Constructing ‘geometric coordinates’ with predefined asymptotic behavior using foliations of constant mean curvature

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The problems

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Problem (Dependence on coordinates)

If we define physical quantities (mass, linear momentum, . . .) for (AE) *manifolds using coordinates, then we have to prove that they do not depend on the chosen coordinate system (or behave correctly under change of coordinates).*

Asymptotically Euclidean manifolds

Definition (Asymptotically Euclidean manifolds)

Let $(\overline{\mathrm{M}}, \overline{g})$ be a Riemannian manifold and $\varepsilon > 0$, $\eta := \varepsilon + 1/2$.
A chart $\overline{x} : \overline{\mathrm{M}} \setminus \overline{K} \to \mathbb{R}^3 \setminus \overline{B_1(0)}$ is called ***asymptotically Euclidean*** if $\overline{K} \subseteq \overline{\mathrm{M}}$ is a compact set and

$$\left.\begin{aligned} \overline{g} - \delta &= \mathcal{O}(r^{-\eta}), & {}^{\delta}\overline{\nabla} - \overline{\nabla} &= \mathcal{O}(r^{-1-\eta}), \\ \overline{\mathrm{Ric}} &= \mathcal{O}(r^{-2-\eta}), & \overline{\mathcal{S}} &= \mathcal{O}(r^{-3-\varepsilon}), \end{aligned}\right\} \tag{AE}$$

where $r := |x|$

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where $r := |x|$, ${}^{\hbar}\overline{g} := \mathrm{d}r^2 + \sinh(r)^2\,\Omega$.

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If there exists an asymptotically Euclidean (hyperbolic) chart $\overline{x}$ for $(\overline{\mathrm{M}}, \overline{g})$, then $(\overline{\mathrm{M}}, \overline{g})$ is called asymptotically Euclidean (hyperbolic).

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$$ {}_\sigma\mathcal{H} \equiv \frac{2}{\sigma} \qquad \text{(AE)} \qquad \text{resp.} \qquad {}_\sigma\mathcal{H} \equiv \frac{2\cosh(\sigma)}{\sinh(\sigma)} \qquad \text{(AH)} $$

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- the surfaces ${}_\sigma\Sigma$ cover $\overline{\mathrm{M}}$ outside of a compact set, i. e. $\overline{\mathrm{M}} \setminus \bigcup_\sigma {}_\sigma\Sigma$ is compact;
- the surfaces ${}_\sigma\Sigma$ are pairwise disjoint.

CMC foliation – Euclidean case

Euclidean setting: $(\overline{\mathrm{M}} = \mathbb{R}^3, \overline{g} = \delta)$

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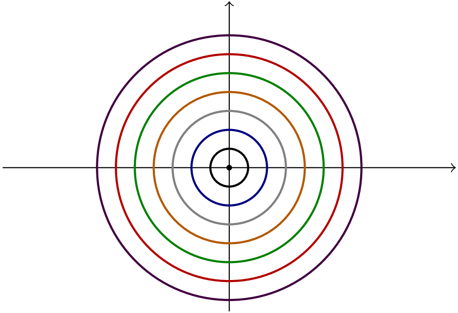

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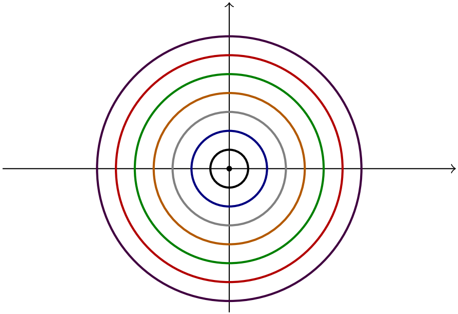

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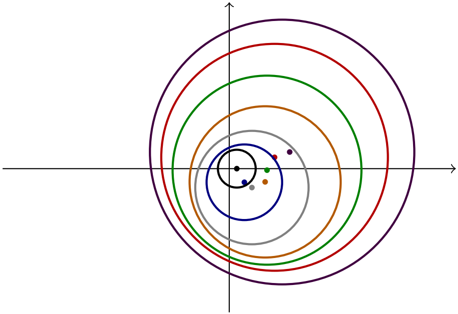

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CMC foliation – Schwarzschildean case

