SIAM J. CONTROL OPTIM.
Vol. 47, No. 5, pp. 2598–2615

A CONSTRUCTIVE APPROACH TO A CLASS OF ERGODIC HJB EQUATIONS WITH UNBOUNDED AND NONSMOOTH COST*

PATRICK CATTIAUX†, PAOLO DAI PRA‡, AND SYLVIE RŒLLY§

Abstract. We consider a class of ergodic Hamilton–Jacobi–Bellman (HJB) equations related to long-time asymptotics of nonsmooth multiplicative functional of diffusion processes. Under suitable ergodicity assumptions on the underlying diffusion, we show existence of these asymptotics and that they solve the related HJB equation in the viscosity sense.

Key words. long-time asymptotics, cluster expansion, viscosity solution, HJB equation

AMS subject classifications. 47D07, 49L25, 60H10, 60J35, 60J60, 93E20

DOI. 10.1137/070698634

1. Introduction.

Let $(x_t)_{t\geq 0}$ be a continuous-time, homogeneous Markov process with infinitesimal generator L. To fix ideas, assume x_t is $\mathbb{R}^d$-valued. Given a function $c : \mathbb{R}^d \to \mathbb{R}$ and $\gamma > 0$, we are interested in obtaining long-time asymptotics of the functional

$$S(T, x) := \log E_x \left[\exp\left(\gamma \int_0^T c(x_t)dt \right) \right],$$

where E_x is the expectation conditioned to $x_0 = x$. Let $\varphi(T, x) = e^{S(T,x)}$. At least at the formal level, φ is a solution of the equation

$$\partial_t \varphi(t, x) = L\varphi(t, x) + \gamma c(x)\varphi(t, x).$$

If a Perron–Frobenius-type theorem holds for the operator $L + \gamma c$, then for T large $\varphi(T, x)$ gets close to $e^{\lambda T} v(x)$, where λ is the largest eigenvalue of $L + \gamma c$, and v is the corresponding strictly positive eigenfunction. In other words, setting $V(x) := \log v(x)$, we obtain

$$S(T, x) = \lambda T + V(x) + o(T),$$

i.e.,

$$\lambda = \lim_{T\to+\infty} \frac{1}{T} \log E_x \left[\exp\left(\gamma \int_0^T c(x_t)dt \right) \right] \tag{1.1}$$

and

$$V(x) = \lim_{T\to+\infty} \left\{ \log E_x \left[\exp\left(\gamma \int_0^T c(x_t)dt \right) \right] - \lambda T \right\}. \tag{1.2}$$

*Received by the editors July 30, 2007; accepted for publication (in revised form) June 5, 2008; published electronically October 13, 2008. This research was supported by the Deutsch-Französische Universität through the doctoral program CDFA 01-06.
http://www.siam.org/journals/sicon/47-5/69863.html
†Institut de Mathématiques, Université Paul Sabatier, 31062 Toulouse cedex 09, France (cattiaux@math.univ-toulouse.fr).
‡Matematica Pura e Applicata, Università di Padova, 31522 Padova, Italy (daipra@math.unipd.it).
§Institut für Mathematik, Universität Potsdam, 14469 Potsdam, Germany (roelly@math.uni-potsdam.de).

Note also that the pair (λ, V) is a solution of the nonlinear equation

$$\lambda = e^{-V} L(e^V) + \gamma c. \tag{1.3}$$

The actual proof of the existence of the limits (1.1) and (1.2) is, in general, not simple, and various assumptions are required. If the empirical measures

$$\mathcal{L}_t := \frac{1}{t}\int_0^t \delta_{x_s} ds$$

of the Markov process obey a large deviation principle with rate function $i(\mu)$ (which is known under fairly general conditions), and $c(\cdot)$ is measurable and bounded (but suitable growth conditions on $c(\cdot)$ may suffice), then the limit (1.1) exists, and

$$\lambda = \sup_{\mu}\left[\int c d\mu - i(\mu)\right], \tag{1.4}$$

where in (1.3) μ varies over probability measures on $\mathbb{R}^d$. The existence of the limit (1.2), i.e., the second-order asymptotics of $S(T,x)$, is a harder problem to solve. For processes taking values in a compact space, where things are simpler, we refer to [7, section 4]. In this paper we consider $\mathbb{R}^d$-valued diffusions of the form

$$dx_t = b(x_t)dt + dB_t, \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and b is a regular drift function. Thus the associated infinitesimal generator is $L = \frac{1}{2}\Delta + b(x)\cdot\nabla$. The first results in this context date back to [6] and [14], where conditions are given for the existence of the solution of (1.3), which takes the form of the Hamilton–Jacobi–Bellman (HJB) equation

$$\begin{aligned}\lambda &= \frac{1}{2}\Delta V(x) + \max_{u\in\mathbb{R}^d}\left[(b(x)+u)\cdot\nabla V(x) + \gamma c(x) - \frac{1}{2}|u|^2\right]\\ &= \frac{1}{2}\Delta V(x) + b(x)\cdot\nabla V(x) + \frac{1}{2}|\nabla V(x)|^2 + \gamma c(x).\end{aligned} \tag{1.6}$$

In [6] it is also shown that, under sufficient ergodicity of $(x_t)_{t\geq 0}$ and if $c(\cdot)$ is bounded and sufficiently smooth, then (1.3) has a solution, which need not be the unique one, for which (1.1) and (1.2) hold. More recent results, which require boundedness from above of $c(\cdot)$ but not smoothness, can be found in [5, Appendix B] or [20, Proposition 6.4].

The case of $c(\cdot)$ unbounded has been recently dealt with in [11] and [10].

In [11] the authors deal with a discrete-time process; it is plausible that many of their proofs can be adapted to continuous time. Their approach is based on a rather sophisticated spectral theory (see also [12]). The translation of their results into our context would allow to prove the existence of the limits (1.1) and (1.2) for any measurable $c(\cdot)$ whose growth at infinity is *strictly less* than quadratic. The assumptions are related to contractivity of the transition operator. In term of continuous-time diffusions, this corresponds to existence of *spectral gap* for the infinitesimal generator of the diffusion, in the space $L^2(m)$, where m is the invariant measure (see assumption A5 below).

In [10] the authors allow quadratic growth for $c(\cdot)$, but require differentiability. Using PDE methods they show, under reasonable conditions on $b(\cdot)$, that (1.3) indeed has multiple solutions, even after identifying solutions that differ by a constant. It is shown in [10] that there exists $\lambda \in \mathbb{R}$ such that the equation

$$\mu = \frac{1}{2}\Delta V + b \cdot \nabla V + \frac{1}{2}|\nabla V|^2 + \gamma c$$

has a (smooth) solution if and only if $\mu \geq \lambda$. Moreover, for $\mu = \lambda$, this solution is unique up to additive constant. Kaise and Sheu also indicate that this λ should be as in (1.1). They do not address the possibility of interpreting *one* solution V as in (1.2).

The main objective of this paper is to propose a totally different approach to the above problems. On one hand we tackle (1.1) and (1.2), for the diffusion (1.5), directly, without relying on properties of (1.3). This makes it easy to avoid any regularity condition on $c(\cdot)$. On the other hand, unlike in [11], we allow $c(\cdot)$ to have quadratic growth. We remark that quadratic growth of $c(\cdot)$ makes a Gaussian concentration property (see assumption A3 below) a natural assumption. This property is by no means implied by existence of spectral gap for the generator (which corresponds to our assumption A5). Our assumptions A1–A6 are discussed in detail in section 3.

