The Strong Cosmic Censorship conjecture in orthogonal Bianchi B perfect fluids and vacuum

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Overview

- The initial value problem and the Strong Cosmic Censorship conjecture
- Perfect fluids, and Bianchi models as an example of cosmological models
- Known and new results in Bianchi class B
- Conclusion

Globally hyberbolic spacetimes

Let (M, g) be a specetime. A Cauchy surface S is a subset which is intersected by every inextendible timelike curve exactly once.

The spacetime (M, g) is *globally hyperbolic* if and only if it admits a Cauchy surface S. In this case, M is homeomorphic to $\mathbb{R} \times S$ [Geroch 1970].

The initial value problem

Given a Riemannian manifold (S, h), a two-tensor K on S and suitable data $\mathcal{F}$ for the non-gravitational fields on S, satisfying the constraint equations

$$\begin{aligned} R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 &= 2\rho(h, \mathcal{F}) \\ \mathrm{div}_h K - \nabla(\mathrm{tr}_h K) &= J(h, \mathcal{F}) \\ C(h, \mathcal{F}) &= 0, \end{aligned}$$

find a spacetime (M, g) which solves the Einstein equation

$$Ein = Ric - \frac{1}{2} Rg = T$$

for the chosen type of matter and correct initial conditions, in particular h the induced metric, K the second fundamental form on S.

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Theorem (Choquet-Bruhat and Geroch 1969 (vacuum case))

There exists a maximal globally hyberbolic spacetime (M, g) with these conditions where $S \to M$ is a Cauchy hypersurface. Up to isometry, the spacetime (M, g) is unique, it is called the maximal globally hyperbolic development.

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Conjecture (Strong Cosmic Censorship)

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- restrict to specific matter models

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Given generic initial data, the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is unbounded in the incomplete directions of causal geodesics in the maximal globally hyperbolic development.

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This conjecture implies Strong Cosmic Censorship in the C^2 sense.
In case $Ric \neq 0$, one can simplify things by considering the scalar $Ric_{\alpha\beta}Ric^{\alpha\beta}$ instead of the Kretschmann scalar.

Our setting I: Perfect fluids

A spacetime (M, g) is called *perfect fluid* if it satisfies the Einstein equation with stress-energy tensor

$$T_{\alpha\beta} = \mu u_\alpha u_\beta + p(g + u_\alpha u_\beta)$$

for a unit timelike vector field u, where the pressure p and the energy density μ satisfy a linear equation of state

$$p = (\gamma - 1)\mu \tag{1}$$

for some constant γ.

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Initial data (S, h, K, μ_0, p_0) have to satisfy the Hamiltonian constraint with $\rho = \mu_0$, the momentum constraint with $J = 0$ and the additional constraint (1).

Our setting II: Cosmological Bianchi models

In cosmological spacetimes, the Cauchy surface S is compact without boundary. For our setting, we choose a *three-dimensional Lie group* $S = G$ and want to find a metric on $\mathbb{R} \times G$ which is left-invariant on every $\{t\} \times G$.

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All information about the Lie group is contained in the structure constants γ_{ij}^k defined by

$$[e_i, e_j] = \gamma_{ij}^k e_k,$$

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- class A (G is unimodular): types I, II, VI_0, VII_0, VIII, IX
- class B (G is non-unimodular): types IV, V, VI_h, VII_h. The topic of this talk, from now on we assume to be in this case.

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The Einstein equations can be translated into a Cauchy problem for the structure constants $\gamma_{ij}^k(t)$ on $\{t\} \times G$.

Expansion normalised variables

By transformation of the structure constants $\gamma_{ij}^{k}(t)$ and introduction of a new time variable τ one obtains a new set of variables with the following properties [Hewitt-Wainwright 1993]:

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- the incomplete direction of causal geodesics corresponds to $\tau \to -\infty$,
- (almost) all Bianchi class B types can be discussed simultaneously, as they correspond to invariant subsets of the full Cauchy problem,
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Only downside:

- the so-called "exceptional" Bianchi B model, type $\mathrm{VI}_{-1/9}$ has to be discussed separately. We restrict our discussion to the remaining ones, called *orthogonal*.

New initial value problem

"Bianchi B + (non-tilted) perfect fluid + new variables" yields

- polynomial evolution equations for the variables $(\Sigma_+, \tilde{\Sigma}, \Delta, \tilde{A}, N_+)$ contained in a compact subset of $\mathbb{R}^5$,
- two polynomial constraint equations,
- two given parameters: γ determines the type of perfect fluid, and the Lie group of some Bianchi types includes a parameter $k = {}^1\!/_h$.