Spatial Schwarzschild solution $(\overline{\mathrm{M}} := \mathbb{R}^3 \setminus B_{\frac{\overline{m}}{2}}(0), \bar{g})$ with mass $\overline{m} \neq 0$,

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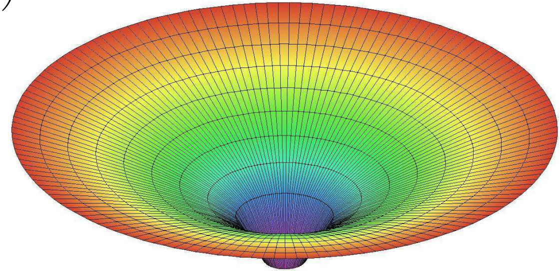

Figure: Schwarzschild as Flamm's paraboloid

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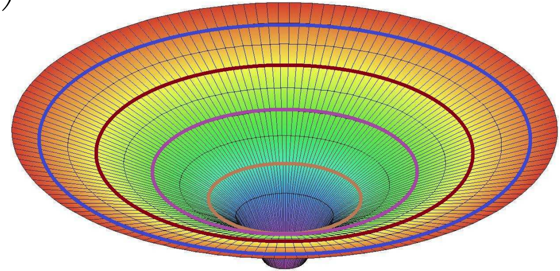

Figure: Schwarzschild as Flamm's paraboloid

There exists a unique CMC foliation $\{_\sigma\Sigma = \mathcal{S}^2_{R(\sigma)}(0)\}_\sigma$.

Existence of the CMC foliation

Theorem ([Huisken and Yau, 1996], [Metzger, 2007], [Huang, 2010], [Eichmair and Metzger, 2012], [N., 2014a])

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Theorem ([Neves and Tian, 2009], [Neves and Tian, 2010], [N., 'tbp])

If $(\overline{\mathrm{M}}, \overline{g})$ is asymptotically hyperbolic with positive mass, then there exists a unique, stable CMC foliation $\{{}_{\sigma}\Sigma\}_{\sigma>\sigma_0}$.

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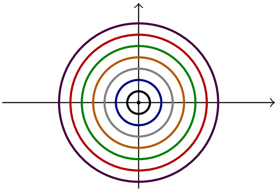

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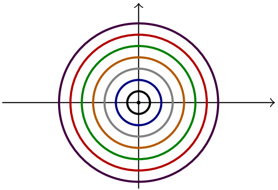

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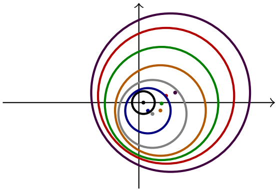

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Question

Can the second step be skipped, i. e. can asymptotic to the Euclidean (hyperbolic) space be characterized ***geometrically*** *by the existence of a suitable CMC foliation?*

Geometric characterization of asymptotically flatness

Theorem ((AE) case [N., 2014])

$(\overline{\mathrm{M}},\overline{g})$ *is asymptotically flat if it possesses a foliation by stable CMC hypersurfaces* ${}_\sigma\Sigma$ *with mean curvature* ${}_\sigma\mathcal{H} \equiv {}^2\!/\!_\sigma$ *(for* $\sigma > \sigma_0$*) and non-vanishing total mass* $\lim \mathrm{m}_H({}_\sigma\Sigma) \neq 0$ *such that* $\overline{\mathrm{Ric}}|_{{}_\sigma\Sigma}$ *decays sufficiently as* $\sigma \to \infty$*.*

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Idea

Construct 'geometric' spherical coordinates satisfying (AE) *resp.* (AH) *using the CMC foliation, i. e. choose 'good' coordinates* $({}_\sigma\varphi, {}_\sigma\vartheta) : {}_\sigma\Sigma \to \mathcal{S}^2_\sigma$ *for each* σ *and define three-dimensional coordinates* $(r, (\varphi, \vartheta)) : \overline{\mathrm{M}} \to (\sigma_0\,;\infty) \times \mathcal{S}^2$ *by*

$$r|_{{}_\sigma\Sigma} :\equiv \sigma, \qquad (\varphi, \vartheta)|_{{}_\sigma\Sigma} := ({}_\sigma\varphi, {}_\sigma\vartheta).$$

Main problem of this idea

Problem (The radial direction)