The method we propose is based on *cluster expansion*, a well-known method in statistical mechanics and combinatorics. Besides the technical advantage of allowing quadratic growth without requiring regularity, our approach has, we believe, other positive aspects as follows:

1. It is considerably simpler than both PDE and spectral methods. Moreover, in principle it allows us to obtain explicit estimates on the limits (1.1) and (1.2) in terms of various parameters related to the drift $b(\cdot)$.
2. It is a very robust method which can be adapted to various modifications of the problem considered here. For example, in (1.1) and (1.2) the integral

$$\int_0^T c(x_t)dt$$

could be replaced by

$$\int_{[0,T]} c(x_t)d\mu(t),$$

where μ could be of the following forms:

   i. μ is a σ-finite periodic measure, for instance, $\mu(dt) = \sum_{k\geq 0}\delta_{k\Delta}(dt)$ for some $\Delta > 0$. In this last case the cost acts only at discrete time.

   ii. μ is a random measure, independent of $(B_t)_{t\geq 0}$, translation invariant, and sufficiently ergodic in law. For instance, we could take $\mu(dt) = \sum_n \delta_{\tau_n}(dt)$, where $(\tau_n)_{n\geq 0}$ are the points of a Poisson process.

Moreover, jump processes, rather than diffusions, should also be treatable.

We also remark that, although in this paper we consider diffusions whose diffusion coefficient is the identity matrix, the uniformly elliptic case could be dealt with after minor modifications. It is worth noticing that the whole content of section 2 is based on assumptions A1–A6 below, which do not refer to any specific form of the Markov process. The fact that the process is a diffusion plays a role in sections 3 and 4.

In the case when $c(\cdot)$ has quadratic growth, $S(t, x)$ could possibly explode in finite time, unless γ is sufficiently small. At the present stage, our results hold for γ in some interval $[0,\overline{\gamma}]$, which is certainly *not optimal*. Note, however, that one could get an explicit expression for $\overline{\gamma}$ (carefully following the proofs) as a function of the constants c_b and K_b appearing in conditions (DC) and (CC) of section 3.

The paper is organized as follows. In section 2 we prove existence of the limits (1.1) and (1.2) under some general conditions (A1–A6 below) on the diffusion process. In section 3 we give explicit sufficient conditions on the drift b for A1–A6 to hold. In section 4 we show that V and λ given by (1.2) and (1.1), respectively, are linked to (1.6); more precisely, we show that V is a viscosity solution of (1.6).

2. Existence of the limits (λ, V). We begin by stating our assumptions on the $\mathbb{R}^d$-valued diffusion

$$dx_t = b(x_t)dt + dB_t. \tag{2.1}$$

A1. Equation (2.1) has, for every deterministic initial condition, a unique strong solution.

A2. There is a $C > 0$ such that

$$|c(x)| \leq C(|x|^2 + 1), \quad x \in \mathbb{R}^d.$$

A3. The process $(x_t)_t$ which is a solution of (2.1) has a unique invariant probability measure $m(dx)$ such that, for some $\beta > 0$,

$$\int e^{\beta|x|^2} m(dx) < +\infty.$$

A4. The transition probability of the process $(x_t)_t$ admits a density $p_t(x,y)$ with respect to the measure m. Furthermore, there exist $K > 0$, $p > 2$, and $t_0 > 0$ such that

$$\sup_{t\geq t_0} \| p_t(.,.) \|_{\mathbb{L}^p(m\otimes m)} \leq K .$$

A5. Let P_t be the semigroup associated with the process $(x_t)_t$. It extends as a continuous semigroup on $\mathbb{L}^2(m)$ and satisfies

$$\forall f \in \mathbb{L}^2(m), \quad \lim_{t\to+\infty} \| P_t f - \int f dm \|_{\mathbb{L}^2(m)} = 0.$$

A6. For all $a > 0$ and all x, there exists $\beta_{a,x} > 0$ such that

$$E_x \left[e^{\beta_{a,x} \int_0^a |x_s|^2 \, ds} \right] < +\infty.$$

We shall say that A6 is uniformly satisfied if for all $a > 0$ there exist $\beta_a > 0$ and a locally bounded function h_a such that for all x,

$$E_x \left[e^{\beta_a \int_0^a |x_s|^2 \, ds} \right] \leq h_a(x).$$

Section 3 will be devoted to giving sufficient conditions for these hypotheses to hold.

Remark 1. Assumption A4 implies that the semigroup P_t is continuous from $\mathbb{L}^2(m)$ into $\mathbb{L}^p(m), p > 2$, for $t \geq t_0$. Hence, according to the Gross hypercontractivity theorem (see, e.g., [1]), m satisfies a defective logarithmic Sobolev inequality. If m is absolutely continuous with respect to the Lebesgue measure, $m(dx) = e^{-V} dx$, and

V is locally bounded, a result by Röckner and Wang says that m satisfies a so-called weak Poincaré inequality; hence, thanks to a result by Aida, m will satisfy a tight log-Sobolev inequality (for all these results see the book of Wang [19]). In particular m will satisfy both a spectral gap inequality, so that Assumption A5 is satisfied, and a Gaussian concentration inequality implying A3.

THEOREM 1. *Under* A1–A6 *there is* $\overline{\gamma} > 0$ *such that for every* $\gamma < \overline{\gamma}$ *the limits* (1.1) *and* (1.2) *exist.*

Furthermore, if A6 *is uniformly satisfied, the convergence in* (1.2) *is uniform on compact sets.*

Proof of Theorem 1. We begin by showing that the limits (1.1) and (1.2) exist along suitable sequences.

PROPOSITION 1. *Under* A1–A6*, for every time-step* $a > 0$ *large enough, there exists* $\gamma(a)$ *such that for all* $\gamma < \gamma(a)$ *and all* $x \in \mathbb{R}^d$*, the limits*

$$\lambda_a = \lim_{n\to+\infty} \frac{1}{an} \log E_x \left[\exp\left(\gamma \int_0^{an} c(x_t)dt\right)\right] \tag{2.2}$$

and

$$V_a(x) = \lim_{n\to+\infty} \left\{\log E_x \left[\exp\left(\gamma \int_0^{an} c(x_t)dt\right)\right] - \lambda_a an\right\} \tag{2.3}$$

exist.

Proof of Proposition 1. The proof is done via a cluster expansion technique. The convergence of the expansion requires us to choose γ small enough and the time-step a large enough.

Define

$$\psi_\gamma(t,x,y) := \log E_{xy} \left[\exp\left(\gamma \int_0^t c(x_s)ds\right)\right], \tag{2.4}$$

where E_{xy} denotes the expectation under the law of the bridge of $(x_s)_{0\le s\le t}$ between x and y, and consider a time-step $a > 0$. Then

(2.5)
$$e^{S(an,x)} = E_x \left[\exp\left(\sum_{k=0}^{n-1} \psi_\gamma(a, x_{ka}, x_{(k+1)a})\right)\right] = \mathbb{E}\left[\exp\left(\sum_{k=0}^{n-1} \phi_\gamma(a, \xi_k, \xi_{k+1})\right)\right],$$

where $\mathbb{E}$ is the expectation with respect to a probability $\mathbb{P}$, and $\xi_0 = x, \xi_1, \ldots, \xi_n$ are random variables that, under $\mathbb{P}$, are independent and identically distributed (i.i.d.) with law $m(dx)$, and

$$\phi_\gamma(a,x,y) = \psi_\gamma(a,x,y) + \log p_a(x,y).$$

A cluster in this context is a subset of $\mathbb{Z}^+ := \{0,1,2,\ldots\}$ of the form $\{k, k+1, \ldots, k+l\}$. We say that two clusters are separated if there is an integer which is strictly larger than all elements of one cluster and strictly smaller that all elements of the other. We denote by $\mathcal{C}$ the set of all clusters, while $\mathcal{C}_n$ denotes the set of clusters contained in $\{0,1,\dots,n-1\}$. The usual cluster expansion procedure yields

