GEOMETRIC FOLIATIONS IN GENERAL RELATIVITY

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1. Lecture 1

1.1. **Introduction.** We consider a spacetime (N, γ) satisfying the Einstein field equations. Recall from [13] that (N, γ) is encapsulated in initial data (M, g, k) consisting of a spacelike hypersurface $M \subset N$ with induced metric g and second fundamental form k. In this context, the scalar curvature R of (M, g) provides a lower bound for the energy density of (N, γ).

If (N, γ) models an isolated gravitational system, (M, g, k) can be chosen to be an asymptotically flat Riemannian manifold in the sense of Definition 1 below. Here and below, we will assume that $k = 0$. A bar indicates that a geometric quantity is computed with respect to the Euclidean metric $\bar{g}$.

Definition 1. *Let (M, g) be a connected complete Riemannian manifold with integrable scalar curvature R. We say that (M, g) is asymptotically flat if there is a non-empty compact subset of M whose complement is diffeomorphic to $\{x \in \mathbb{R}^3 : |x|_{\bar{g}} > 1/2\}$ such that, in this so-called asymptotically flat chart, $g = \bar{g} + \sigma$ where*

$$|\sigma|_{\bar{g}} + |x|_{\bar{g}}\, |\bar{D}\sigma|_{\bar{g}} + |x|_{\bar{g}}^2\, |\bar{D}^2\sigma|_{\bar{g}} = O(|x|_{\bar{g}}^{-\tau})$$

for some $\tau \in (1/2, 1]$.

We usually fix an asymptotically flat chart and use it as a reference. We use B_r, $r > 1/2$, to denote the connected, bounded subset of M whose boundary corresponds to $S_r(0) = \{x \in \mathbb{R}^3 : |x|_{\bar{g}} = r\}$ with respect to this chart.

Definition 2. *The mass of a an asymptotically flat Riemannian manifold (M, g) is given by*

$$m = \frac{1}{16\,\pi} \lim_{\lambda\to\infty} \lambda^{-1} \int_{S_\lambda(0)} \sum_{i,j=1}^{3} x^i\,(\partial_j g_{ij} - \partial_i g_{jj})\,\mathrm{d}\bar{\mu}.$$

The mass is a geometric invariant that measures the total gravitational energy of the initial data set; see [3, 1]. It is positive if the scalar curvature of (M, g) is non-negative and if (M, g) is not isometric to flat $\mathbb{R}^3$; see [28].

Definition 3 ([27])**.** *Let (M, g) be an asymptotically flat Riemannian manifold with positive mass. The Hamiltonian center of mass of (M, g) is given by $C = (C^1, C^2, C^3)$ where*

$$C^\ell = \frac{1}{16\,\pi\, m} \lim_{\lambda\to\infty} \lambda^{-1} \int_{S_\lambda(0)} \sum_{i,j=1}^{3} x^\ell\, x^j\,(\partial_i g_{ij} - \partial_j g_{ii}) - \sum_{i=1}^{3} (x^i\, g_{i\ell} - x^\ell\, g_{ii})\,\mathrm{d}\bar{\mu} \tag{1}$$

provided the limits on the left-hand side exist.

Remark 4 ([18])**.** *The center of mass exists if (M, g) satisfies the so-called Regge-Teitelboim conditions*

$$|\hat{g}|_{\bar{g}} + |x|_{\bar{g}}\, |\bar{D}\hat{g}|_{\bar{g}} + |x|_{\bar{g}}^2\, |\bar{D}^2\hat{g}|_{\bar{g}} = O\left(|x|_{\bar{g}}^{-1-\tau}\right) \qquad \textit{and} \qquad \hat{R} = O(|x|_{\bar{g}}^{-7/2-\tau})$$

where

$$\hat{g}(x) = g(x) - g(-x) \qquad \textit{and} \qquad \hat{R}(x) = R(x) - R(-x).$$

Example 5. *Spatial Schwarzschild* (M_m, g_m) *of mass* $m > 0$ *is given by*

$$M_m = \{x \in \mathbb{R}^3 : |x|_{\bar{g}} \geq m/2\} \qquad \text{and} \qquad g_m = (1 + m/2\,|x|_{\bar{g}}^{-1})^4\,\bar{g}.$$

It models initial data for a static black hole with mass m.

A central goal in mathematical relativity is to understand the relationship between the asymptotic geometry of (M, g) and the global physical properties of (M, g).

Definition 6. *We say that a family* $\{\Sigma(s) : s \in (0, 1)\}$ *of spheres* $\Sigma(s) \subset M$ *forms an asymptotic foliation of* (M, g) *if there is a smooth function* $u : M \to (0, \infty)$ *with the following properties.*

- $u \to 0$ *as* $x \to \infty$.
- $\Sigma(s) = \{x \in M : u(x) = s\}$.
- *Every* $s \in (0, 1)$ *is a regular value of* u.

Note that an asymptotic foliation provides an asymptotic coordinate system of (M, g). We will study geometric foliations that

- are defined in a canonical way,
- detect the mass of (M, g),
- detect the center of mass or the asymptotic energy distribution of (M, g).

1.2. **Special surfaces in initial data sets.** Let $\Sigma \subset M$ be a closed surface with outward normal ν, area element $\mathrm{d}\mu$, second fundamental form h, traceless second fundamental form $\mathring{h}$, and mean curvature H. The Hawking mass of Σ is defined by

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\,\pi}}\left(1 - \frac{1}{16\,\pi}\int_\Sigma H^2\,\mathrm{d}\mu\right).$$

It provides a measure for the strength of the gravitational field in the domain enclosed by Σ; see [17]. It plays an important part in the proof of the Riemannian Penrose inequality; see [19].

The following proposition shows that the quantity $m_H(\Sigma)$ is meaningless unless Σ is in some way special.

Proposition 7. *Let* (M_m, g_m) *be spatial Schwarzschild of mass* $m > 0$. *There holds*

$$\sup_{\Sigma \subset M_m} m_H(\Sigma) = \infty \qquad \text{and} \qquad \inf_{\Sigma \subset M_m} m_H(\Sigma) = -\infty$$

where the supremum and infimum are taken over all embedded spheres $\Sigma \subset M$.

We will study the existence, uniqueness, and asymptotic positioning of two classes of surfaces that are well-adapted to the Hawking mass.

