# Symplectic formalism and the covariant phase space on Scalar Electrodynamics

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**Junior scientist Andrejewski Days: 100 years of General Relativity**

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# Outline

- **PART I: The geometry of Classical Mechanics**
  - Symplectic formulation of Hamiltonian Mechanics
  - Symmetries and conserved quantities
- **PART II: Covariant phase space on field theories**
  - (Pre)–Symplectic structure and boundary conditions
  - Symmetries and conserved quantities
- **PART III: Scalar Electrodynamics**
  - Lagrangian, gauge symmetries and field equations
  - Sympectic structure and boundary conditions
  - Symmetries and conserved quantities

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# PART I

The geometry of Classical Mechanics

# Symplectic formulation of Hamiltonian Mechanics

- Any classical system with $n$ degrees of freedom is characterized by a Lagrangian

$$\mathcal{L} = \mathcal{L}(q^i, \dot{q}^i, t),$$

where the coordinates $q^i = q^i(t)$. We introduce $n$ covectors given by

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}^i}.$$

- Locally, the phase space $\Gamma$ of the system is descripted by

$$\left(\{q^i\},\ \{p_i\}\right), \quad i = 1, 2, \cdots, n.$$

- The Hamiltonian of the system is a smooth function on $\Gamma$,

$$\mathcal{H} : \Gamma \to \mathbb{R}, \quad \mathcal{H} := p_i \dot{q}^i - \mathcal{L},$$

and the dynamics of the system is descripted by

$$\boxed{\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}}.$$

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# Symplectic formulation of Hamiltonian Mechanics

- The phase space $\Gamma$ seems to have some intrinsic geometrical structure related to the form of Hamilton equations.
- Let's introduce arbitrary coordinates $x^\mu(q^i, p_i)$, $\mu = 1, \cdots, 2n$ on $\Gamma$.The evolution in the new coordinates is

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial q^i}\dot{q}^i + \frac{\partial x^\mu}{\partial p_i}\dot{p}_i = \frac{\partial x^\mu}{\partial q^i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial x^\mu}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q^i}$$

$$= \left(\frac{\partial x^\mu}{\partial q^i}\frac{\partial x^\nu}{\partial p_i} - \frac{\partial x^\mu}{\partial p_i}\frac{\partial x^\nu}{\partial q^i}\right)\frac{\partial \mathcal{H}}{\partial x^\nu} = \omega^{\mu\nu}\frac{\partial \mathcal{H}}{\partial x^\nu};$$

and thus,

$$\dot{x}^\mu = \omega^{\mu\nu}\frac{\partial \mathcal{H}}{\partial x^\nu}, \quad \omega^{\mu\nu} := \left(\frac{\partial x^\mu}{\partial q^i}\frac{\partial x^\nu}{\partial p_i} - \frac{\partial x^\mu}{\partial p_i}\frac{\partial x^\nu}{\partial q^i}\right) = \{x^\mu, x^\nu\}.$$

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# Symplectic formulation of Hamiltonian Mechanics

- Properties of $\omega^{\mu\nu}$:

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \; ; \tag{1}$$

$$\det(\omega^{\mu\nu}) = \left| \frac{\partial x^{\mu}}{\partial(q^i, p_j)} \right|^2 \neq 0 \; ; \tag{2}$$

$$\partial_{[\mu}\omega_{\nu\rho]} = 0. \tag{3}$$

- In particular, if $(x^1, \cdots, x^{2n}) = (q^1, \cdots, q^n, p_1, \cdots, p_n)$,

$$\omega^{\mu\nu} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0} \end{pmatrix},$$

  and we return to Hamilton Equations.

- If $f : \Gamma \to \mathbb{R}$ and $g : \Gamma \to \mathbb{R}$, we redefine the Poisson bracket $\{f, g\}$ in terms of $\omega$:

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- Fixing $g$, $\{f, g\}$ is a derivation on $f$ along the vector

$$X_g^\mu := \omega^{\mu\nu} \frac{\partial g}{\partial x^\nu}.$$

This vector field is called Hamiltonian vector field.