Infinitesimal distance (lapse) function between the leaves of $\{\mathcal{S}^2_\sigma\}_\sigma$ *is constant* 1*, i. e.* $\delta(\partial_r, \partial_r) \equiv 1$*. But, in the above construction*

$$\overline{g}(\partial_r, \partial_r)|_{\sigma\Sigma} = \textit{lapse function between the leaves of } \{_{\sigma'}\Sigma\}_{\sigma'} = 1 + \mathcal{O}(\sigma^{-\varepsilon}).$$

$$\xrightarrow[]{\textit{(in general)}} \quad \overline{g} - \delta = \mathcal{O}(r^{-\varepsilon}) \textit{ and not } \overline{g} - \delta = \mathcal{O}(r^{-\frac{1}{2}-\varepsilon}) \textit{ in this chart.}$$

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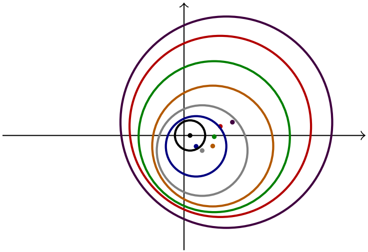

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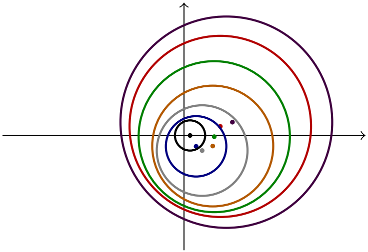

$\Longrightarrow$ choose the centers of the spheres more carefully

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Idea

We use the lapse function, i. e. $u := \overline{g}(\partial_\sigma \varphi, \nu)$ *where* φ *is any parametrization of the foliation.*

Understanding the lapse function

Let $\varphi : (-\eta\,;\eta) \times \mathcal{S}^2_r(0) \to \mathbb{R}^3$ be smooth with $\varphi(0,\cdot) = \mathrm{id}\,|_{\mathcal{S}^2_r(0)}$.

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Geometric characterization (for $\eta = 0$):

$$u^* \equiv u_0 = \fint_{\mathcal{S}^2_r(0)} u\,\mathrm{d}\mathcal{H}^2, \qquad \Delta\,u^t = \frac{-2}{r^2}u^t, \qquad u = u^* + u^t + u^d.$$

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1. Choose a complete orthogonal system $\{{}_\sigma f_i\}_{i=1}^\infty$ of $\mathrm{L}^2({}_\sigma\Sigma)$ of eigenfunctions of the Laplace operator, i. e. ${}_\sigma\Delta\,{}_\sigma f_i = -{}_\sigma\lambda_i\,{}_\sigma f_i$ with ${}_\sigma\lambda_i \leq {}_\sigma\lambda_{i+1}$ and $\sup|{}_\sigma f_i| = 1$;

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4. fix some σ_1 and define the *centers* ${}_\sigma z := \left(\int_{\sigma_1}^{\sigma} {}_\varsigma u^i \,\mathrm{d}\varsigma\right)_{i=1}^{3}$.

Optimizing the coordinates

Problem

We cannot choose the coordinates of one ${}_{\sigma}\Sigma$ *independently of the ones for the other surfaces* $\{{}_{\varsigma}\Sigma\}_{\varsigma}$*, as their* σ*-derivative has to satisfy some decay assumption.*

Final step

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Do not choose coordinates (e. g. conformally) mapping ${}_\sigma\Sigma$ to some Euclidean sphere, i. e. $\overline{x}({}_\sigma\Sigma) = \mathcal{S}^2_\sigma({}_\sigma z)$, but choose 'geometric' functions h^1, h^2, h^3 on ${}_\sigma\Sigma$ as the components of the chart, i. e. $\overline{x}^i|_\Sigma := h^i$. Then prove that these depend regulary enough on σ and map ${}_\sigma\Sigma$ to a surfaces near to a Euclidean sphere.

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The coordinates are

$$\overline{x} : \overline{\mathrm{M}} \to \mathbb{R}^3 : p \mapsto \sigma(p)\left({}_\sigma f^1, {}_\sigma f^2, {}_\sigma f^3\right) + z(\sigma(p)),$$

where $p \in {}_{\sigma(p)}\Sigma$ and ${}_\sigma z$ is the center of ${}_\sigma\Sigma$ as defined before.

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Thank you for your attention!

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