1.2.1. *Stable constant mean curvature surfaces.* A closed surface $\Sigma \subset M$ is called a stable constant mean curvature surface if it passes the second derivative test for area among all volume-preserving variations. Such surfaces are potential candidates to have least area for the volume they enclose. Their mean curvature equals a scalar and they satisfy the stability inequality

$$\int_\Sigma |h|^2\,f^2 + Ric(\nu,\nu)\,f^2\,\mathrm{d}\mu \leq \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu$$

for all $f \in C^\infty(\Sigma)$ with

$$\int_\Sigma f\,\mathrm{d}\mu = 0.$$

Proposition 8 ([14])**.** *Let (M,g) be an asymptotically flat Riemannian manifold with non-negative scalar curvature and $\Sigma\subset M$ be a stable constant mean curvature sphere. There holds $m_H(\Sigma)\geq 0$.*

1.2.2. *Area-constrained Willmore surfaces.* A closed surface $\Sigma\subset M$ is called an area-constrained Willmore surface if it passes the first derivative test for the Hawking mass among all area-preserving variations; see [22] where area-constrained Willmore surfaces are called surfaces of Willmore type. Such surfaces are potential candidates to have a maximal amount of Hawking mass for their area. They satisfy the area-constrained Willmore equation

$$\Delta H+(|\mathring{h}|^2+Ric(\nu,\nu)+\kappa)\,H=0 \tag{2}$$

for some Lagrange parameter $\kappa\in\mathbb{R}$. In this context, recall that the first variation of the Willmore energy

$$\mathcal{W}(\Sigma)=\frac{1}{4}\int_\Sigma H^2\,\mathrm{d}\mu$$

along a normal variation with initial speed $f\in C^\infty(\Sigma)$ is given by

$$\delta\mathcal{W}(\Sigma)(f)=-\frac{1}{2}\int_\Sigma f\,\Delta H+f\,|\mathring{h}|^2\,H+f\,Ric(\nu,\nu)\,H\,\mathrm{d}\mu.$$

On the one hand, area-constrained Willmore surfaces are expected be fine-tuned to the Hawking mass. On the other hand, the area-cosntrained Willmore equation allows for more analytical flexibility:

- (2) is a fourth-order quasi-linear equation so that standard tools such as the maximum principle are not available.
- A closed minimal surface $\Sigma\subset M$ satisfies the second derivative test for the Hawking mass among area-preserving variations. It does not necessarily satisfy the second derivative test for area among volume-preserving variations.
- There are infinitely many (area-constrained) embedded stable Willmore surfaces in $\mathbb{R}^3$ that are not congruent to each other; see [4]. By contrast, every immersed stable constant mean curvature surface in $\mathbb{R}^3$ is a sphere; see [2].

1.3. **Asymptotic foliations by stable constant mean curvature spheres.** Let (M,g) be an asymptotically flat Riemannian manifold with positive mass.

Theorem 9 ([25])**.** *There exists $H_0>0$ and a family*

$$\{\Sigma(H):H\in(0,H_0)\}, \tag{3}$$

where $\Sigma(H)\subset M$ is a stable constant mean curvature sphere with mean curvature H, that forms a foliation of the complement of a compact subset of M.

Theorem 9 was proved in [20] under the assumption that (M,g) is asymptotic to Schwarzschild; see Definition 25. We also note the important previous works [18, 24]. Theorem 9 has been extended to a spacetime setting in [9]. The following proof is from the author's recent joint work with M. Eichmair [15].

1.3.1. *Heuristics.* Since g is asymptotic to $\bar g$, the results in [2] suggest that large stable constant mean curvature spheres in (M,g) are perturbations of round coordinate spheres. The Euclidean area is invariant under rigid motions. We therefore use a Lyapunov-Schmidt reduction instead of the implicit function theorem; see also [10, 6].

1.3.2. *Spherical harmonics.* Recall that the eigenvalues of the operator

$$-\bar{\Delta}: H^2(S_1(0)) \to L^2(S_1(0))$$

are given by

$$\{\ell\,(\ell+1) : \ell = 0\,, 1\,, 2, \dots\}. \tag{4}$$

The corresponding eigenspaces $\Lambda_\ell(S_1(0))$ are finite-dimensional and form an orthonormal basis for $L^2(S_1(0))$. $\Lambda_0(S_1(0))$ consists of constant functions and $\Lambda_1(S_1(0))$ is spanned by the coordinate functions $y \mapsto \bar{g}(y, e_i)$, $i = 1,\, 2,\, 3$.

1.3.3. *Notation.* Given $\xi \in \mathbb{R}^3$ and $\lambda > 1$, we abbreviate

$$S_{\xi,\lambda} = S_\lambda(\lambda\,\xi) = \{x \in \mathbb{R}^3 : |x - \lambda\,\xi|_{\bar{g}} = \lambda\}.$$

Given $u \in C^\infty(S_{\xi,\lambda})$, we define $\Sigma_{\xi,\lambda}(u)$ to be the Euclidean graph of u over $S_{\xi,\lambda}$. In the estimate (5), $\bar{D}$, the dash, and $\bar{\nabla}$ denote differentiation with respect to $\xi \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$, and $x \in S_{\xi,\lambda}$, respectively.

1.3.4. *Lyapunov-Schmidt reduction.*

Proposition 10. *Let $\delta \in (0, 1/2)$. There are constants $\lambda_0 > 1$ and $\epsilon > 0$ such that for every $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar{g}} < 1 - \delta$ and $\lambda > \lambda_0$ there exists a function $u_{\xi,\lambda} \in C^\infty(S_{\xi,\lambda})$ such that the following holds. $u_{\xi,\lambda} \perp \Lambda_1(S_{\xi,\lambda})$ and, as $\lambda \to \infty$,*

$$\begin{aligned} \lambda^{-1}\,|u_{\xi,\lambda}| + |\bar{\nabla} u_{\xi,\lambda}|_{\bar{g}} + \lambda\,|\bar{\nabla}^2 u_{\xi,\lambda}|_{\bar{g}} &= o(\lambda^{-1/2}),\\ \lambda^{-1}\,(\bar{D}u)|_{(\xi,\lambda)} &= o(\lambda^{-1/2}),\\ u'|_{(\xi,\lambda)} &= o(\lambda^{-1/2}) \end{aligned} \tag{5}$$

uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar{g}} < 1 - \delta$. The surface $\Sigma_{\xi,\lambda} = \Sigma_{\xi,\lambda}(u_{\xi,\lambda})$ has the properties

- *$H \in \Lambda_0(S_{\xi,\lambda}) \oplus \Lambda_1(S_{\xi,\lambda})$,*
- *$\operatorname{vol}(\Sigma_{\xi,\lambda}) = \frac{4\pi}{3}\lambda^3$.*

Proof. Let $\mathcal{G}$ be the space of Riemannian metrics on $\{y \in \mathbb{R}^3 : 1 - \delta/2 < |y|_{\bar{g}} < 3\}$ equipped with the C^2-topology. We consider the map

$$\Theta_{\xi,\lambda} : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{given by} \qquad \Theta_{\xi,\lambda}(y) = \lambda\,(\xi + y).$$

Note that $\Theta_{\xi,\lambda}(S_1(0)) = S_{\xi,\lambda}$. The rescaled metric $g_{\xi,\lambda} = \lambda^{-2}\,\Theta^*_{\xi,\lambda}\,g$ satisfies

$$||g_{\xi,\lambda} - \bar{g}||_{\mathcal{G}} = o(\lambda^{-1/2}\,|1 - |\xi||_{\bar{g}}^{-1/2}) = o(\lambda^{-1/2}\,\delta^{-1/2}).$$

Let $\alpha \in (0, 1)$, $k \geq 0$ be an integer, and $\Lambda_{0,k}(S_1(0))$ and $\Lambda_{1,k}(S_1(0))$ be the constants and first spherical harmonics viewed as subspaces of $C^{k,\alpha}(S_1(0))$, respectively. We define the smooth map

$$T : \Lambda_{1,2}(S_1(0))^\perp \times \mathcal{G} \to [\Lambda_{0,0}(S_1(0)) \oplus \Lambda_{1,0}(S_1(0))]^\perp \times \mathbb{R}$$

by

$$T(u,\, g) = \Big(\operatorname{proj}_{[\Lambda_{0,0}(S_1(0)) \oplus \Lambda_{1,0}(S_1(0))]^\perp} H,\, \operatorname{vol}(\Sigma_{1,0}(u))\Big)$$

where all geometric quantities are with respect to $\Sigma_{1,0}(u)$ and the metric g. Note that

$$(DT)|_{(0,\,\bar{g})}(u,\, 0) = \Big(\operatorname{proj}_{[\Lambda_{0,0}(S_1(0)) \oplus \Lambda_{1,0}(S_1(0))]^\perp}(-\bar{\Delta} u - 2\,u),\, -4\,\pi\, \operatorname{proj}_{\Lambda_0(S_1(0))}\, u\Big).$$