$$\exp\left(\sum_{k=0}^{n-1}\phi_\gamma(a,\xi_k,\xi_{k+1})\right)=\prod_{k=0}^{n-1}\left[\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)+1\right]$$

$$=\sum_{\tau\subseteq\{0,1,\dots,n-1\}}\prod_{k\in\tau}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)=\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n\\ \text{separated}}}\prod_{i=1}^{p}\prod_{k\in\tau_i}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)$$

$$=\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n\\ \text{separated}}}q_{\tau_1}q_{\tau_2}\cdots q_{\tau_p},$$

where

$$q_{\tau_i}:=\prod_{k\in\tau_i}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right),$$

and we have used the fact that any subset of $\{0,\dots,n-1\}$ is a union of p separated clusters for some $p\geq 0$, and these clusters can be rearranged in $p!$ ways. The key fact is that if τ_i and τ_j are separated clusters, then q_{τ_i} and q_{τ_j} are independent. Thus, by (2.5),

$$e^{S(an,x)}=\sum_{p\geq 0}\frac{1}{p!}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n\\ \text{separated}}}\mathbb{E}(q_{\tau_1})\mathbb{E}(q_{\tau_2})\cdots\mathbb{E}(q_{\tau_p}).$$

The logarithm of the above expression can be rewritten as

$$S(an,x)=\sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\sum_{p\geq 0}\sum_{\substack{\tau_1,\dots,\tau_p\in\mathcal{C}_n\\ \tau_1\cup\cdots\cup\tau_p=\tau}}a_p(\tau_1,\tau_2,\dots,\tau_p)\mathbb{E}(q_{\tau_1})\mathbb{E}(q_{\tau_2})\cdots\mathbb{E}(q_{\tau_p}):=\sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\Gamma_\tau,$$

where the coefficients $a_p(\tau_1,\dots,\tau_p)$ come from the Taylor expansion of the logarithm (see [13, page 492]). Now note that Γ_τ depends on x if and only if $0\in\tau$, i.e., $\tau=\{0,1,\dots,m\}$ for some m. In what follows, we write Γ_m in place of $\Gamma_{\{0,1,\dots,m\}}$. Thus

$$\sum_{\tau\in\mathcal{C}_n,\tau\neq\emptyset}\Gamma_\tau=\sum_{m=0}^{n-1}\Gamma_m+\sum_{i=1}^{n-1}\sum_{\substack{\tau\in\mathcal{C}_n,\tau\not\ni 0\\ i\in\tau}}\frac{1}{|\tau|}\Gamma_\tau=\sum_{m=0}^{n-1}\Gamma_m+(n-1)\sum_{\substack{\tau\in\mathcal{C}_n,\tau\not\ni 0\\ 1\in\tau}}\frac{1}{|\tau|}\Gamma_\tau,\tag{2.6}$$

where we used the fact that, for $0\notin\tau$, Γ_τ is invariant by translation and permutation of τ. Thus, at a formal level, the limits (2.2) and (2.3) should be given by

$$\lambda_a=\frac{1}{a}\sum_{\substack{\tau\in\mathcal{C},\tau\not\ni 0\\ 1\in\tau}}\frac{1}{|\tau|}\Gamma_\tau,\tag{2.7}$$

$$V_a(x)=\sum_{m=0}^{+\infty}\Gamma_m-a\lambda_a.\tag{2.8}$$

As usual (see, e.g., [4]), the convergence of the above sums will follow from the following strong cluster estimates:

$$\exists\rho<1\quad\forall\tau\in\mathcal{C}\text{ with }0\notin\tau,\quad\mathbb{E}(q_\tau)|\leq\rho^{|\tau|}\tag{2.9}$$

and

$$\forall \tau \in \mathcal{C} \text{ with } \tau \ni 0, \quad |\mathbb{E}(q_\tau)| \le C(x)\rho^{|\tau|}, \tag{2.10}$$

where $C(\cdot)$ is a locally bounded function of x.

Thus, we have only to prove the estimates (2.9) and (2.10) for γ sufficiently small. We begin by proving (2.9). By the generalized Hölder inequality in [15, Lemma 5.2], we have

(2.11)

$$|\mathbb{E}(q_\tau)| = \left|\mathbb{E}\left[\prod_{k\in\tau}\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)\right]\right| \le \prod_{k\in\tau}\mathbb{E}^{1/2}\left[\left(e^{\phi_\gamma(a,\xi_k,\xi_{k+1})}-1\right)^2\right] = \rho^{|\tau|},$$

where

$$\rho := \mathbb{E}^{1/2}\left[\left(e^{\phi_\gamma(a,\xi_1,\xi_2)}-1\right)^2\right].$$

We now show that ρ can be made strictly less than 1 by choosing a sufficiently large and γ small enough:

$$\begin{aligned}
\rho^2 &= \mathbb{E}\left(\left(e^{\psi_\gamma(a,\xi_1,\xi_2)}p_a(\xi_1,\xi_2)-1\right)^2\right)\\
&= \int_{\mathbb{R}^{2d}}\left[E_{xy}\left(e^{\gamma\int_0^a c(x_s)ds}\right)p_a(x,y)-1\right]^2 m(dx)m(dy)\\
&= \int_{\mathbb{R}^{2d}}\left[E_{xy}\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a(x,y)+(p_a(x,y)-1)\right]^2 m(dx)m(dy)\\
&\le 2\int_{\mathbb{R}^{2d}}E_{xy}^2\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a^2(x,y)m(dx)m(dy)\\
&\quad +2\int_{\mathbb{R}^{2d}}(p_a(x,y)-1)^2 m(dx)m(dy)\\
&=: 2I_1(a,\gamma)+2I_2(a).
\end{aligned}$$

We first analyze $I_1(a,\gamma)$. For any $\varepsilon \in\,]0,1[$, by the Hölder inequality,

$$\begin{aligned}
I_1(a,\gamma) &= \left[\int_{\mathbb{R}^{2d}}E_{xy}^{2/\varepsilon}\left(e^{\gamma\int_0^a c(x_s)ds}-1\right)p_a(x,y)m(dx)m(dy)\right]^{\varepsilon}\\
&\quad\cdot\left[\int_{\mathbb{R}^{2d}}p_a^{\frac{2-\varepsilon}{1-\varepsilon}}(x,y)m(dx)m(dy)\right]^{1-\varepsilon}\\
&\le E_m^{\varepsilon}\left(|e^{\gamma\int_0^a c(x_s)ds}-1|^{2/\varepsilon}\right)\parallel p_a\parallel^{2-\varepsilon}_{\mathbb{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(m\otimes m)},
\end{aligned}$$

where E_m denotes the expectation under the law of $(x_t)_t$ with initial measure m. Thanks to assumption A4, for a large enough and ε small enough (such that $\frac{2-\varepsilon}{1-\varepsilon}\le p$), $\parallel p_a\parallel^{2-\varepsilon}_{\mathbb{L}^{\frac{2-\varepsilon}{1-\varepsilon}}(m\otimes m)} < +\infty$. To control $J(a,\gamma) := E_m\big(|e^{\gamma\int_0^a c(x_s)ds}-1|^{2/\varepsilon}\big)$, we represent $e^{\gamma\int_0^a c(x_s)ds}-1$ as $\gamma\int_0^a c(x_s)ds\int_0^1 e^{u\gamma\int_0^a c(x_s)ds}du$ and obtain

$$\begin{aligned}
J(a,\gamma) &= \gamma^{2/\varepsilon}E_m\left(\left|\int_0^a c(x_s)ds\right|^{2/\varepsilon}\left(\int_0^1 e^{u\gamma\int_0^a c(x_s)ds}du\right)^{2/\varepsilon}\right)\\
&\le \gamma^{2/\varepsilon}E_m^{1/2}\left(\left|\int_0^a c(x_s)ds\right|^{4/\varepsilon}\right)E_m^{1/2}\left(e^{\frac{4\gamma}{\varepsilon}\int_0^a c(x_s)ds}\right).
\end{aligned}$$