- The inverse of $\omega^{\mu\nu}$, $\omega_{\mu\nu}$ is called symplectic structure.
- A symplectic manifold is a pair $(\mathcal{M}, \omega)$ such that $\omega$ satisfies (1), (2) and (3).
- Note that (3) implies that $\omega_{\mu\nu}$ is a closed non degenerate 2-form.
- Darboux's Theorem: Let $(\Gamma, \omega)$ be a symplectic manifold. Then, for each point $p \in \Gamma$, there exists a neighbourhood of $p$ and a chart $(\{q^i\}, \{p_i\})$ such that

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i = \mathrm{d}\left(p_i \mathrm{d}q^i\right).$$

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# Symmetries and conserved quantities

- **Symmetry?** $\Phi : \Gamma \to \Gamma$ smooth and invertible, that takes a *solution* and produces *another*.
- **Solution?** Curve $\gamma : I \subseteq \mathbb{R} \to \Gamma$, $t \mapsto \gamma(t)$ such that

$$\dot{\gamma} = X_{\mathcal{H}}, \quad \dot{x}^{\mu} = \omega^{\mu\nu} \frac{\partial \mathcal{H}}{\partial x^{\nu}}.$$

- If $\{\Phi_s\}_{s\in\mathbb{R}}$ is a **monoparametric family of symmetries** and $p \in \Gamma$, let's consider the curve $s \mapsto \gamma_p(s) := \Phi_s(p)$. The **tangent vector**

$$\xi := \left.\frac{\mathrm{d}\Phi_s(p)}{\mathrm{d}s}\right|_{s=0}$$

is called an infinitesimal transformation of $\Phi$ at $p$.

- The fields $X_{\mathcal{H}}$ and $\xi$ generate a 2-dimensional submanifold on $\Gamma$, and must be coordinate vector fields:

$$0 = [\xi, X_{\mathcal{H}}]^{\mu} = X^{\mu}_{\xi(\mathcal{H})} + (\pounds_{\xi}\omega^{\mu\nu}) \frac{\partial \mathcal{H}}{\partial x^{\nu}} \Rightarrow \boxed{\pounds_{\xi}\, \omega = 0}$$

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# PART II

Covariant Phase Space on Field Theories

# Covariant Phase Space on Field Theories

- Consider a smooth 4-dimensional lorentzian manifold $\mathcal{M}$ with the topology of $\Sigma \times \mathbb{R}$ and $\Sigma \equiv \mathbb{R}^3$.
- $\mathcal{M}$ is equipped with a stationary and globally hyperbolic metric $g_{ab}$ such that Cauchy surfaces are diffeomorphic to $\Sigma$.
- On this spacetime, consider a dynamical theory for a collection of fields $\phi^\alpha(x)$, where $\alpha$ labels the fields. We denote

$$\mathcal{F} := \{\phi^\alpha : \mathcal{M} \to \mathcal{T}^{(k,l)\alpha}_{\mathcal{M}} \,|\, \phi^\alpha \text{ satisfy some boundary conditions}\}.$$

- $\mathcal{F}$ has the structure of an infinite-dimensional manifold. Functions on $\mathcal{F}$ are functionals $f : \mathcal{F} \to \mathbb{R}$.
- Dynamics is specified by some action $\mathcal{S}_V$, defined over any measurable region $V \subset \mathcal{M}$:

$$\mathcal{S}_V(\phi^\alpha) = \int_V \mathcal{L}\left(\phi^\alpha, \nabla_a \phi^\alpha, \nabla_a \nabla_b \phi^\alpha, \cdots\right) \mathrm{dV}.$$

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# Symmetries and conserved quantities

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- Is there some *conserved quantity* associated with $\xi$?