Since kernel of the operator

$$-\bar{\Delta} - 2 : C^{2,\alpha}(S_1(0)) \to C^{0,\alpha}(S_1(0))$$

is given by $\Lambda_{1,2}(S_1(0))$, $(DT)|_{(0,\bar{g})}(\,\cdot\,,0) : \Lambda_{1,2}(S_1(0))^\perp \to [\Lambda_{0,0}(S_1(0)) \oplus \Lambda_{1,0}(S_1(0))]^\perp \times \mathbb{R}$ is an isomorphism. The assertions follow from this and the implicit function theorem. □

To capture the variational nature of the constant mean curvature equation on the families of surfaces $\{\Sigma_{\xi,\lambda} : |\xi|_{\bar{g}} < 1-\delta\}$ from Proposition 10, we consider the reduced area function

$$G_\lambda : \{\xi \in \mathbb{R}^3 : |\xi|_{\bar{g}} < 1-\delta\} \to \mathbb{R} \qquad \text{given by} \qquad G_\lambda(\xi) = \lambda^{-1}\,|\Sigma_{\xi,\lambda}|.$$

Lemma 11. *Given $\delta \in (0, 1/2)$, there is $\lambda_0 > 1$ such that for every $\lambda > \lambda_0$ and $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar{g}} < 1-\delta$ the following holds. The sphere $\Sigma_{\xi,\lambda}$ has constant mean curvature if and only if ξ is a critical point of G_λ.*

2. Lecture 2

2.0.1. *Computing the reduced area function.*

Lemma 12. *Let $\delta \in (0, 1/2)$ and $a \in \mathbb{R}^3$ with $|a|_{\bar{g}} = 1$. There holds, as $\lambda \to \infty$,*

$$\operatorname{div} a = \frac{1}{2}\,\bar{D}_a \bar{\operatorname{tr}}\sigma + O(\lambda^{-1-2\tau}), \tag{6}$$

$$g(D_\nu a, \nu) = \frac{1}{2}\,(\bar{D}_a\sigma)(\bar{\nu},\bar{\nu}) + O(\lambda^{-1-2\tau})$$

on $S_{\xi,\lambda}$ uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar{g}} < 1-\delta$. Moreover,

$$\nu - \bar{\nu} = -\frac{1}{2}\,\sigma(\bar{\nu},\bar{\nu})\,\bar{\nu} - \sum_{\alpha=1}^{2} \sigma(\bar{\nu}, f_\alpha)\, f_\alpha + O(\lambda^{-2\tau}), \tag{7}$$

$$\mathrm{d}\mu = \left[1 + \frac{1}{2}[\bar{\operatorname{tr}}\sigma - \sigma(\bar{\nu},\bar{\nu})] + O(\lambda^{-2\tau})\right] \mathrm{d}\bar{\mu}.$$

Here, $\{f_1,\, f_2\}$ is a local Euclidean orthonormal frame for $TS_{\xi,\lambda}$.

Proof. We sketch the proof of (6) and (7).

There holds

$$\operatorname{div}(a) = \sum_{i,j=1}^{3} g^{ij}\, g(D_{e_i} a, e_j) = \sum_{i,k=1}^{3} a^k\, \Gamma^i_{ik} + O(|x|_{\bar{g}}^{-1-2\tau}) = \frac{1}{2}\,\bar{D}_a \bar{\operatorname{tr}}\sigma + O(|x|_{\bar{g}}^{-1-2\tau}).$$

Note that $|x|_{\bar{g}}^{-1} = O(\lambda^{-1})$ on $S_{\xi,\lambda}$.

Let $g_t = g + t\,\sigma$ and ν_t the unit normal of $S_{\xi,\lambda}$ with respect to g_t. Differentiating $g_t(\nu_t,\nu_t) = 1$, we obtain

$$\sigma(\bar{\nu},\bar{\nu}) + 2\,\bar{g}(\dot{\nu},\bar{\nu}) = 0.$$

□

Lemma 13. *Let $\delta \in (0, 1/2)$. There holds, as $\lambda \to \infty$ on $S_{\xi,\lambda}$ uniformly for all $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar{g}} < 1-\delta$,*

$$H(\Sigma_{\xi,\lambda}) = H(S_{\xi,\lambda}) - \Delta(S_{\xi,\lambda})u_{\xi,\lambda} - |h(S_{\xi,\lambda})|^2\, u_{\xi,\lambda} - Ric(\nu(S_{\xi,\lambda}), \nu(S_{\xi,\lambda}))\, u_{\xi,\lambda} + o(\lambda^{-5/2}).$$

In the following two lemmas, we compute an asymptotic expansion of G_λ as $\lambda \to \infty$.

Lemma 14. *Let $a \in \mathbb{R}^3$ with $|a|_{\bar{g}} = 1$. There holds, as $\lambda \to \infty$,*

$$(\bar{D}_a G_\lambda)|_\xi = \frac{1}{2}\int_{S_{\xi,\lambda}} \left[\bar{D}_a \bar{\operatorname{tr}}\sigma - (\bar{D}_a\sigma)(\bar{\nu},\bar{\nu}) - 2\,\lambda^{-1}\,\bar{\operatorname{tr}}\sigma\,\bar{g}(a,\bar{\nu})\right] \mathrm{d}\bar{\mu} + o(1)$$

uniformly for all $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}<1-\delta$.

Proof. Using that $\operatorname{vol}(\Sigma_{\xi,\lambda})$ does not depend on $\xi\in\mathbb{R}^3$, we obtain

$$(\bar D_a G_\lambda)|_\xi=\int_{\Sigma_{\xi,\lambda}}[H-2\,\lambda^{-1}]\,g(a+\lambda^{-1}\,(\bar D_a u)|_{(\xi,\lambda)}\,\bar\nu,\nu)\,\mathrm{d}\mu. \tag{8}$$

By Lemma 13, Lemma 12, and (5),

$$H(\Sigma_{\xi,\lambda})=H(S_{\xi,\lambda})+o(\lambda^{-3/2})=2\,\lambda^{-1}+o(\lambda^{-3/2}).$$

In conjunction with (5) and (8), we find

$$(\bar D_a G_\lambda)|_\xi=\int_{\Sigma_{\xi,\lambda}}[H-2\,\lambda^{-1}]\,g(a,\nu)\,\mathrm{d}\mu+o(1). \tag{9}$$

The first variation formula implies that

$$\begin{aligned}\int_{\Sigma_{\xi,\lambda}}[H-2\,\lambda^{-1}]\,g(a,\nu)\,\mathrm{d}\mu&=\int_{\Sigma_{\xi,\lambda}}[\operatorname{div}a-g(D_\nu a,\nu)-2\,\lambda^{-1}\,g(a,\nu)]\,\mathrm{d}\mu\\&=\int_{S_{\xi,\lambda}}[\operatorname{div}a-g(D_\nu a,\nu)-2\,\lambda^{-1}\,g(a,\nu)]\,\mathrm{d}\mu+o(1).\end{aligned} \tag{10}$$