By Jensen's inequality,

$$J(a,\gamma) \le \gamma^{2/\varepsilon} a^{2/\varepsilon-1/2} E_m^{1/2}\left(\int_0^a |c(x_s)|^{4/\varepsilon} ds\right)\left[\frac{1}{a}\int_0^a E_m\left(e^{\frac{4\gamma}{\varepsilon}ac(x_s)}\right)ds\right]^{1/2}$$
$$\le (\gamma a)^{2/\varepsilon}\left[\int |c(x)|^{4/\varepsilon} m(dx)\right]^{1/2}\left[\int e^{\frac{4\gamma a}{\varepsilon}c(x)} m(dx)\right]^{1/2}.$$

The first integral term on the right-hand side in the above inequality is finite due to assumptions A2 and A3. For the same reason, if $\gamma a < \frac{\varepsilon}{4C}\beta$, the last integral term of the right-hand side is finite. Then, for γa small enough, $J(a,\gamma)$ and $I_1(a,\gamma)$ are as small as we want.

We now prove that $I_2(a)$ goes to 0 as $a \to +\infty$.

Lemma 1. *If* A4 *and* A5 *are satisfied,* $\lim_{a\to+\infty}\int_{\mathbb{R}^{2d}}(p_a(x,y)-1)^2 m(dx)m(dy) = 0$.

Proof. By assumption A5, the semigroup P_t is a contraction on $\mathbb{L}^2(m)$, and

$$\lim_{t\to+\infty}\int\left|P_t f(x) - \left(\int f dm\right)\right|^2 m(dx) = 0.$$

Notice also that, for $a > b > 0$ and m almost all y,

$$p_a(x,y) = P_{a-b}p_b(.,y)(x)$$

in $\mathbb{L}^2(m)$ for all rational times a and b and, by invariance of m, $\int p_b(x,y)m(dx) = 1$. Consider now an increasing sequence $(a_n)_{n\ge 0}$ such that $a_n \to +\infty$. We have to show that, for any such sequence,

$$\lim_n \int_{\mathbb{R}^{2d}} (p_{a_n}(x,y)-1)^2 m(dx)m(dy) = 0. \tag{2.12}$$

It is not restrictive to assume $a_1 > t_0$, where t_0 is the constant in assumption A4. For (m almost all) fixed y,

$$\int (p_{a_n}(x,y)-1)^2\, m(dx) = \int (P_{a_n-a_1}p_{a_1}(.,y)(x)-1)^2\, m(dx) \to 0$$

by assumption A5. But thanks to assumption A4, the sequence

$$y \mapsto \int (p_{a_n}(x,y)-1)^2\, m(dx)$$

is uniformly integrable, which implies (2.12) by the Vitali convergence theorem. □

We can now conclude that for γa small enough and for a large enough, the cluster estimate ρ is smaller than 1, which completes the proof of (2.9).

For the proof of (2.10), we proceed in the same way, just observing that the first factor in the right-hand side of (2.11) is now dependent on x. The additional term to control is $E_x\left[e^{q\gamma\int_0^a c(x_s)\,ds}\right]$ for some large $q > 1$. This can be done using A2 and A6. Thus, we completed the proof of Proposition 1.

To complete the proof of Theorem 1, we shall show why the limits (2.2) and (2.3) do not depend on the time-step a, yielding the limits (1.1) and (1.2). So we choose once and for all some convenient a and consider the corresponding set of convenient γ's, yielding for each γ a λ obtained thanks to Proposition 1. For large T we choose n such that $a(n-1) \le T < an$.

Notice that

$$S^-(an,x) \le S(T,x) \le S^+(an,x),$$

where

$$e^{S^-(an,x)} := E_x\left[e^{\gamma\int_0^{a(n-1)} c(x_s)ds}\, e^{-\gamma\int_{a(n-1)}^{an} c^-(x_s)ds}\right]$$

and

$$e^{S^+(an,x)} := E_x\left[e^{\gamma\int_0^{a(n-1)} c(x_s)ds}\, e^{\gamma\int_{a(n-1)}^{an} c^+(x_s)ds}\right].$$

Both $S^-(an,x)$ and $S^+(an,x)$ can be calculated using the same cluster expansion, except that now we have to replace $\psi_\gamma(a, x_{a(n-1)}, x_{an})$ with a function ψ^- (resp., ψ^+) obtained by replacing c with $-c^-$ (resp., c^+). We thus obtain a similar decomposition $S^-(an,x) = \sum_{T\in\mathcal{C}_n, T\neq\emptyset} \Gamma_T^-$ with $\Gamma_T^- = \Gamma_T$ if $n-1 \notin T$ and with Γ_T^- obviously modified if $n-1 \in T$. In particular, in the decomposition (2.6) we see that, in the first sum, the only modified term is Γ_{n-1}. But since $-c^-$ also satisfies A2, estimates similar to those in (2.9) and (2.10) hold true for both S^- and S^+, whose difference goes to 0 as n goes to infinity. This yields the desired result.

3. Some properties of diffusion processes and their invariant measures. In this section we provide explicit conditions on the drift $b(\cdot)$ for assumptions A1 and A3–A6 to hold for the diffusion process in (2.1). Our main result, Theorem 2 below, will be stated in terms of the following two drift conditions:

(3.1) Condition (DC) $\exists c_b > 0$ and $\exists R \ge 0$ s.t. for $|x| \ge R$, $\quad b(x)\cdot x \le -c_b|x|^2.$

The second condition is usually called a "curvature condition." Assume $b \in C^1$, and for $\xi \in \mathbb{R}^d$ recall the notation

$$\langle \nabla_\xi b(x), \xi\rangle = \sum_{i,j} \xi_i\, \partial_i b_j(x)\, \xi_j. \tag{3.2}$$

The curvature condition is then as follows:

(3.3) Condition (CC) $\quad \exists K_b \in \mathbb{R}$ s.t. $\forall\ x$ and all ξ, $\quad \langle \nabla_\xi b(x), \xi\rangle \le K_b|\xi|^2.$

In what follows, L denotes the generator $b(x)\cdot\nabla + \frac{1}{2}\Delta$, and P_t denotes the associated semigroup.

THEOREM 2. *Let $b \in C^1$. If* (DC) *holds, then Assumptions* A1 *and* A3 *are satisfied, and assumption* A6 *is uniformly satisfied.*

If (DC) *and* (CC) *are both satisfied with constants c_b, K_b such that $c_b > 2K_b$, then assumptions* A4 *and* A5 *are satisfied.*

Remark 2. The condition $c_b > 2K_b$ is a mild condition. Indeed, if we replace (DC) by a stronger (but more symmetric) condition, namely,

$$(b(x) - b(y))\,\cdot\,(x-y)\ \le -c_b\,|x-y|^2$$

(if $b = -\nabla V$, this is a convexity assumption), then we may choose $K_b < 0$.

It is also worth noticing that if we reinforce (DC), assuming the condition

$$\lim_{|x|\to\infty} b(x)\cdot\frac{x}{|x|^2} = -\infty,$$

then we may always choose $c_b > 2K_b$ (if K_b is finite, of course). In this situation it can be shown (see [19, Corollary 5.7.7]) that the semigroup P_t is superbounded.

Proof of Theorem 2. We divide the proof into several steps.