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- Thus, there also exists a scalar function (the conserved quantity) $\mathcal{C}_\xi$ such that

$$\mathsf{d}\mathcal{C}_\xi = \theta_\xi.$$

- If $\xi \in \mathrm{Ker}(\omega)$, then $\xi$ is an infinitesimal symmetry.

# Symmetries and conserved quantities

- A smooth vector field on $\Gamma$, $\xi : \Gamma \to T\Gamma$ is called an infinitesimal canonical transformation if

$$\boxed{\pounds_\xi \, \omega = 0}$$

- Is there some *conserved quantity* associated with $\xi$?

$$\pounds_\xi \, \omega = (\mathsf{d}\omega)\,(\xi,\cdot,\cdot) + \mathsf{d}\,(\omega(\xi,\cdot)) = \mathsf{d}\,(\omega(\xi,\cdot))\,,$$

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# PART III

## Scalar Electrodynamics

# Scalar Electrodynamics (SED)

- Consider a complex scalar field $\Phi$ with mass $m$ and charge $e$, and a Maxwell field $F_{ab} := 2\partial_{[a}A_{b]}$ in $\mathcal{M}$.
- The Lagrangian of the theory takes the form

$$\mathcal{L}(\Phi, A_a) = (\mathcal{D}_a\Phi)(\mathcal{D}^a\Phi)^* - m^2|\Phi|^2 - \frac{1}{4}F_{ab}F^{ab},$$

where

$$\mathcal{D}_a\Phi = (\nabla_a + ieA_a)\,\Phi, \quad (D^a\Phi)^* = (\nabla^a - ieA^a)\,\Phi^*.$$

- $\mathcal{L}$ is invariant under

$$\begin{pmatrix} \Phi \\ A_a \end{pmatrix} \mapsto \begin{pmatrix} e^{-ie\lambda}\Phi \\ A_a + \nabla_a\lambda \end{pmatrix}, \quad \lambda : \mathcal{M} \to \mathbb{R}.$$

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- **Field Equations:**

$$\Box\Phi + \left[m^2 - e^2 A^a A_a + ie\nabla_a A^a + 2ieA^a\nabla_a\right]\Phi = 0,$$

$$\nabla_c\nabla^d A^c - \Box A^d = ie\left[\Phi\nabla^a\Phi^* - \Phi^*\nabla^a\Phi\right] + 2e^2\Phi\Phi^* A^a.$$

- The covariant phase space is, thus,

$$\Gamma = \{\phi^\alpha := (\Phi, A_a) | \Phi \text{ and } A_a \text{ satisfy field eq.}\}.$$

- Since the action $\mathcal{S} = \int \sqrt{-g}\, d^4x\, \mathcal{L}$ is of first order, the symplectic structure takes the form

$$\omega(X,Y) = \int_\Sigma \mathcal{J}_1^a(X,Y)dS_a + \int_\Sigma \mathcal{J}_2^a(X,Y)dS_a,$$

where $\Sigma$ is any Cauchy surface in $\mathcal{M}$ with boundary $\partial\Sigma$.

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# Field equations and Symplectic structure

- $X$ and $Y$ are solutions to linearized field equations around some solution $\phi^\alpha$:

$$X^\alpha = \delta\phi^\alpha = (\delta\Phi, \delta A_a) := (\Psi, \alpha_a).$$

- Let's take $X = (\Psi_1, \alpha_a^1)$ and $Y = (\Psi_2, \alpha_a^2)$. The currents of $\omega$ are

$$\mathcal{J}_1^a(X,Y) = 2ieA^a\left(\Psi_1\Psi_2^* - \Psi_2\Psi_1^*\right) + ie\Phi\left(\Psi_2^*\alpha_1^a - \Phi_1^*\alpha_2^a\right)$$

$$-ie\Phi^*\left(\Psi_2\alpha_1^a - \Psi_1\alpha_2^a\right),$$

and

$$\mathcal{J}_2^a(X,Y) = \Psi_2\nabla^a\Psi_1^* - \Psi_1\nabla^a\Psi_2^* + \Psi_2^*\nabla^a\Psi_1 - \Psi_1^*\nabla^a\Psi_2 + \alpha_2^b\nabla_b\alpha_1^a$$