The assertion follows from this, Lemma 12, and the divergence theorem. □

Lemma 15. *Let $\delta\in(0,1/2)$. There holds, as $\lambda\to\infty$, uniformly for all $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}<1-\delta$,*

$$\begin{aligned}G_\lambda(\xi)&=G_\lambda(0)+4\,\pi\,m\,|\xi|^2_{\bar g}+o(1)\ \text{ and}\\(\bar D G_\lambda)|_\xi&=8\,\pi\,m\,\xi+o(1).\end{aligned}$$

Proof. Let $a\in\mathbb{R}^3$ with $|a|_{\bar g}=1$. Note that

$$\begin{aligned}\bar D_a\operatorname{tr}\sigma-(\bar D_a\sigma)(\bar\nu,\bar\nu)=\bar g(a,\bar\nu)\,\left[\bar D_{\bar\nu}\bar{\operatorname{tr}}\sigma-(\bar{\operatorname{div}}\sigma)(\bar\nu)\right]&+\sum_{\alpha,\beta=1}^2\bar g(a,f_\alpha)(\bar D_{f_\alpha}\sigma)(f_\beta,f_\beta)\\&+\sum_{\alpha=1}^2\left[\bar g(a,\bar\nu)\,(\bar D_{f_\alpha}\sigma)(\nu,f_\alpha)-\bar g(a,f_\alpha)\,(\bar D_{f_\alpha}\sigma)(\bar\nu,\bar\nu)\right].\end{aligned}$$

Using Lemma 14 and integration by parts, we have

$$\begin{aligned}(\bar D_a G_\lambda)|_\xi&=\frac12\int_{S_{\xi,\lambda}}\left[\bar g(a,\bar\nu)\,\left[\bar D_{\bar\nu}\bar{\operatorname{tr}}\sigma-(\bar{\operatorname{div}}\,\sigma)(\bar\nu)\right]+\sigma(\bar\nu,a)-\bar g(a,\bar\nu)\,\bar{\operatorname{tr}}\sigma\right]\mathrm{d}\bar\mu\\&\quad+o(1)\\&=\frac12\,\lambda^{-1}\int_{S_{\xi,\lambda}}\left[\bar g(a,x-\lambda\,\xi)\,\left[\bar D_{\bar\nu}\bar{\operatorname{tr}}\sigma-(\bar{\operatorname{div}}\,\sigma)(\bar\nu)\right]+\sigma(\bar\nu,a)-\bar g(a,\bar\nu)\,\bar{\operatorname{tr}}\sigma\right]\mathrm{d}\bar\mu\\&\quad+o(1).\end{aligned}$$

Moreover,

$$\begin{aligned}&\bar{\operatorname{div}}\left(\sum_{j=1}^3\left[[\bar D_{e_j}\bar{\operatorname{tr}}\sigma-(\bar{\operatorname{div}}\,\sigma)(e_j)]\,\bar g(a,\lambda^{-1}\,x-\,\xi)\,+\lambda^{-1}\,[\sigma(a,e_j)-\bar g(a,e_j)\,\bar{\operatorname{tr}}\sigma]\right]e_j\right)\\&\quad=-R\,\bar g(a,\lambda^{-1}\,x-\,\xi)+O(|x|_{\bar g}^{-2-2\,\tau}).\end{aligned}$$

Using the divergence theorem and that the scalar curvature is integrable, we find that

$$(11)\quad \begin{aligned}(\bar D_a G_\lambda)|_\xi = \frac{1}{2}\,\bar g(a,\xi)\int_{S_{2\lambda}(0)} &\left[(\bar{\operatorname{div}}\,\sigma)(\bar\nu) - \bar D_{\bar\nu}\operatorname{tr}\sigma\right]\mathrm{d}\bar\mu\\ &+\frac{1}{2}\,\lambda^{-1}\int_{S_{2\lambda}(0)}\left[\bar g(a,x)\left[\bar D_{\bar\nu}\operatorname{tr}\sigma - (\bar{\operatorname{div}}\,\sigma)(\bar\nu)\right] + \sigma(\bar\nu,a) - \bar g(a,\bar\nu)\,\bar{\operatorname{tr}}\sigma\right]\mathrm{d}\bar\mu\\ &+ o(1).\end{aligned}$$

Using that the scalar curvature is integrable again, we have

$$\int_{S_{2\lambda}(0)}\left[(\bar{\operatorname{div}}\,\sigma)(\bar\nu) - \bar D_{\bar\nu}\bar{\operatorname{tr}}\sigma\right]\mathrm{d}\bar\mu = 16\,\pi\,m + o(1)$$

and

$$\lambda^{-1}\int_{S_{2\lambda}(0)}\left[\bar g(a,x)\left[\bar D_{\bar\nu}\bar{\operatorname{tr}}\sigma - (\bar{\operatorname{div}}\,\sigma)(\bar\nu)\right] + \sigma(\bar\nu,a) - \bar g(a,\bar\nu)\,\bar{\operatorname{tr}}\sigma\right]\mathrm{d}\bar\mu = o(1).$$

In fact, if (M,g) satisfies the Regge-Teitelboim conditions, the last integral equals $\bar g(C,a)+o(1)$. □

2.0.2. *Existence of large stable constant mean curvature spheres.*

Proof of Theorem 9. Let $\delta = 1/2$. Lemma 15 implies that, for every $\lambda > 1$ sufficiently large, G_λ is strictly radially increasing on $\{\xi\in\mathbb{R}^3 : |\xi|_{\bar g} = 1/2\}$. In particular, G_λ has a critical point $\xi(\lambda)\in\mathbb{R}^3$ with $|\xi(\lambda)|_{\bar g} < 1/2$. According to Lemma 11, $\Sigma(\lambda) = \Sigma_{\xi(\lambda),\lambda}$ is a constant mean curvature sphere.

By (4), we find that

$$(12)\quad \int_{\Sigma(\lambda)}|\nabla f|^2 - |h|^2 f^2 - Ric(\nu,\nu)\,f^2\mathrm{d}\mu \geq 2\,\lambda^{-2}\int_{\Sigma(\lambda)} f^2\,\mathrm{d}\mu$$

for every $f\in[\Lambda_0(S_{\xi(\lambda),\lambda})\oplus\Lambda_1(S_{\xi(\lambda),\lambda})]^\perp$ provided that $\lambda>1$ is sufficiently large. Using that

$$\bar D^2 G_\lambda = 8\,\pi\,m\ \mathrm{Id} - o(1),$$

we have

$$\int_{\Sigma(\lambda)}|\nabla f|^2 - |h|^2 f^2 - Ric(\nu,\nu)\,f^2\,\mathrm{d}\mu \geq \lambda^{-3}\,[8\,\pi\,m - o(1)]\int_{\Sigma(\lambda)} f^2\,\mathrm{d}\mu$$

for every $f\in\Lambda_1(S_{\xi(\lambda),\lambda})$. In particular, $\Sigma(\lambda)$ is stable.