Step 1. A1 *holds under* (DC). Since $b \in C^1$, it is locally Lipschitz; thus existence and strong uniqueness are ensured up to the explosion time, starting from any x. Define now $\psi(x) = 1 + |x|^2$. By condition (DC) it is easy to check that $L\psi \leq C\psi$ for some $C > 0$. By applying Ito's rule to $\psi(x_t)$ up to the exit time of the level sets of ψ (in the same spirit as in [16, Théorème 2.2.19]), it follows that the explosion time is a.s. infinite.

Step 2. A3 *holds under* (DC). Existence and uniqueness of the invariant measure m under (DC) follow, for instance, from [17, Theorem 7.4.21]. We prove here that, if (DC) holds, then for all $\beta < c_b$, $\int e^{\beta|y|^2}\, m(dy) < +\infty$. Since (DC) holds, there exists $D > 0$ such that $b(x) \cdot x \leq -c_b|x|^2 + D$ for all x; we set $c = c_b$ in the proof of this step.

Let g_n be a smooth nondecreasing concave function defined on $\mathbb{R}^+$ such that $g_n(u) = u$ if $u \leq n-1$ and $g_n(u) = n$ if $u \geq n$ (such a function exists). Let $f_n(x) = \exp(\beta\, g_n(|x|^2))$ for $\beta < c$.

Then $\nabla f_n(x) = 2\beta f_n(x) g_n'(|x|^2) x$ and

$$\Delta f_n(x) = 2\beta f_n(x)\left(2g_n''(|x|^2)|x|^2 + 2\beta(g_n')^2(|x|^2)|x|^2 + d g_n'(|x|^2)\right)$$

so that

$$\begin{aligned} L f_n(x) &= \beta\, f_n(x)\, \left((2g_n''(|x|^2)|x|^2 + d g_n'(|x|^2)) + 2g_n'(|x|^2)(\beta g_n'(|x|^2)|x|^2 + b(x)\cdot x)\right) \\ &\leq \beta\, f_n(x)(d + 2D - 2(c-\beta)|x|^2) \\ &\leq \beta(d+2D)e^{\beta\frac{d+2D}{c-\beta}} \;-\; \beta(d+2D)\, f_n(x) \end{aligned}$$

since

$$d + 2D - 2(c-\beta)|x|^2 \leq -(c-\beta)|x|^2 \leq -(d+2D)$$

for $|x|^2 \geq \frac{d+2D}{c-\beta}$.

In short, there exist c_1 and c_2 positive constants such that for all n, $L f_n \leq c_1 - c_2 f_n$.

Define $h_n(s) = E_x\left[e^{\beta g_n(|x_s|^2)}\right]$. Ito's formula yields

$$h_n(t) \leq h_n(0) + c_1 t - c_2 \int_0^t h_n(s)ds,$$

and hence by applying Gronwall's lemma, we obtain

$$E_x\left[e^{\beta g_n(|x_t|^2)}\right] \leq \frac{c_1}{c_2} \,+\, e^{-c_2 t}\, e^{\beta g_n(|x|^2)}. \tag{3.4}$$

Integrating (3.4) with respect to the invariant measure m yields

$$(1 - e^{-c_2 t}) \int e^{\beta g_n(|y|^2)} m(dy) \leq \frac{c_1}{c_2}.$$

We may thus choose t large enough for $e^{-c_2 t} \leq 1/2$ and then use the monotone convergence theorem with $n \to +\infty$ in order to obtain $\int e^{\beta|y|^2} m(dy) < +\infty$ for $\beta < c_b$.

Step 3. A6 *is uniformly satisfied under* (DC). Using Ito's formula up to the exit time T_M of the ball of center 0 and radius M, we have

$$E_x\left[e^{\theta\,|x_{t\wedge T_M}|^2}\right] = e^{\theta|x|^2} + E_x\left[\int_0^{t\wedge T_M} \left(2\theta\, b(x_s)\cdot x_s + d\theta + 2\theta^2|x_s|^2\right)\, e^{\theta\,|x_s|^2}\, ds\right].$$

In particular, if condition (DC) holds with $c_b > \theta$, the integrand in the right-hand side is nonpositive for large values of $|x_s|$, and hence we can let M go to infinity in order to show that there exists some constant κ (depending on (DC) and θ) such that

$$E_x\left[e^{\theta\,|x_t|^2}\right] < e^{\theta|x|^2} + \kappa t.$$

Accordingly, using the Jensen inequality, we obtain

$$\begin{aligned} E_x\left[e^{\int_0^a \beta\,|x_s|^2\,ds}\right] &= E_x\left[e^{\frac{1}{a}\int_0^a a\,\beta\,|x_s|^2\,ds}\right] \\ &\le \frac{1}{a}\, E_x\left[\int_0^a e^{a\,\beta\,|x_s|^2}\, ds\right] < +\infty \end{aligned}$$

as soon as $a\beta < c_b$. The proof is completed.

Step 4. A4 *holds if* (DC) *and* (CC) *are both satisfied and if* $c_b > 2K_b$. Since $b \in C^1$, Malliavin calculus shows that the law of x_t is absolutely continuous w.r.t. the Lebesgue measure for all initial conditions x and all $t > 0$. Hence m is also absolutely continuous w.r.t. the Lebesgue measure, and it can be shown that dm/dy is a.e. positive. Thus, the existence of $p_t(x, y)$ follows. The proof of the integrability condition stated in A4 relies on a beautiful Harnack inequality derived by Wang (see [19, Theorem 2.5.2]),

$$(P_t f(x))^\alpha \;\le\; P_t f^\alpha(y)\, \exp\left(\frac{\alpha}{2(\alpha-1)}\, K_b(1-e^{-2K_b t})^{-1}\, |x-y|^2\right), \tag{3.5}$$

holding for $t > 0$, $\alpha > 1$, all (x, y), and all nonnegative continuous and bounded f, with the convention $K_b(1 - e^{-2K_b t})^{-1} = 1/2t$ if $K_b \le 0$ (see also [1, Lemma 7.5.4] if $\alpha = 2$).

Using (3.5), we show that for all $p > 2$, $p_t(.,.) \in \mathbb{L}^p(m \otimes m)$ for all t such that

$$c_b \;>\; \frac{K_b\, p\,(p-1)}{1-e^{-2K_b t}}.$$

In particular, if $K_b \le 0$, then for all $p > 2$ there exists t_p such that $p_t(.,.) \in \mathbb{L}^p(m\otimes m)$ for $t \ge t_p$, while for $K_b > 0$ such a t_p exists provided $c_b > K_b\, p\,(p-1)$. We shall first derive an upper bound for the density.

Let $\alpha > 1$, $D_t := \{x \in \mathbb{R}^d, |x| \le \gamma(t)\}$ for some increasing function γ going to ∞, and let f be nonnegative and bounded. Integrating the Harnack inequality for P_t with respect to $m(dy)$ on D_t and denoting

$$\kappa(t) = \frac{\alpha}{2(\alpha-1)}\, K_b(1-e^{-2K_b t})^{-1},$$

we get

$$\begin{aligned} ((P_t f)(x))^\alpha &\le \int_{D_t} (P_t f^\alpha)(y)\, e^{\kappa(t)|x-y|^2}\, m(dy)/m(D_t) \\ &\le \int f^\alpha(y)\, (P_t^*(\mathbb{1}_{D_t}(.)\, e^{\kappa(t)|x-.|^2}))(y)\, m(dy)/m(D_t) \\ &\le e^{2\kappa(t)(|x|^2+\gamma^2(t))} \int f^\alpha(y)\; m(dy)/m(D_t) \end{aligned}$$

since

$$\| \mathbb{I}_{D_t}(.)\, e^{\kappa(t)|x-.|^2} \|_\infty \leq e^{2\kappa(t)(|x|^2+\gamma^2(t))}.$$