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# Linearized problem and infinitesimal gauge transformation

- Explicitly, the perturbations $(\Psi, \alpha_a)$ satisfy

$$\left[\Box + m^2 - e^2 A^a A_a + ie\nabla_a A^a + 2ieA^a\nabla_a\right]\Psi = \left[2e^2 A^a \alpha_a - ie\nabla_a \alpha^a - 2ie\alpha^a\nabla_a\right]\Phi;$$

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- In particular, the infinitesimal gauge tranformation

$$X_G := \left(-ie\lambda\Phi, \nabla_a\lambda\right),$$

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# Kernel of $\omega$

- Computing $\omega\left(X_G, Y\right)$ and integrating by parts, we obtain

$$\omega(X_G, Y) = 2\oint_{\partial\Sigma} \lambda \nabla_{[a}\alpha^2_{b]} t^a n^b dS,$$

where the integral is over $\Sigma$, with time-like normal $n^a$ and $t^a$ is spacelike normal to $\partial\Sigma$. If $\lambda \to 0$ at $\partial\Sigma \subset i^0$, $X_G \in \text{Ker}(\omega)$.

- **Quiz:** Does Ker$(\omega)$ include all local symmetries of the theory? **Answer:** No! Explicitly,

$$\pounds_{X_G}\omega = (\mathrm{d}\omega)(X_G, \cdot, \cdot) + \mathrm{d}(\omega(X_G, \cdot)) = 0.$$

- The conserved quantity $\mathcal{C}_{X_G}$ is such that

$$\mathrm{d}\mathcal{C}_{X_G}(Y) = \omega(X_G, Y) \quad \Rightarrow \quad \mathcal{C}_{X_G}(A) = 2\oint_{\partial\Sigma} \lambda \nabla_{[a}A_{b]} t^a n^b dS,$$

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**Particular case: $m = 0$**

- *What happens at null infinity?* We consider the conformal transformation

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega : \tilde{\mathcal{M}} \to \mathbb{R}, \quad \Omega > 0,$$

where $\tilde{g}_{ab}$ represents the physical metric, and $g_{ab}$ the unphysical one.

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$$\Omega^2\tilde{\Phi}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\Omega - 2\Omega\tilde{\Phi}\tilde{g}^{ab}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega.$$

# How does $\tilde{\omega}(\tilde{X}, \tilde{Y})$ change?

- The volume element in the physical picture is

$$d\tilde{S}_a = \tilde{t}_a d\tilde{S} = \tilde{t}_a |\tilde{h}|^{1/2} \, d^3\tilde{x}, \quad \tilde{t}^a \tilde{t}_a = -1, \quad h_{ab} = g_{ab}|_\Sigma.$$

- Under the transformation $\tilde{h}_{ab} = \Omega^2 h_{ab}$, and taking $t^a t_a = -1$,

$$\tilde{h} = \Omega^6 h \quad \Rightarrow \quad |\tilde{h}|^{1/2} = \Omega^3 |h|^{1/2};$$

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- By direct calculation, it can be shown that

$$\boxed{\tilde{\mathcal{J}}^a_j(\tilde{X}, \tilde{Y}) = \Omega^{-4} \mathcal{J}^a_j(X, Y); \quad j = 1, 2.}$$

- Thus, if the integration is over space-like Cauchy surfaces, the symplectic structure results **conformally invariant**.