We have

$$H(\Sigma(\lambda)) = 2\,\lambda^{-1} + o(\lambda^{-3/2}) \qquad\text{and}\qquad H(\Sigma(\lambda))' = -2\,\lambda^{-2} + o(\lambda^{-5/2}).$$

It follows that $\lambda\mapsto H(\Sigma(\lambda))$ is strictly decreasing on (λ_0,∞) provided that $\lambda_0>1$ is sufficiently large. By Lemma 15, $\xi(\lambda) = o(1)$. Moreover, $\bar D G'_\lambda = o(\lambda^{-1})$ as $\lambda\to\infty$ uniformly for all $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g} < 1/2$. Differentiating the equation $(\bar D G_\lambda)|_{\xi(\lambda)} = 0$, we find that

$$\xi'(\lambda) = [(\bar D^2 G_\lambda)|_{\xi(\lambda)}]^{-1}\,(\bar D G'_\lambda)|_{\xi(\lambda)} = o(\lambda^{-1}).$$

Consequently,

$$(\lambda\,y + u_{\xi(\lambda),\lambda}\,y + \lambda\,\xi(\lambda))' = y + o(1),$$

$y\in S_1(0)$. In particular, the family $\{\Sigma(\lambda) : \lambda > \lambda_0\}$ is transversal. □

2.0.3. *Asymptotic positioning.* The geometric center of mass $C_{CMC} = (C^1_{CMC}, C^2_{CMC}, C^3_{CMC})$ of (M,g) is given by

$$(13)\quad C^\ell_{CMC} = \lim_{H\to 0}|\Sigma(H)|^{-1}\int_{\Sigma(H)} x^\ell\,\mathrm{d}\mu$$

provided the limit on the right-hand side exists.

Theorem 16 ([18, Theorem 1])**.** *Suppose that (M,g) is an asymptotically flat Riemannian manifold with positive mass that satisfies the Regge-Teitelboim conditions. Then the limits in* (1) *and* (13) *exist and $C = C_{CMC}$.*

Theorem (16) has been proven under weaker assumptions in [25]. It has been generalized to a spacetime setting in [9]. In [15], we provide a proof based on the identity (11).

3. Lecture 3

Let (M,g) be an asymptotically flat Riemannian manifold with positive mass and non-negative scalar curvature.

Theorem 17 ([15])**.** *There exists $r > 1/2$ with the following property. Every stable constant mean curvature sphere $\Sigma \subset M$ that encloses B_r satisfies $\Sigma = \Sigma(H)$ for some $H \in (0, H_0)$.*

Remark 18. *Theorem 17 shows that quantities associated to the foliation $\{\Sigma(H) : H \in (0,H_0)\}$ are canonical.*

Theorem 17 was proved in [26] if (M,g) is asymptotic to Schwarzschild and in [23] if $\tau = 1$. Previous results have been obtained in [20, 18]. The assumption that Σ encloses B_r cannot be dropped; see [8]. We note that stronger results are available if (M,g) is asymptotic to Schwarzschild and if the scalar curvature satisfies a growth condition; see [7, 11, 10, 6]. If (M,g) is spatial Schwarzschild, all embedded constant mean curvature spheres have been classified in [5]. It has been shown in [12] that the spheres $\Sigma(H)$ bound isoperimetric regions.

3.0.1. *Christodoulou-Yau estimate.*

Proposition 19. *Let $\Sigma \subset M$ be a stable constant mean curvature sphere. There holds*

$$\frac{2}{3}\int_\Sigma |\mathring{h}|^2 \,\mathrm{d}\mu \leq 16\,\pi - \int_\Sigma H^2 \,\mathrm{d}\mu.$$

Proof. By the uniformization theorem, we may choose a conformal diffeomorphism $\psi : \Sigma \to S_1(0)$ with

$$\int_\Sigma \psi \,\mathrm{d}\mu = 0.$$

In particular, there exists $u \in C^\infty(\Sigma)$ with $\bar g(\nabla_{f_\alpha}\psi, \nabla_{f_\beta}\psi) = u^2\,\delta_{\alpha\beta}$ for every local orthonormal frame $\{f_1,\, f_2\}$ of Σ. Note that

$$\int_\Sigma \bar g(\nabla\psi,\nabla\psi)\,\mathrm{d}\mu = 2\int_\Sigma u^2\,\mathrm{d}\mu = 2\int_\Sigma \sqrt{\det{(\nabla\psi)^t\,\nabla\psi}}\,\mathrm{d}\mu = 8\,\pi.$$

Since Σ is stable, we have

$$\int_\Sigma |h|^2 + Ric(\nu,\nu)\,\mathrm{d}\mu = \sum_{i=1}^3 \int_\Sigma |h|^2\,\bar g(\psi,e_i)^2 + Ric(\nu,\nu)\,\bar g(\psi,e_i)^2 \mathrm{d}\mu \leq 8\,\pi.$$

The assertion follows from this, the Gauss equation

$$|h|^2 + Ric(\nu,\nu) = \frac{1}{2}\,|\mathring{h}|^2 + \frac{3}{4}\,H^2 + \frac{1}{2}\,R - K,$$

and the Gauss-Bonnet theorem, using that $R \geq 0$. ∎

3.0.2. *Curvature estimates.* Let $\Sigma\subset M$ be a closed surface. We define the area radius $\lambda(\Sigma)$ of Σ by $4\,\pi\,\lambda(\Sigma)^2=|\Sigma|$ and the inner radius $\rho(\Sigma)$ by

$$\rho(\Sigma)=\sup\{r>1/2:B_r\cap\Sigma\}=\emptyset.$$

Let $\{\Sigma_\ell\}_{\ell=1}^\infty$ be a sequence of stable constant mean curvature spheres $\Sigma_\ell\subset M$ enclosing B_1 with $\rho(\Sigma_\ell)\to\infty$. By Proposition 19, $H=O(\lambda(\Sigma_\ell)^{-1})$.

Lemma 20. [26] *There holds, as $\ell\to\infty$,*

$$|x|_{\bar g}^2\,|\mathring{h}|^2=O(|x|_{\bar g}^{-2\,\tau})+O\bigg(\int_{\Sigma_\ell}|\mathring{h}|^2\,\mathrm{d}\mu\bigg).$$

Proof. We only sketch the argument. By the Simons' identity

$$\Delta h=\nabla^2H+h*h*h+h*Rm+DRm*1=h*h*h+h*Rm+DRm*1.$$

More precisely,

$$-|\mathring{h}|\,\Delta|\mathring{h}|=O(|\mathring{h}|^4)+O(H^2\,|\mathring{h}|^2)+O(|x|_{\bar g}^{-2-\tau}\,|\mathring{h}|^2)+O(|x|_{\bar g}^{-2-2\,\tau}\,|\mathring{h}|).$$

In conjunction with the Michael-Simon-Sobolev inequality,

$$\bigg(\int_{\Sigma_\ell}u^2\,\mathrm{d}\mu\bigg)^{\frac12}=O(1)\int_{\Sigma_\ell}|\nabla u|\,\mathrm{d}\mu+O(1)\int_{\Sigma_\ell}H\,u\,\mathrm{d}\mu$$

where $u\in C^\infty(\Sigma_\ell)$, we obtain

$$\int_{B_{|x|_{\bar g}/4}(x)\cap\Sigma_\ell}|\mathring{h}|^4\,\mathrm{d}\mu\le O(|x|_{\bar g}^{-2})\int_{B_{|x|_{\bar g}/2}(x)\cap\Sigma_\ell}|\mathring{h}|^2\,\mathrm{d}\mu.$$

The assertion now follows from Moser iteration. □

3.0.3. *Hawking mass estimate.* Using the inequality

$$\int_{\Sigma_\ell}H^2\,\mathrm{d}\mu\le 16\,\pi,$$

we see that

$$\int_{\Sigma_\ell}\bar H^2\le 16\,\pi+O(\rho(\Sigma_\ell)^{-\tau}).$$

In particular, $\lambda(\Sigma_\ell)^{-1}\,\Sigma_\ell$ converges to a round sphere in Haussdorff distance. In particular, $\sup_{x\in\Sigma_\ell}|x|_{\bar g}=O(\lambda(\Sigma_\ell))$.