From here on, we denote this initial value problem by (BPF).

Asymptotic behaviour as $\tau \to -\infty$

Theorem (Hewitt-Wainwright 1993)

Assume either vacuum or inflationary matter, i. e. either $\mu = 0$ or $\gamma \in \left[0, \frac{2}{3}\right)$, $\mu > 0$. Let $\Gamma = (\Sigma_+, \tilde{\Sigma}, \Delta, \tilde{A}, N_+)(\tau)$ be a solution to (BPF). As $\tau \to -\infty$ all accumulation points of Γ are contained in the Kasner parabola $\mathcal{K}$ given by

$$\Sigma_+^2 + \tilde{\Sigma} = 1 \qquad \Delta = \tilde{A} = N_+ = 0.$$

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Proposition (R. tbp)

Assume either $\mu = 0$ or $\gamma \in \left[0, \frac{2}{3}\right)$, $\mu > 0$. Then for every solution to (BPF) there exists $\mathrm{s} \in [-1, 1]$ such that

$$\lim_{\tau \to -\infty} (\Sigma_+, \tilde{\Sigma}, \Delta, \tilde{A}, N_+)(\tau) = (\mathrm{s}, 1 - \mathrm{s}^2, 0, 0, 0).$$

Strong cosmic censorship in perfect fluids (not vacuum)

Theorem (R. tpb)

Assume $\mu > 0$ *and* $\gamma > 0$*. Then for every solution to (BPF), the scalar* $Ric_{\alpha\beta}Ric^{\alpha\beta}$ *is unbounded as* $\tau \to -\infty$*.*

Strong cosmic censorship in perfect fluids (not vacuum)

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Assume $\mu > 0$ *and* $\gamma > 0$. *Then for every solution to (BPF), the scalar* $Ric_{\alpha\beta}Ric^{\alpha\beta}$ *is unbounded as* $\tau \to -\infty$.

Idea of the proof: From the stress-energy tensor for perfect fluids, one computes $Ric_{\alpha\beta}$ and

$$Ric_{\alpha\beta}Ric^{\alpha\beta} = \mu^2 + 3p^2 = (1 + 3(\gamma - 1)^2)\mu^2.$$

The evolution equations (BPF) give

$$\mu^2 = c_{\mathrm{ini}} \exp(-6\gamma\tau),$$

with a constant c_{ini} depending only on the initial data.

The Kretschmann scalar

For vacuum $\mu = 0$ and for the missing matter case $\mu > 0$, $\gamma = 0$, one computes the Kretschmann scalar

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} + 4 Ric_{\alpha\beta} Ric^{\alpha\beta} - R^2.$$

The scalar and Ricci curvature term are constant, but the Weyl tensor $C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}$ is not. For points on the Kasner parabola, it reads

$$C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = c_{\text{ini}} (2\,\mathsf{s} - 1)^2 (\mathsf{s} + 1) \exp(-12\gamma\tau)$$

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Conclusion: The Kretschmann scalar can only remain bounded for $\mathrm{s} = -1$ (Taub 1) or $\mathrm{s} = 1/2$ (Taub 2).
It remains to show that the set of Bianchi B models converging to one of those points is "non-generic" in a suitable sense.

Convergence to the Taub points and additional results

Theorem (R. tbp)

Assume either $\mu = 0$ *or* $\gamma \in \left[0, \frac{2}{3}\right)$, $\mu > 0$. *Then the following holds:*

1. *The only solution to (BPF) converging to the point* $(-1, 0, 0, 0, 0)$ *as* $\tau \to -\infty$ *is the constant solution.*
2. *Every solution to (BPF) converging to the point* $({}^1/_2, {}^3/_4, 0, 0, 0)$ *as* $\tau \to -\infty$ *is contained in the set*

$$3\Sigma_+^2 = \tilde{\Sigma}, \qquad \Sigma_+ N_+ = \Delta, \tag{2}$$

   which consists of locally rotationally symmetric (LRS) Bianchi II and VI_{-1} *models.*

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1. *The only solution to (BPF) converging to the point $(-1, 0, 0, 0, 0)$ as $\tau \to -\infty$ is the constant solution.*
2. *Every solution to (BPF) converging to the point $(1/2, 3/4, 0, 0, 0)$ as $\tau \to -\infty$ is contained in the set*

$$3\Sigma_+^2 = \tilde{\Sigma}, \qquad \Sigma_+ N_+ = \Delta, \tag{2}$$

   which consists of locally rotationally symmetric (LRS) Bianchi II and VI_{-1} models.

