\to\infty} \limsup_{h\downarrow 0} \limsup_{n\to\infty} \int_{X_n^2((h,\hat{t}-\hat{s}))\ge\alpha h} X_n^2((h,\hat{t}-\hat{s}))dP = 0\,,$$

where $(h, \hat{t} - \hat{s}) \in \mathbf{T}^\nu$. This is now a consequence of the uniform integrability of $\{X_n^2(\mathbf{t})\}$ for each $\mathbf{t} \in \mathbf{T}^\nu$.

Thus it remains to prove formula (9).

If B is a block (parallelepiped) in $\mathbb{Z}^\nu$, then by (8),

$$(11) \quad E\left\{(\sum_{\mathbf{u}\in B} \xi(\mathbf{u}))^2\right\} = \sum_{\mathbf{u}\in B} E\{\xi^2(\mathbf{u})\}\,,$$

and if ξ_0 has a fourth moment then

$$E\left\{\big(\sum_{\mathbf{u}\in B} \xi(\mathbf{u})\big)^4\right\} = \sum_{\mathbf{u}\in B} E\{\xi^4(\mathbf{u})\} + 4 \sum_{\substack{\mathbf{u}_1,\mathbf{u}_2\in B\\ \mathbf{u}_1<\mathbf{u}_2}} E\left\{\xi(\mathbf{u}_1)\xi^3(\mathbf{u}_2)\right\}$$

$$+6 \sum_{\substack{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\in B\\ \mathbf{u}_1,\mathbf{u}_2<\mathbf{u}_3}} E\left\{\xi(\mathbf{u}_1)\xi(\mathbf{u}_2)\xi^2(\mathbf{u}_3)\right\}$$

Suppose first that $|\xi(0)|$ is bounded by q with probability 1. Then

$$\begin{aligned} E\left\{(\sum_{\mathbf{u}\in B}\xi(\mathbf{u}))^4\right\} &\leq q^4|B| + 4q^4\frac{|B|^2}{2^\nu} + 6q^4\frac{|B|^2}{2^\nu} \\ &\leq K_\nu q^4\cdot|B|^2\,, \end{aligned} \tag{12}$$

where $K_\nu = 1 + \frac{10}{2^\nu}$.

By Cairoli's maximal inequality ([7], Theorem 2.2)

$$E\left\{\max_{|\mathbf{k}|\leq n}\left|S(\mathbf{k})\right|^\gamma\right\} \leq (\frac{\gamma}{\gamma-1})^{\gamma\nu}\max_{|\mathbf{k}|\leq n}E\left\{\left|S(\mathbf{k})\right|^\gamma\right\},\gamma>1\,.$$

Hence by (11)

$$E\left\{\max_{|\mathbf{k}|\leq n}S^2(\mathbf{k})\right\} \leq 2^{2\nu}n^\nu E\{\xi^2(0)\} \tag{13}$$

In the same way it follows from (8) that

$$E\left\{\max_{|\mathbf{k}|\leq n}S^4(\mathbf{k})\right\} \leq (\frac{4}{3})^{4\nu}K_\nu n^{2\nu}q^4.$$

For $c>0$, we define

$$\xi_c(\mathbf{t}) = \begin{cases} \xi(\mathbf{t}) & \text{if } |\xi(\mathbf{t})|\leq c, \\ 0 & \text{if } |\xi(\mathbf{t})|>c \end{cases}$$

Let

$$\eta_c(\mathbf{t}) = \xi_c(\mathbf{t}) - E(\xi_c(\mathbf{t})/\mathcal{P}(\mathbf{t}-\mathbf{1})),$$

$$\delta_c(\mathbf{t}) = \xi(\mathbf{t}) - \eta_c(\mathbf{t}) = \xi(\mathbf{t}) - \xi_c(\mathbf{t}) - E(\xi(\mathbf{t})-\xi_c(\mathbf{t})/\mathcal{P}(\mathbf{t}-\mathbf{1})).$$

Evidently $\xi(\mathbf{t}) = \eta_c(\mathbf{t}) + \delta_c(\mathbf{t})$.