If

$$\theta(t,x) = \left(e^{2\kappa(t)\gamma^2(t)}/m(D_t)\right) e^{2\kappa(t)|x|^2},$$

we thus have

$$((P_t f)(x))^\alpha \leq \theta(t) \int f^\alpha(y)\, m(dy). \tag{3.6}$$

Applying the previous inequality with a continuous approximation of $f_N(z) = p_t^\beta(x,z)\, \mathbb{I}_{\{p_t(x,z)\leq N\}}$ and then taking limits, we have

$$\left(\int p_t^{1+\beta}(x,z)\, \mathbb{I}_{\{p_t(x,z)\leq N\}}\, m(dz)\right)^\alpha \leq \theta(t,x) \int p_t^{\alpha\,\beta}(x,y)\, \mathbb{I}_{\{p_t(x,y)\leq N\}}\, m(dy);$$

i.e., by letting N go to ∞ and choosing $1+\beta = \alpha\,\beta$ and hence $\beta = 1/(\alpha-1)$ we obtain

$$\int p_t^{\frac{\alpha}{\alpha-1}}(x,y)\, m(dy) \leq \theta^{1/(\alpha-1)}(t,x). \tag{3.7}$$

By Step 2, the right-hand side in (3.7) is in $\mathbb{L}^1(m)$ provided

$$c_b > \frac{2\kappa(t)}{\alpha-1} = \frac{\alpha K_b}{(\alpha-1)^2(1-e^{-2K_b t})}. \tag{3.8}$$

In particular if $p > 2$, define $1 < \alpha = p/(p-1) < 2$. Hence $p_t(.,.) \in \mathbb{L}^p(m\otimes m)$ provided

$$c_b > \frac{K_b\, p\,(p-1)}{1-e^{-2K_b t}}.$$

Step 5. A5 *holds if* (DC) *and* (CC) *are both satisfied, and* $c_b > K_b$. (Note that the condition needed here is weaker than $c_b > 2K_b$.) It is well known that the $L^2(m)$-contractivity stated in assumption A5 is implied by *hypercontractivity* of P_t, which means that for all $1 < p < q < +\infty$ there exists $t_{p,q}$ such that for $t \geq t_{p,q}$, P_t is a bounded operator from $\mathbb{L}^p(m)$ into $\mathbb{L}^q(m)$ with norm equal to 1. The fact that $c_b > K_b$ implies hypercontractivity of P_t is shown in [19, Theorem 5.7.3, Corollary 5.7.2, and Theorem 5.7.1].

4. The limiting function as viscosity solution. We consider the function

$$\varphi(t,x) := E_x\left[\exp\left(\gamma\int_0^t c(x_s)ds\right)\right]. \tag{4.1}$$

We have shown that (under some assumptions we shall assume to be in force below), for γ sufficiently small, the limits

$$\lambda := \lim_t \frac{1}{t} \log \varphi(t,x) \tag{4.2}$$

and

$$V(x) := \lim_t [\log \varphi(t,x) - \lambda t] \tag{4.3}$$

exist uniformly over compact sets. We want to show that V is a *viscosity solution* of the HJB equation (1.6) or, equivalently, that $v(x) := e^{V(x)}$ is a viscosity solution of the linear equation

$$-\left[\frac{1}{2}\Delta v + b \cdot \nabla v + \gamma c v\right] + \lambda v = 0. \tag{4.4}$$

We first prove that $\varphi(T-t,x)$ is a continuous viscosity solution of a suitable evolution equation. Then by using (4.3) we show that (4.4) holds.

This problem has been dealt with in [9] in a much more general setting. However, the assumptions given in [9] are not satisfied here, due to the unboundedness of c. Thus, some modifications of their proof are needed.

PROPOSITION 2. *Assume that conditions* A1–A5 *are satisfied and that condition* (DC) *is satisfied, so that condition* A6 *is uniformly satisfied (in particular, the strong Feller property holds). Moreover, let $\overline{\gamma}$ be as in Theorem* 1*, and assume $\gamma < \overline{\gamma}$ (hence the limits* (4.2) *and* (4.3) *exist). Then $v(\cdot)$ is continuous and is a viscosity solution of* (4.4).

Proof. Step 1. *Continuity of $\varphi(t,x)$.* We first establish continuity in x.

First note that, according to the proofs in section 2, for $\gamma < \bar{\gamma}$, one can find some $\delta \in]1, \frac{\bar{\gamma}}{\gamma}[$ and some function $h_\gamma(t,x)$, which is bounded on compact sets such that

$$E_x\left[\exp\left(\gamma\delta \int_0^t |c(x_s)| ds\right)\right] \le h_\gamma(t,x) \tag{4.5}$$

for all $t > 0$ and $x \in \mathbb{R}^d$.

Note that, for $0 < \epsilon < t$,

$$\begin{aligned}
&|\varphi(t,x) - \varphi(t,y)| \\
&\quad = \left| E_x\left[\varphi(t-\epsilon, x_\epsilon) \exp\left(\gamma \int_0^\epsilon c(x_s) ds\right)\right] \right. \\
&\qquad \left. - E_y\left[\varphi(t-\epsilon, x_\epsilon) \exp\left(\gamma \int_0^\epsilon c(x_s) ds\right)\right] \right| \\
&\quad \le E_x\left[\left|e^{\gamma \int_0^\epsilon c(x_s) ds} - 1\right| \varphi(t-\epsilon, x_\epsilon)\right] + E_y\left[\left|e^{\gamma \int_0^\epsilon c(x_s) ds} - 1\right| \varphi(t-\epsilon, x_\epsilon)\right] \\
&\qquad + \left| E_x[\varphi(t-\epsilon, x_\epsilon)] - E_y[\varphi(t-\epsilon, x_\epsilon)] \right|.
\end{aligned} \tag{4.6}$$

We begin by estimating the first term in the right-hand side of (4.6). By the Hölder inequality,

$$\begin{aligned}
&E_x\left[\left|e^{\gamma \int_0^\epsilon c(x_s) ds} - 1\right| \varphi(t-\epsilon, x_\epsilon)\right] \\
&\qquad \le \left\{E_x\left[\left|e^{\gamma \int_0^\epsilon c(x_s) ds} - 1\right|^p\right]\right\}^{1/p} \left\{E_x\left[\exp\left(\gamma\delta \int_0^t |c(x_s)| ds\right)\right]\right\}^{1/\delta},
\end{aligned} \tag{4.7}$$

where $p = \frac{\delta}{\delta-1}$. Our goal is to show that the left-hand side of (4.7) goes to 0 as $\epsilon \to 0$, uniformly in x varying in a compact set. By (4.5), the second factor in the right-hand side of (4.7) is locally bounded. Thus, it is enough to show that

$$E_x\left[\left|e^{\gamma \int_0^\epsilon c(x_s)ds} - 1\right|^p\right]$$

goes to zero uniformly in compact sets. By the inequality $|e^x - 1| \leq |x|e^{|x|}$, the Cauchy–Schwarz inequality, and Jensen's inequality,

$$\begin{aligned}
E_x\left[\left|e^{\gamma \int_0^\epsilon c(x_s)ds} - 1\right|^p\right]^2 &\leq \gamma^{2p} E_x\left[\left(\int_0^\epsilon c(x_s)ds\right)^p e^{\gamma p \int_0^\epsilon |c(x_s)|ds}\right]^2 \\
&\leq \gamma^{2p} E_x\left[\left(\int_0^\epsilon c(x_s)ds\right)^{2p}\right] E_x\left[e^{2\gamma p \int_0^\epsilon |c(x_s)|ds}\right] \\
&\leq \epsilon^{2p-1}\gamma^{2p} \int_0^\epsilon E_x\left[|c(x_s)|^{2p}\right] ds \; E_x\left[e^{2\gamma p \int_0^\epsilon |c(x_s)|ds}\right] \\
&\leq \epsilon^{2p-2}\gamma^{2p} \int_0^\epsilon E_x\left[|c(x_s)|^{2p}\right] ds \int_0^\epsilon E_x\left[e^{2\gamma p \epsilon |c(x_s)|}\right] ds. \qquad (4.8)
\end{aligned}$$

Since $p > 1$, it is enough to show that the two integrals in (4.8) are locally bounded. This follows easily from the assumption that $c(\cdot)$ has quadratic growth (see A2, where the constant C is defined), and from the proof of the first part of Theorem 2, as soon as $2\gamma p C\varepsilon < c_b$, that holds true for ε small enough. Indeed we get some exponential integrability which is strong enough to control both terms.

