Mass-like invariants for asymptotically hyperbolic manifolds

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Plan

1. Asymptotically hyperbolic manifolds
2. The case $\tau = n$
3. Asymptotic invariants for $\tau > 0$

Plan

Why asymptotically hyperbolic?

- Cauchy surfaces for asymptotically anti-de Sitter (adS^{n+1}) spacetimes,
- Most natural boundary condition "at infinity" after asympt. Euclidean?
- adS/CFT correspondence (conformally compact initial data).

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- ... such that $|\Phi_* g - b|_b(x) = O(e^{-\tau r})$ (and derivatives up to order 2).

Dependence on the chart?

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W.l.o.g., one can write

$\Phi_* g = dr^2 + \sinh^2 r \left[\sigma_{\mathbb{S}^{n-1}} + \mathbf{m} e^{-\tau r} + O(e^{-(\tau+1)r})\right]$ with $\mathbf{m} \in \Gamma(S^2 T^* \mathbb{S}^{n-1})$.

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- In the Euclidean case: $\Phi_2 \circ \Phi_1^{-1} = A + O(r^{-(\tau-1)})$.

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The mass vector for $\tau = n$ (X. Wang)

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If Φ_1 and Φ_2 satisfy $\Phi_2 \circ \Phi_1^{-1} = A + O(e^{-(n+1)r})$ with $A \in O^+(n,1)$, then

$$\mathbf{p}_{\Phi_1} = A.\mathbf{p}_{\Phi_2}.$$

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Consequences:

- The number $|\mathbf{p}|^2_\eta = -p_0^2 + \sum_i p_i^2$ does not depend on Φ!
- But $p_0 = \int_{\mathbb{S}^{n-1}} \mathrm{tr}^\sigma \mathbf{m}\, d\sigma$ alone does depend on Φ.
- Remark: **p** also contains a suitable notion of AH center of mass, fitting with the one from CMC-foliations (Cederbaum-C-Sakovich '15).

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Example: Kottler metrics for $n = 3$

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- We find that $\mathbf{m} = 3M\sigma$, thus $\mathbf{p} = C(M, 0, 0, 0)$.

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- Example: $(\tau = n)$, we have $\mathbb{V} = \mathbb{R}^{n,1}$, $\Psi(\mathbf{m}) = \mathbf{p}$.

Results for $n = 3$

Theorem (C-Dahl-Gicquaud '15)

For all $\tau \geq 2$, $\tau \in \mathbb{N}$, there is a unique map $\Psi_\tau : \Gamma(S^2 T^* \mathbb{S}^{n-1}) \to \mathbb{V}_\tau$ which is intertwining:

$$\Psi_\tau(\mathbf{m}) = \left(P \in \mathbb{V}_\tau^* \mapsto \int_{\mathbb{S}^2} P(1, x^1, x^2, x^3) \operatorname{tr}^\sigma \mathbf{m} \, d\sigma \right),$$

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- For $\tau = 2$, $\mathbb{V}_2 = \mathbb{R}$, and $\int_{\mathbb{S}^2} \mathrm{tr}^\sigma \mathbf{m} \, d\sigma$ is an asymptotic invariant!

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$$\overline{\Psi}_\tau : g \mapsto \mathbb{V}^*_\tau$$

defined on the set of metrics $g \in \mathrm{AH}(\frac{\tau}{2} + \epsilon)$ such that $e^{(\tau-2)r} F_\tau(g)$ is integrable on $\mathbb{H}^3 \setminus \overline{B}_R$, where F_τ is a geometric differential operator. Moreover, we have $\overline{\Psi}_\tau(g) = \Psi_\tau(g)$ for $g \in \mathrm{AH}(\tau)$.

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- The operators F_τ are defined by $F_\tau(g) := \Delta_g \mathrm{Scal}_g - C(\tau)\mathrm{Scal}_g$ for $\tau \neq 3$, and $F_3 = \mathrm{Scal}$.

Open questions

- Case $n \geq 3$ (Work in progress)
- Complex intertwinning maps (in progress)
- Multilinear in $\mathbf{m}$ intertwinning maps (Gauss-Bonnet-Chern masses)

Representation theory for $O(n,1)$

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Figure : Nick Hobgood, http://commons.wikimedia.org/wiki/File:Amphiprion_ocellaris_%28Clown_anemonefish%29_Nemo.jpg