We now prove a refined estimate for the Willmore energy.

Lemma 21 ([11])**.** *There holds*

$$16\,\pi-\int_{\Sigma_\ell}H^2\,\mathrm{d}\mu\le O(\lambda(\Sigma_\ell)^{-1}).\tag{14}$$

Proof. Let $\Sigma_\ell'\subset M$ be the minimizing hull of Σ_ℓ. Note that

$$16\,\pi-\int_{\Sigma_\ell}H^2\,\mathrm{d}\mu\le 16\,\pi-\int_{\Sigma_\ell'}H^2\,\mathrm{d}\mu.\tag{15}$$

Moreover, there holds $\lambda(\Sigma_\ell)=(1+o(1))\,\lambda(\Sigma_\ell')$. By [19],

$$\sqrt{\frac{|\Sigma_\ell'|}{16\,\pi}}\bigg(1-\frac{1}{16\,\pi}\int_{\Sigma_\ell'}H^2\,\mathrm{d}\mu\bigg)\le m.$$

□

Corollary 22. *There holds*

$$|x|_{\bar g}^2\,|\mathring{h}|^2 = O(|x|_{\bar g}^{-2\,\tau}) + O(\lambda(\Sigma_\ell)^{-1}).$$

3.0.4. *Convergence to a coordinate sphere.* Let $x_\ell \in \Sigma_\ell \cap S_{\rho(\Sigma_\ell)}(0)$. Passing to a subsequence, we may assume that there is $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar g} = 1$ and

$$\lim_{\ell\to\infty} |x_\ell|_{\bar g}^{-1}\,x_\ell = -\xi. \tag{16}$$

Lemma 23. *The surfaces $\frac{1}{2}\,H(\Sigma_\ell)\,\Sigma_\ell$ converge to $S_1(\xi)$ in C^1 in $\mathbb{R}^3$.*

Proof. We may assume that $\xi = e_3$. Moreover, we may assume that $|x_\ell|^{-1}\,x_\ell = -e_3$ for every $\ell > 1$.

Let $\gamma_\ell > 0$ be largest such that there is a smooth function $u_\ell : \{y \in \mathbb{R}^2 : |y|_{\bar g} \leq \gamma_\ell\} \to \mathbb{R}$ with

$$\begin{aligned} &\circ \quad |(\bar\nabla u_\ell)(y)| \leq 1,\\ &\circ \quad (y, \rho(\Sigma_\ell) + u_\ell(y)) \in \Sigma_\ell \end{aligned} \tag{17}$$

for all $y \in \mathbb{R}^2$ with $|y|_{\bar g} \leq \gamma_\ell$. Clearly, $(\bar\nabla u_\ell)(0) = 0$. It follows that

$$4\,|(y, \rho(\Sigma_\ell) + u_\ell(y))|_{\bar g} \geq |y|_{\bar g} + \rho(\Sigma_\ell)$$

and

$$|(\bar\nabla^2 u_\ell)(y)|_{\bar g} \leq 8\,|\bar h((y, \rho(\Sigma_\ell) + u_\ell(y)))|_{\bar g}.$$

Moreover,

$$|\bar h|_{\bar g} = \frac{1}{2}\,|H(\Sigma_\ell)| + O(|x|_{\bar g}^{-1-\tau}) + O(|x|_{\bar g}^{-1}\,\lambda(\Sigma_\ell)^{-1/2}) = \frac{1}{2}\,|H(\Sigma_\ell)| + O(|x|_{\bar g}^{-3/2}).$$

Integrating,

$$|(\bar\nabla u_\ell(y)|_{\bar g} \leq 4\,|y|_{\bar g}\,H(\Sigma_\ell) + O(\rho(\Sigma_\ell)^{-1/2}).$$

It follows that $\frac{1}{2}\,H_\ell\,\gamma_\ell \geq \frac{1}{16}$ for all ℓ sufficiently large. The assertion follows. □

3.0.5. *Uniqueness of large stable constant mean curvature spheres.* We need the following decay estimate.

Lemma 24 ([20])**.** *Let $q > 2$. There holds*

$$\rho(\Sigma_\ell)^{q-2} \int_{\Sigma_\ell} |x|_{\bar g}^{-q}\,\mathrm{d}\bar\mu \leq O(1).$$

Proof. This follows from an application of the first variation formula. □

Proof of Theorem 17. Suppose, for a contradiction, that the conclusion of Theorem 17 fails. It follows that there is a sequence $\{\Sigma_\ell\}_{\ell=1}^\infty$ of stable constant mean curvature spheres $\Sigma_\ell \subset \mathbb{R}^3$ enclosing $B_1(0)$ with $\rho(\Sigma_\ell) \to \infty$ and $\Sigma_\ell \neq \Sigma(H)$ for every $H \in (0, H_0)$.

Let $a \in \mathbb{R}^3$ with $|a|_{\bar g} = 1$. Clearly,

$$\int_{\Sigma_\ell} H\,g(a,\nu)\,\mathrm{d}\mu = H(\Sigma_\ell)\,\int_{\Sigma_\ell} g(a,\nu)\,\mathrm{d}\mu.$$

On the one hand, arguing as in Lemma 12, we see that

$$g(a,\nu)\,\mathrm{d}\mu = [\bar g(a,\bar\nu) + \bar g(a,\bar\nu)\,\bar{\operatorname{tr}}\sigma + O(|x|_{\bar g}^{-2\,\tau})]\,\mathrm{d}\bar\mu.$$

Moreover, by the divergence theorem,

$$\int_{\Sigma_\ell} \bar g(a,\bar\nu)\,\mathrm{d}\bar\mu = 0.$$

Using Lemma 24, we obtain

$$H(\Sigma_\ell)\int_{\Sigma_\ell} g(a,\nu)\,d\mu = \frac{1}{2}\int_{\Sigma_\ell} \bar H\,\bar g(a,\bar\nu)\,\bar{\mathrm{tr}}\sigma\,\mathrm{d}\bar\mu + o(1).$$

On the other hand, by the first variation formula, we have

$$\int_{\Sigma_\ell} H\,g(a,\nu)\,\mathrm{d}\mu = \int_{\Sigma_\ell} [\operatorname{div} a - g(D_\nu a,\nu)]\,\mathrm{d}\mu.$$

As in Lemma 12,

$$[\operatorname{div} a - g(D_\nu a,\nu)]\,\mathrm{d}\mu = \frac{1}{2}[\bar D_a \operatorname{tr}\sigma - (\bar D_a\sigma)(\bar\nu,\bar\nu) + O(|x|_{\bar g}^{-1-2\tau})]\,\mathrm{d}\bar\mu.$$

In conjunction with Lemma 24, we find

$$\int_{\Sigma_\ell} [\operatorname{div} a - g(D_\nu a,\nu)]\,\mathrm{d}\mu = \frac{1}{2}\int_{\Sigma_\ell} [\bar D_a \bar{\mathrm{tr}}\sigma - (\bar D_a\sigma)(\bar\nu,\bar\nu)]\,\mathrm{d}\bar\mu + o(1).$$