It remains to deal with the last term in (4.6). It is enough to show that, for given $\epsilon > 0$, the map

$$x \mapsto E_x[\varphi(t-\epsilon, x_\epsilon)] \qquad (4.9)$$

is continuous in x. For this purpose, we realize all diffusion starting from any $x \in \mathbb{R}^d$ in the same probability space. We denote by $X_t(x)$ the diffusion starting from x, and denote by E the expectation in this probability space. Thus

$$E_x[\varphi(t-\epsilon, x_\epsilon)] = E[\varphi(t-\epsilon, X_\epsilon(x))].$$

By (4.5),

$$E[\varphi^\delta(t-\epsilon, X_\epsilon(x))]$$

is locally bounded in x. This implies that, for any ball B, the family of random variables

$$(\varphi(t-\epsilon, X_\epsilon(x)))_{x\in B}$$

is uniformly integrable. Thus, letting $\varphi_M(x) := \varphi(t-\epsilon, x)\mathbf{1}_{[0,M]}(|\varphi(t-\epsilon, x)|)$ ($\mathbf{1}_A$ is the indicator function of the set A), we have that

- for every $M > 0$, $E[\varphi_M(X_\epsilon(x))]$ is continuous in x by the strong Feller property; and
- $E[\varphi(t-\epsilon, X_\epsilon(x))] - E[\varphi_M(X_\epsilon(x))]$ goes to zero as $M \to +\infty$ uniformly in $x \in B$ for any ball B.

From these two statements, continuity of (4.9) follows.

To get joint continuity in (t, x) just observe that, by the integrability condition (4.5), we can differentiate in t $\varphi(t, x)$ and show that this derivative is locally bonded. Thus $\varphi(t, x)$ is locally Lipschitz in t, locally uniformly in x. This, together with continuity in x, implies joint continuity.

Step 2. *Viscosity solution of the parabolic equation.* In what follows we introduce the upper-semicontinuous (resp., lower-semicontinuous) extension c^* (resp., c_*) of $c(\cdot)$:

$$c^*(x) := \limsup_{y \to x} c(y), \quad c_*(x) := \liminf_{y \to x} c(y).$$

Moreover, let $v_T(t, x) := \varphi(T - t, x)$. We now show that v_T is a viscosity solution (in $[0, T]$) of the parabolic equation

$$-\left(\partial_t v_T + b \cdot \nabla v_T + \frac{1}{2}\Delta v_T + \gamma c v_T\right) = 0. \tag{4.10}$$

Since v_T is continuous, this amounts to showing that the following two properties hold true:

i. (supersolution property). Let $(t, x) \in [0, T) \times \mathbb{R}^d$ and let $\psi : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $\psi(t, x) = v_T(t, x)$ and $v_T - \psi$ has a local maximum at (t, x) (there may be no such function). Then

$$-\left(\partial_t \psi(t, x) + b(x) \cdot \nabla \psi(t, x) + \frac{1}{2}\Delta \psi(t, x) + \gamma c^*(x) v_T(t, x)\right) \le 0.$$

ii. (subsolution property). Let $(t, x) \in [0, T) \times \mathbb{R}^d$ and let $\psi : [0, T) \times \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $\psi(t, x) = v_T(t, x)$ and $v_T - \psi$ has a local minimum at (t, x). Then

$$-\left(\partial_t \psi(t, x) + b(x) \cdot \nabla \psi(t, x) + \frac{1}{2}\Delta \psi(t, x) + \gamma c_*(x) v_T(t, x)\right) \ge 0.$$

$v_T - \psi$ has a *strict* local extreme in (t, x). Indeed, if $v_T - \psi$ has a local extreme at (t, x) and $\tilde{\psi}(s, y) := \psi(s, y) \pm \left[(s - t)^2 + |x - y|^4\right]$ (where the sign depends on whether we are dealing with a maximum or a minimum), then $v_T - \tilde{\psi}$ has a *strict* local extreme in (t, x), and ψ and $\tilde{\psi}$ have the same first space and time derivatives and second space derivatives at (t, x). We now observe the following identities:

$$\begin{aligned}
\varphi(t, x) &= 1 - \int_0^t \frac{d}{ds} E_x \left[\exp\left(\int_s^t \gamma c(x_\tau) d\tau\right)\right] ds \\
&= 1 + \gamma \int_0^t E_x \left[c(x_s) \exp\left(\gamma \int_s^t c(x_\tau) d\tau\right)\right] ds \\
&= 1 + \gamma \int_0^t E_x \left[c(x_s) \varphi(t - s, x_s)\right] ds,
\end{aligned}$$

where all steps are justified by (4.5). It follows that, for $\epsilon > 0$,

$$\varphi(t, x) - E_x[\varphi(t - \epsilon, x_\epsilon)] = \gamma E_x \left[\int_0^\epsilon c(x_s) \varphi(t - s, x_s) ds\right].$$

By a change $t \mapsto T-t$ of the time variable, we get

$$v_T(t,x) - E_x[v_T(t+\epsilon, x_\epsilon)] = \gamma E_x \left[\int_0^\epsilon c(x_s) v_T(t+s, x_s) ds \right]. \tag{4.11}$$

Now we use (4.11) to prove that v_T has the subsolution property. The supersolution property is proved in the same way. Note that both properties are local, so it is not restrictive to assume the test functions ψ to have compact support.

So let ψ be a smooth function with compact support such that $\psi(t,x) = v_T(t,x)$ and $v_T - \psi$ has a local minimum at (t,x). We first claim that

$$\limsup_{\epsilon \to 0} \frac{v_T(t,x) - E_x[v_T(t+\epsilon, x_\epsilon)]}{\epsilon} \leq \limsup_{\epsilon \to 0} \frac{\psi(t,x) - E_x[\psi(t+\epsilon, x_\epsilon)]}{\epsilon}. \tag{4.12}$$

This is done by a simple localization. Let $\rho > 0$ be such that $v_T(s,y) \geq \psi(s,y)$ for $(s,y) \in [t-\rho, t+\rho] \times B(x,\rho)$. Then, for $|\epsilon| < \rho$,

$$\begin{aligned}
&\frac{v_T(t,x) - E_x[v_T(t+\epsilon, x_\epsilon)]}{\epsilon} \\
&= E_x \left[\frac{v_T(t,x) - v_T(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| \leq \rho} \right] + E_x \left[\frac{v_T(t,x) - v_T(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| > \rho} \right] \\
&\leq E_x \left[\frac{\psi(t,x) - \psi(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| \leq \rho} \right] + E_x \left[\frac{v_T(t,x) - v_T(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| > \rho} \right] \\
&= E_x \left[\frac{\psi(t,x) - \psi(t+\epsilon, x_\epsilon)}{\epsilon} \right] \\
&\quad - E_x \left[\frac{\psi(t,x) - \psi(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| > \rho} \right] + E_x \left[\frac{v_T(t,x) - v_T(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| > \rho} \right].
\end{aligned}$$