Additional results about the asymptotic behaviour:

- For limit points with $-1 < s < 1/2$, there is a similar restriction on the possible orbits.
- Limit points $1/2 < s < 1$ act as a source.

Conclusion

Theorem (R. tbp)

Consider initial data to Einstein's equation for orthogonal Bianchi B cosmological models with a non-tilted perfect fluid or vacuum. If the initial data is not of Bianchi type LRS II or LRS VI_{-1}*, then the maximal globally hyperbolic development is inextendible.*

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Thank you for your attention!

The evolution equations

$$\begin{aligned}
\Sigma_+' &= (q-2)\Sigma_+ - 2\tilde{N} \\
\tilde{\Sigma}' &= 2(q-2)\tilde{\Sigma} - 4\Sigma_+\tilde{A} - 4\Delta N_+ \\
\Delta' &= 2(q+\Sigma_+ - 1)\Delta + 2(\tilde{\Sigma} - \tilde{N})N_+ \\
\tilde{A}' &= 2(q+2\Sigma_+)\tilde{A} \\
N_+' &= (q+2\Sigma_+)N_+ + 6\Delta.
\end{aligned}$$

with

$$\begin{aligned}
q &= \frac{3}{2}(2-\gamma)(\Sigma_+^2 + \tilde{\Sigma}) + \frac{1}{2}(3\gamma - 2)(1 - \tilde{A} - \tilde{N}), \\
\tilde{N} &= \frac{1}{3}(N_+^2 - k\tilde{A})
\end{aligned}$$

and constraints $\tilde{\Sigma} \geq 0$, $\tilde{A} \geq 0$, $\tilde{N} \geq 0$ and

$$\begin{aligned}
0 &= \tilde{\Sigma}\tilde{N} - \Delta^2 - \Sigma_+^2\tilde{A} \\
\Omega &= 1 - \Sigma_+^2 - \tilde{\Sigma} - \tilde{A} - \tilde{N}.
\end{aligned}$$

Invariant subsets

Notation	Restrictions
$B(\mathrm{VI}_h)$	$k = 1/h < 0$, $\tilde{A} > 0$
$B^{\pm}(\mathrm{VII}_h)$	$k = 1/h > 0$, $\tilde{A} > 0$, $N_+ > 0$ or $N_+ < 0$
$B^{\pm}(\mathrm{IV})$	$k = 0$, $\tilde{A} > 0$, $N_+ > 0$ or $N_+ < 0$
$B(\mathrm{V})$	$k = 0$, $\tilde{A} > 0$, $\Sigma_+ = \Delta = N_+ = 0$
$B^{\pm}(\mathrm{II})$	$\tilde{A} = 0$, $N_+ > 0$ or $N_+ < 0$
$B(\mathrm{I})$	$\tilde{A} = \Delta = N_+ = 0$

Table : Bianchi invariant sets with higher symmetry.

Notation	Class of models	Restrictions
$S^{\pm}(\mathrm{II})$	LRS Bianchi II	$\tilde{A} = 0$, $3\Sigma_+^2 = \tilde{\Sigma}$, $\Delta^2 = \Sigma_+^2 N_+^2$
$S(\mathrm{VI}_{-1})$	LRS Bianchi VI_{-1}	$k = -1$, $\tilde{A} > 0$, $3\Sigma_+^2 = \tilde{\Sigma}$, $\Delta = \Sigma_+ N_+$
$S(\mathrm{VI}_h)$	Bianchi VI_h, $n^{\alpha}{}_{\alpha} = 0$	$\Delta = N_+ = 0$, $3\Sigma_+^2 + k\tilde{\Sigma} = 0$, $\tilde{A} > 0$
$S(\mathrm{V})$	Bianchi V FRW	$k = 0$, $\Sigma_+ = \tilde{\Sigma} = \Delta = N_+ = 0$
$S^{\pm}(\mathrm{VII}_h)$	Bianchi VII_h FRW	$\Sigma_+ = \tilde{\Sigma} = \Delta = 0$, $k\tilde{A} = N_+^2 > 0$