Exercises

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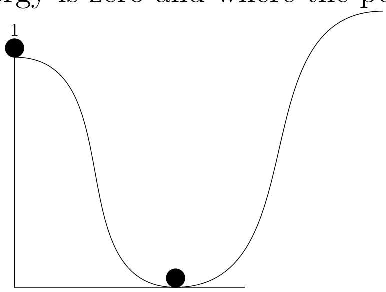
Part I

Problem 1: Calculate the potential energy, kinetic enery and total energy of a particle moving under gravity in the curve given by the figure. Assume that the particle is at rest in the point 1. Locate the points where the kinetic energy is zero and where the potential enegy has the minimum value.



Problem 2: Calculate the total electrostatic energy of:

- A ball of constant density, radius R and charge Q.
- A spherical shell of constant surface density, radius R and charge Q.

Prove that in the case of the shell, that the integral of the energy density $|E|^2$ over the sphere of radius R is zero.

Problem 3: Using Maxwell equation, prove the identity

$$\frac{\partial e}{\partial t} + \nabla \cdot S = -J \cdot E, \tag{1}$$

where the energy density of the electromagnetic field is given by

$$e = \frac{1}{8\pi}\left(|E|^2 + |B|^2\right)$$

and the Poynting vector is given by

$$S = \frac{c}{4\pi} E \times B.$$

Problem 4: Prove that the electromagnetic energy momentum tensor satisfies the dominant energy condition.

Problem 5: Using the conservation of energy, prove the uniqueness of solution of the initial value problem for the Maxwell equations.

Problem 6: Prove that the linear momentum is a Lorentz vector.

Problem 7: Using the dominant energy condition on lightcones, prove that the electromagnetic field propagate with velocity less or equal to the speed of light.

Part II

Problem 8: Construct explicitly initial data for spherical symmetric (and time symmetric) star with constant energy density. Prove that if the energy density is positive then the total energy is also positive.

Problem 9: Write down explicitly the initial data for the Kerr metric with parameters m, a corresponding to a slice $t = constant$ in the standart Boyer-Lindquist coordinates. For these initial data calculate the energy and the linear momentum.

- For the case where $m > |a|$ prove that the manifold is complete with two asymptotically flat ends. The energy is the same at both ends.
- For the case $m = |a|$, prove that the manifold is complete with one asymptotically flat end and one cylindrical end.
- For the case $m < |a|$ prove that the manifold is singular.

Problem 10: Consider a conformally flat and asymptotically flat metric $h_{ij} = \psi^4 \delta_{ij}$ on $\mathbb{R}^3$. Assume that h_{ij} has $R \geq 0$ insider a compact ball and $R = 0$ outside it.

- Prove that the funtion ψ has an expansion near infinity of the form

$$\psi = a + \frac{b}{r} + O(r^{-2}) \tag{2}$$

- Prove that the energy is given by

$$E = 2ab \tag{3}$$

- Using the maximum principle for the Laplacian prove that $E \geq 0$.

Problem 11: Calculate explicitly the energies for the 3-ends in the Brill-Lindquist initial data. Interpret the difference between them as interaction energy between the black holes and prove that in the large separation distance the Newton interaction energy is recovered.

Problem 12: Assuming axial symmetry, construct initial data that are pure vacuum, have topology of $\mathbb{R}^3$ and have non-zero, positive, total mass. These data represent pure gravitational waves.

Part III

Problem 13: Construct explicitly the Lorentz transformation that correspond to a spin transformation.

Problem 14: Prove that the imaginary part of the Nester-Witten form is the exterior derivative of a 1-form.

Problem 15: Write explicitly in components the principal part of Witten equations

$$\mathcal{D}_{AA'}\lambda^A, \tag{4}$$

for flat space and argue that it is an elliptic system of equations for the spinor components.

Part IV

Problem 16: Prove that the standard time symmetric initial data for the Schwarzschild black hole has a closed, stable minimal surface.

Problem 17: Prove the following statement: if a metric h is asymptotically flat on the manifold N with $R \geq 0$ and negative mass M then there exists a conformally rescaled metric $\tilde{h}$ such that $\tilde{R} \geq 0$ on N and $\tilde{R} > 0$ outside a compact set, and the mass of $\tilde{h}$ is negative.

Problem 18: Give an explicit example of an asymptotically flat Riemannian metric with non-zero mass such that there exists a complete minimal surface between two planes.