If we denote by

$$S_c(\mathbf{k}) = \sum_{\mathbf{t}\leq\mathbf{k}}\eta_c(\mathbf{t}), R_c(\mathbf{k}) = \sum_{\mathbf{t}\leq\mathbf{k}}\delta_c(\mathbf{t}), \mathbf{k}\in\mathbb{Z}_+^\nu(0)\,,$$

we obtain that

$$S(\mathbf{k}) = S_c(\mathbf{k}) + R_c(\mathbf{k})\,.$$

Therefore

$$\frac{1}{n^\nu}\max_{|\mathbf{k}|\leq n}S^2(\mathbf{k}) \leq \frac{2}{n^\nu}\max_{|\mathbf{k}|\leq n}S_c^2(\mathbf{k}) + \frac{2}{n^\nu}\max_{|\mathbf{k}|\leq n}R_c^2(\mathbf{k})\,.$$

This, together with the inequality

$$E_\alpha(X+Y) \leq 2E_{\alpha/2}(X) + 2E_{\alpha/2}(Y),$$

implies that

$$\begin{aligned} E_\alpha\left\{\frac{1}{n^\nu}\max_{|\mathbf{k}|\leq n}S^2(\mathbf{k})\right\} &\leq 2E_{\alpha/2}\left\{\frac{2}{n^\nu}\max_{|\mathbf{k}|\leq n}S_c^2(\mathbf{k})\right\} \\ &+ 2E_{\alpha/2}\left\{\frac{2}{n^\nu}\max_{|\mathbf{k}|\leq n}R_c^2(\mathbf{k})\right\}\,. \end{aligned} \tag{14}$$

Applying the formula

$$E_\alpha\{\xi\} \le \frac{1}{\alpha} E\{\xi^2\}, \xi \ge 0\,,$$

we find that

$$\begin{aligned} 2E_{\alpha/2}\left\{\frac{2}{n^\nu}\max_{|\mathbf{k}|\le n} S_c^2(\mathbf{k})\right\} &\le \frac{4}{\alpha} E\left\{\frac{4}{n^{2\nu}}\max_{|\mathbf{k}|\le n} S_c^4(\mathbf{k})\right\} \\ &\le \frac{1}{\alpha} 16 K_\nu \big(\frac{4}{3}\big)^{4\nu}(2c)^4 = \frac{1}{\alpha}\overline{K}_\nu c^4\,, \end{aligned} \tag{15}$$

where $\overline{K}_\nu = 16^2 K_\nu \big(\frac{4}{3}\big)^{4\nu}$.

Now by (13)

$$\begin{aligned} 2E_{\alpha/2}\left\{\frac{2}{n^\nu}\max_{|\mathbf{k}|\le n} R_c^2(\mathbf{k})\right\} &\le 2E\left\{\frac{2}{n^\nu}\max_{|\mathbf{k}|\le n} R_c^2(\mathbf{k})\right\} \\ &\le 4\cdot 2^{2\nu} E\{\delta_c^2(o)\}\,. \end{aligned} \tag{16}$$

By lemma 1 p. 184 from [2] , for any two σ-algebras $\mathcal{A}_1 \subset \mathcal{A}_2$ and a random variable X with $E\{X^2\} < \infty$ the following inequality holds :

$$E\{[X - E(X/\mathcal{A}_2)]^2\} \le E\{[X - E(X/\mathcal{A}_1)]^2\}.$$

Hence taking $\mathcal{A}_1$ trivial we get that

$$E\{\delta_c^2(0)\} \le E\{(\xi(0) - \xi_c(0))^2\} = E_{c^2}\{\xi^2(0)\} \tag{17}$$

Combining (14)-(17) we find that

$$E_\alpha\left\{\frac{1}{n^\nu}\max_{|\mathbf{k}|\le n} S^2(\mathbf{k})\right\} \le \frac{1}{\alpha}\overline{K}_\nu c^4 + 4\cdot 2^{2\nu} E_{c^2}\{\xi^2(0)\}\,.$$

Since $E\xi^2(0) < \infty$ we get the desired formula (9).

Theorem 3 is proved.

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