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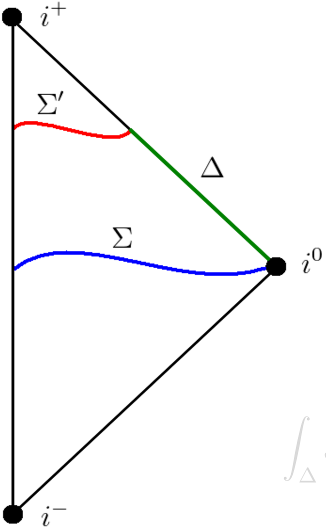
# Conformal Symplectic structure




- $\omega$ does not depend on the Cauchy surface. In particular, one can take

$$\Sigma^* := \Sigma' \cup \Delta, \ \Delta := S^2 \times I, \ I \subset \mathbb{R}.$$

- $\nabla^a \mathcal{J}_a = 0$ on $\mathcal{M}$.
- **Charge flux:**

$$\int_\Delta \mathcal{J}^a(X_G, Y) dS_a = \mathrm{d}\mathcal{Q}|_{i^0}(Y) - \mathrm{d}\mathcal{Q}|_{\mathcal{C}}(Y)$$

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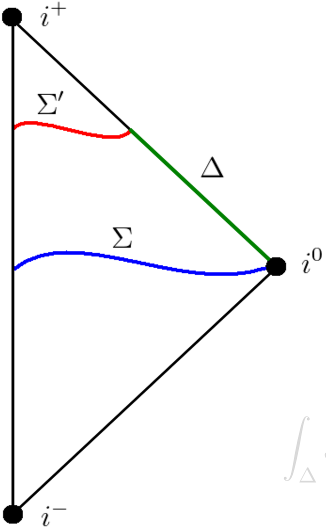


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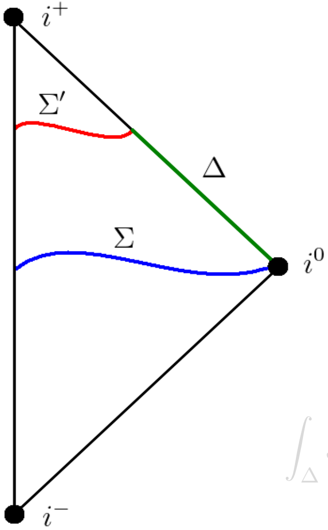


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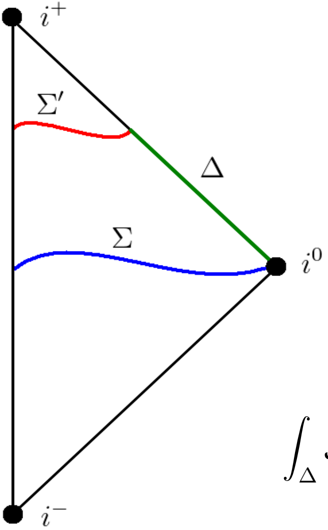


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- Let $k^a$ be a Killing vector field in the spacetime background $(\mathcal{M}, g_{ab})$. Then,

$$\pounds_k \, g_{ab} = 0.$$

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$$X^\alpha = (\dot{\Phi}, \dot{A}_a) := (\pounds_k \, \Phi, \pounds_k \, A_a)$$

  satisfies the linearized SED equations, using the fact that $\pounds$ conmutes with $\nabla_a$ along the Killing field.
- $\pounds_X \, \omega = 0 \quad \Rightarrow X$ **is a symmetry**.
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$$\mathcal{L} = \nabla^a \Phi \nabla_a \Phi - m^2 \Phi^2, \quad \Box \Phi + m^2 \Phi = 0, \quad (+,-,-,-).$$

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# Final Remarks and future work

- We presented a covariant formulation to describe the dynamics of field theories, a notion of covariant phase space and a symplectic structure.
- We have seen how this covariant framework can be used to derive eventually usefull results in field theories.
- As future perspectives, we want to obtain some general results in general relativity, using this covariant framework, particularly in vacuum spacetimes, as for example:
  - give a correct and general definition of angular momentum at null infinity;
  - study all the transformations that leave $\omega$ invariant, and see if this transformations belong to the BMS group of symmetries or not.

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Thank you for your attention!