Using these estimates and integration by parts, we conclude that

$$\begin{aligned} 0 = &\int_{\Sigma_\ell} [\bar D_{\bar\nu}\bar{\mathrm{tr}}\sigma - (\bar{\operatorname{div}}\,\sigma)(\bar\nu)]\,\bar g(a,\bar\nu)\,\mathrm{d}\bar\mu + \frac{1}{2}\,H(\Sigma_\ell)\int_{\Sigma_\ell} [\sigma(a,\bar\nu) - \bar{\mathrm{tr}}\sigma\,\bar g(a,\bar\nu)]\,\mathrm{d}\bar\mu \\ &+ O\Big(\int_{\Sigma_\ell} |\mathring{\bar h}|_{\bar g}\,|\sigma|_{\bar g}\,\mathrm{d}\bar\mu\Big) + o(1). \end{aligned}$$

Note that

$$\int_{\Sigma_\ell} |\mathring{\bar h}|_{\bar g}\,|\sigma|_{\bar g}\,\mathrm{d}\bar\mu = o(1).$$

Let $z_\ell \in \Sigma_\ell$ with $\bar\nu(z_\ell) = -|x_\ell|_{\bar g}^{-1}\,x_\ell$ and

$$\xi_\ell = \frac{1}{2}\,H(\Sigma_\ell)\,z_\ell - \bar\nu(z_\ell).$$

It follows from Lemma 23 that $\xi_\ell \to \xi$. We define the map $E_\ell : \Sigma_\ell \to \mathbb{R}^3$ by

$$E_\ell = \bar\nu(\Sigma_\ell) - \frac{1}{2}\,H(\Sigma_\ell)\,x + \xi_\ell.$$

Using Lemma 23 and the curvature estimates, we have

$$\bar\nabla E_\ell = O(|x|_{\bar g}^{-3/2}) \tag{18}$$

and, consequently, $E_\ell = O(|x|_{\bar g}^{-1/2})$. We obtain

$$\begin{aligned} 0 = &\int_{\Sigma_\ell} [\bar D_{\bar\nu}\operatorname{tr}\sigma - (\bar{\operatorname{div}}\,\sigma)(\bar\nu)]\,\bar g\big(a, \tfrac{1}{2}\,H(\Sigma_\ell)\,x - \xi_\ell\big)\,\mathrm{d}\bar\mu + \frac{1}{2}\,H(\Sigma_\ell)\int_{\Sigma_\ell} [\sigma(a,\bar\nu) - \bar g(a,\bar\nu)\,\bar{\mathrm{tr}}\sigma]\,\mathrm{d}\bar\mu \\ &+ o(1). \end{aligned}$$

As in the proof of Theorem 9, using the divergence theorem and that R is integrable, we find

$$\begin{aligned} 0 = \bar g(a,\xi_\ell) &\int_{S_{H(\Sigma_\ell)^{-1}}(0)} (\bar{\operatorname{div}}\,\sigma)(\bar\nu) - \bar D_{\bar\nu}\bar{\mathrm{tr}}\sigma\,\mathrm{d}\bar\mu \\ &+ \frac{1}{2}\,H(\Sigma_\ell)\int_{S_{H(\Sigma_\ell)^{-1}}(0)} \bar g(a,x)\,\big[\bar D_{\bar\nu}\bar{\mathrm{tr}}\sigma - (\bar{\operatorname{div}}\,\sigma)(\bar\nu)\big] + \sigma(\bar\nu,a) - \bar g(a,\bar\nu)\,\bar{\mathrm{tr}}\sigma\,\mathrm{d}\bar\mu \end{aligned}$$

$$+ o(1)$$

so that

$$0 = 16\,\pi\, m\, \bar g(a,\xi).$$

It follows that $\xi = 0$. By local uniqueness of the implicit function theorem, we have $\Sigma_\ell = \Sigma_{\tilde\xi_\ell,\lambda_\ell}$ for suitable $\tilde\xi_\ell \in \mathbb{R}^3$ and $\lambda_\ell \in \mathbb{R}$ with $\tilde\xi_\ell \to 0$ and $\lambda_\ell \to \infty$. By Lemma 11, we have $\tilde\xi_\ell = \xi(\lambda_\ell)$, a contradiction. ☐

4. Lecture 4

4.1. **Asymptotic foliations by area-constrained Willmore spheres.** Let (M,g) be a Riemannian manifold that is asymptotically flat. Area-constrained Willmore spheres are more sensitive to the local geometry of (M,g). We therefore require stronger decay assumptions on the metric g.

Definition 25. *We say that (M,g) is asymptotic to Schwarzschild with mass $m>0$ if, in the asymptotically flat chart, $g = g_m + \sigma$ where*

$$|\sigma|_{\bar g} + |x|_{\bar g}\,|\bar D\sigma|_{\bar g} + |x|_{\bar g}^2\,|\bar D^2\sigma|_{\bar g} = O(|x|_{\bar g}^{-2}).$$

Theorem 26 ([16])**.** *Let (M,g) be asymptotic to Schwarzschild with mass $m>0$ and non-negative scalar curvature. There exists a family $\{\Sigma(\kappa) : \kappa \in (0,\kappa_0)\}$ of area-constrained Willmore spheres $\Sigma(\kappa) \subset M$ with Lagrange parameter κ that sweeps out the complement of a compact subset of M.*

Remark 27. *The assumption that $R \geq 0$ cannot be dropped. Understanding large area-constrained Willmore spheres in general asymptotically flat manifolds with non-negative scalar curvature appears to be beyond the reach of the methods presented here.*

Theorem 26 has been proved in [22] under stronger decay assumptions on both the metric g and the scalar curvature R.

4.1.1. *Lyapunov-Schmidt reduction.* Recall that, given $\xi \in \mathbb{R}^3$ and $\lambda > 1$,

$$S_{\xi,\lambda} = S_\lambda(\lambda\,\xi) = \{x \in \mathbb{R}^3 : |x - \lambda\,\xi|_{\bar g} = \lambda\}.$$

Moreover, recall that $\Sigma_{\xi,\lambda}(u)$ is the Euclidean graph of $u \in C^\infty(S_{\xi,\lambda})$ over $S_{\xi,\lambda}$.

Proposition 28. *Let $\delta \in (0,1/2)$. There are constants $\lambda_0 > 1$ and $\epsilon > 0$ such that for every $\xi \in \mathbb{R}^3$ with $|\xi|_{\bar g} < 1-\delta$ and $\lambda > \lambda_0$ there exists a function $u_{\xi,\lambda} \in C^\infty(S_{\xi,\lambda})$ such that the following holds. Let $\Sigma_{\xi,\lambda} = \Sigma_{\xi,\lambda}(u_{\xi,\lambda})$. There holds*

$$|\Sigma_{\xi,\lambda}| = 4\,\pi\,\lambda^2.$$

Moreover, $\Sigma_{\xi,\lambda}$ is an area-constrained Willmore sphere if and only if ξ is a critical point of the reduced Willmore energy $G_\lambda : \{\xi \in \mathbb{R}^3 : |\xi|_{\bar g} < 1-\delta\}$ given by

$$G_\lambda(\xi) = \lambda^{-2}\left(\int_\Sigma H^2\,\mathrm{d}\mu - 16\,\pi - 32\,\pi\, m\,\lambda^{-1}\right).$$

4.1.2. *Computing the reduced Willmore energy.* By scaling, we may assume from now on that $m=2$. We use a tilde to indicate that a geometric quantity is computed with respect to the metric $\tilde g = g_2$.