Thus, in order to obtain (4.12), it is enough to show that the last two terms go to zero as $\epsilon \to 0$. We deal only with the last (the others being easier),

$$\begin{aligned}
\left| E_x \left[\frac{v_T(t,x) - v_T(t+\epsilon, x_\epsilon)}{\epsilon} \mathbb{1}_{|x_\epsilon - x| > \rho} \right] \right| &\leq \frac{2}{\epsilon} E_x \left[e^{\gamma \int_0^T |c(x_s)| ds} \mathbb{1}_{|x_\epsilon - x| > \rho} \right] \\
&\leq \frac{2}{\epsilon} E_x \left[e^{\gamma \delta \int_0^T |c(x_s)| ds} \right] E_x (\mathbb{1}_{|x_\epsilon - x| > \rho})^{1 - \frac{1}{\delta}},
\end{aligned}$$

which goes to zero as $\epsilon \to 0$ since, by small time estimates (see, e.g., [18]), $E_x(\mathbb{1}_{|x_\epsilon - x| > \rho}) = o(\epsilon)$. This establishes (4.12). On the other hand, by Ito's rule,

$$\lim_{\epsilon \to 0} \frac{\psi(t,x) - E_x[\psi(t+\epsilon, x_\epsilon)]}{\epsilon} = - \left(\partial_t \psi(t,x) + b(t,x) \cdot \nabla \psi(t,x) + \frac{1}{2} \Delta \psi(t,x) \right). \tag{4.13}$$

Putting together (4.11), (4.12), and (4.13), the subsolution property follows from

$$\liminf_{\epsilon \to 0} \frac{1}{\epsilon} E_x \left[\int_0^\epsilon c(x_s) v_T(t+s, x_s) ds \right] \geq c_*(x) v_T(t,x), \tag{4.14}$$

where the above convergence is again controlled by small time estimates and the fact that v_T is continuous.

Step 3. *Conclusion.* Letting $\tilde{v}_T(t,x) := v_T(t,x) e^{-\lambda(T-t)}$, it is easily checked that $\tilde{v}_T$ is a viscosity solution of

$$- \left(\partial_t \tilde{v}_T + b \cdot \nabla \tilde{v}_T + \frac{1}{2} \Delta \tilde{v}_T + \gamma c v_T \right) + \lambda \tilde{v} = 0. \tag{4.15}$$

Moreover, $\tilde{v}_T(t,x) \to v(x)$ as $T \to +\infty$ uniformly on compact sets. In particular, v is continuous. We now sketch a standard argument to show that v is a viscosity solution of (4.4).

Let $x \in \mathbb{R}^d$, and let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $v(x) = \psi(x)$ and $v-\psi$ has a local minimum at x. Fix $t > 0$, and define $\tilde{\psi}(s,y) := \psi(y) - |y-x|^4 - (s-t)^2$. Note that $v - \tilde{\psi}$ has a *strict* local minimum at (t,x), and

$$\partial_t \tilde{\psi}(t,x) + b(t,x) \cdot \nabla \tilde{\psi}(t,x) + \frac{1}{2}\Delta\tilde{\psi}(t,x) = b(x) \cdot \nabla\psi(x) + \frac{1}{2}\Delta\psi(x). \tag{4.16}$$

A simple exercise in uniform convergence shows that there is a sequence $(t_n, x_n) \to (t,x)$ as $n \to +\infty$ such that $\tilde{v}_n - \tilde{\psi}$ has a local minimum at (t_n, x_n). Therefore, since $\tilde{v}$ is a viscosity solution of (4.15),

$$-\left(\partial_t \tilde{\psi}(t_n,x_n) + b(x_n) \cdot \nabla\tilde{\psi}(t_n,x_n) + \frac{1}{2}\Delta\tilde{\psi}(t_n,x_n) + \gamma c_*(x_n)\tilde{v}_n(x_n)\right) + \lambda \tilde{v}_n(x_n) \geq 0. \tag{4.17}$$

Letting $n \to +\infty$ and using (4.16) and lower-semicontinuity of c_*, we obtain the subsolution property for (4.4). The supersolution property is obtained in the same way. ∎

Acknowledgment. The authors thank an anonymous referee for bringing to their attention the reference [12].

REFERENCES

[1] C. Ané, S. Blachère, D. Chafai, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses 10, Société Mathématique de France, Paris, 2000.

[2] D. Bakry, M. Ledoux, and Z. Qian, *Logarithmic Sobolev Inequalities, Poincaré Inequalities and Heat Kernel Bounds*, unpublished manuscript, available online at http://www.lsp.ups-tlse.fr/Fp/Qian/BLQ.pdf (1997).

[3] V. I. Bogachev, N. V. Krylov, and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations, 26 (2001), pp. 2037–2080.

[4] P. Dai Pra and S. Rœlly, *An existence result for infinite-dimensional Brownian diffusions with non-regular and non-Markovian drift*, Markov Process. Related Fields, 10 (2004), pp. 113–136.

[5] J. Feng and T. G. Kurtz, *Large Deviations for Stochastic Processes*, Math. Surveys Monogr. 131, American Mathematical Society, Providence, RI, 2006.

[6] W. H. Fleming and W. M. McEneaney, *Risk-sensitive control on an infinite time horizon*, SIAM J. Control Optim., 33 (1995), pp. 1881–1915.

[7] W. Fleming, S. J. Sheu, and M. Soner, *On the existence of the dominant eigenfunction and its application to the large deviation properties of an ergodic Markov process*, Stochastics, 22 (1987), pp. 187–199.

[8] H. Föllmer, *Time reversal on Wiener space*, in Stochastic Processes—Mathematics and Physics, Lecture Notes in Math. 1158, Springer, 1986, pp. 119–129.

[9] A. Gulisashvili and J. A. Van Casteren, *Non-autonomous Kato Classes and Feynman-Kac Propagators*, World Scientific, Hackensack, NJ, 2006.

[10] H. Kaise and S. J. Sheu, *On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control*, Ann. Probab., 34 (2006), pp. 284–320.

[11] I. Kontoyiannis and S. P. Meyn, *Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes*, Electron. J. Probab., 10 (2005), pp. 61–123.

[12] I. Kontoyiannis and S. P. Meyn, *Spectral theory and limit theorems for geometrically ergodic Markov processes*, Ann. Appl. Probab., 13 (2003), pp. 304–362.

[13] R. Kotecký and D. Preiss, *Cluster expansions for abstract polymer models*, Comm. Math. Phys., 103 (1986), pp. 491–498.

[14] W. M. McEneaney and K. Ito, *Infinite time-horizon risk-sensitive systems with quadratic growth*, in Proceedings of the 36th IEEE Conference on Decision and Control, IEEE Press, Piscataway, NJ, 1997, pp. 3413–3418.

[15] R. A. Minlos, A. Verbeure, and V. Zagrebnov, *A quantum crystal model in the light-mass limit: Gibbs states*, Rev. Math. Phys., 12 (2000), pp. 981–1032.

[16] G. Royer, *Une initiation aux inégalités de Sobolev logarithmiques*, Cours Spécialisés 5, Société Mathématique de France, Paris, 1999.

[17] D. W. Stroock, *Probability Theory: An Analytic View*, Cambridge University Press, Cambridge, UK, 1999.

[18] S. R. S. Varadhan, *Diffusion processes in a small time interval*, Comm. Pure Appl. Math., 20 (1967), pp. 659–685.

[19] F. Y. Wang, *Functional Inequalities, Markov Processes, and Spectral Theory*, Science Press, Beijing, 2004.

[20] L. Wu, *Uniformly integrable operators and large deviations for Markov processes*, J. Funct. Anal., 172 (2000), pp. 301–376.