Lemma 29. *There holds*

$$G_\lambda(\xi) = 64\,\pi + \frac{32\,\pi}{1-|\xi|_{\bar g}^2} - 48\,\pi\,|\xi|_{\bar g}^{-1}\log\frac{1+|\xi|_{\bar g}}{1-|\xi|_{\bar g}} - 128\,\pi\log(1-|\xi|_{\bar g}^2) + 2\,\lambda\int_{\mathbb{R}^3\setminus B_\lambda(\lambda\,\xi)} R\, d\bar v + O(\lambda^{-1}).$$

Proof. We only sketch the argument.

In the first step, by an explicit calculation, we estimate

$$\int_{S_{\xi,\lambda}} \tilde{H}^2 \,\mathrm{d}\tilde{\mu}.$$

Second, we estimate

$$\int_{S_{\xi,\lambda}} H^2 \,\mathrm{d}\mu - \int_{S_{\xi,\lambda}} \tilde{H}^2 \,\mathrm{d}\tilde{\mu}.$$

To this end, note that

$$\int_{S_{\xi,\lambda}} H^2 \,\mathrm{d}\mu = 16\,\pi + 2\int_{S_{\xi,\lambda}} |\mathring{h}|^2 \,\mathrm{d}\mu + 2\int_{S_{\xi,\lambda}} (2\,\mathrm{Ric}(\nu,\nu) - R)\,\mathrm{d}\mu. \tag{19}$$

We have

$$\int_{S_{\xi,\lambda}} |\mathring{\tilde{h}}|^2_{\bar{g}} \,\mathrm{d}\tilde{\mu} = 0 \qquad \text{and} \qquad \int_{S_{\xi,\lambda}} |\mathring{h}|^2 \,\mathrm{d}\mu = O(\lambda^{-4}).$$

Next, recall that the Einstein tensor

$$E = \mathrm{Ric} - \frac{1}{2}\,R\,g$$

is divergence free. Let $Z = (1+|x|_{\bar{g}}^{-1})^{-2}\,\lambda^{-1}\,(x-\lambda\,\xi)$ and note that $Z=\tilde{\nu}$ on $S_{\xi,\lambda}$. By the divergence theorem,

$$\int_{S_{\xi,\lambda}} E(Z,\nu)\,\mathrm{d}\mu = -\int_{\mathbb{R}^3\setminus B_\lambda(\lambda\,\xi)} \left[\frac{1}{2}\,g(E,\mathcal{D}Z) - \frac{1}{6}\,(\operatorname{div} Z)\,R\right]\mathrm{d}v + 8\,\pi\,m\,\lambda^{-1}. \tag{20}$$

Here,

$$\mathcal{D}Z = \mathcal{L}_Z g - \frac{1}{3}\,\mathrm{tr}(\mathcal{L}_Z g)\,g$$

is the conformal Killing operator. Using that $\mathcal{D}Z = O(\lambda^{-1}\,|x|_{\bar{g}}^{-1})$ and $\tilde{R}=0$, we see that the relevant contribution of the perturbation σ is given by

$$\frac{1}{6}\int_{\mathbb{R}^3\setminus B_\lambda(\lambda\,\xi)} (\operatorname{div} Z)\,R\,\mathrm{d}v = \frac{1}{2}\,\lambda^{-1}\int_{\mathbb{R}^3\setminus B_\lambda(\lambda\,\xi)} R\,\mathrm{d}\bar{v} + O(\lambda^{-3}).$$

Finally, we estimate

$$\int_{\Sigma_{\xi,\lambda}} H^2 \,\mathrm{d}\mu - \int_{S_{\xi,\lambda}} H^2 \,\mathrm{d}\mu. \tag{21}$$

To this end, let $W = -\Delta H - H\,(|\mathring{h}|^2 + Ric(\nu,\nu))$ and Q be the linearization of W. We compute $\tilde{W}(S_{\xi,\lambda})$ explicitly in terms of spherical harmonics. Using that

$$\tilde{Q}(S_{\xi,\lambda})(u_{\xi,\lambda}) - \tilde{W}(S_{\xi,\lambda}) - 2\,\kappa\,\lambda^{-1} = \tilde{W}(\Sigma_{\xi,\lambda}) - 2\,\kappa\,\lambda^{-1} + O(\lambda^{-5}) = W(\Sigma_{\xi,\lambda}) - \kappa\,H(\Sigma_{\xi,\lambda}) + O(\lambda^{-5}),$$

that $W(\Sigma_{\xi,\lambda}) - \kappa\,H(\Sigma_{\xi,\lambda}) \in \Lambda_1(S_{\xi,\lambda})$, and that

$$\tilde{Q}(S_{\xi,\lambda})(u_{\xi,\lambda}) = -\bar{\Delta}^2\,u_{\xi,\lambda} - 2\,\lambda^{-2}\,\bar{\Delta}u_{\xi,\lambda} + O(\lambda^{-5}),$$

we obtain an expansion for $u_{\xi,\lambda}$ in terms of spherical harmonics. We then estimate (21) using the first and second variation for the Willmore energy. □

Proof of Theorem 26. Note that

$$64\,\pi + \frac{32\,\pi}{1-|\xi|^2_{\bar{g}}} - 48\,\pi\,|\xi|_{\bar{g}}^{-1}\log\frac{1+|\xi|_{\bar{g}}}{1-|\xi|_{\bar{g}}} - 128\,\pi\log(1-|\xi|^2_{\bar{g}}) \to \infty$$

as $|\xi|_{\bar g}\to 1$. Using that $R\geq 0$ and that $R=O(|x|_{\bar g}^{-4})$, we see that $G_\lambda(\xi)>G_\lambda(0)$ for every $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}=1-\delta$ provided that $\delta>0$ is sufficiently small and that $\lambda>1$ is sufficiently large. In particular, G_λ has a critical point for every $\lambda>1$ sufficiently large. □

Remark 30. *We can construct Riemannian manifolds that are asymptotic to Schwarzschild which have local concentrations of negative scalar curvature and such that, given $\delta>0$, for infinitely many values of λ, G_λ has no critical point $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}<1-\delta$.*

4.1.3. *Asymptotic positioning.*

Theorem 31. *Let (M,g) be asymptotic to Schwarzschild with mass $m>0$ and non-negative scalar curvature satisfying*

$$R(x)=R(-x)\qquad \text{and}\qquad \sum_{i=1}^3 x^i\,\partial_i(|x|_{\bar g}^2\,R(x))\leq 0.$$

Then the family $\{\Sigma(\kappa):\kappa\in(0,\kappa_0)\}$ forms a foliation of the complement of a compact subset of M.

Remark 32. *The weakest possible assumption on the scalar curvature that guarantees that the family $\{\Sigma(\kappa):\kappa\in(0,\kappa_0)\}$ forms a foliation is not known.*

Lemma 29 suggests that the asymptotic positioning of the family $\{\Sigma(\kappa):\kappa\in(0,\kappa_0)\}$ is determined by the asymptotic distribution of scalar curvature in a nonlinear way. In the special case where (M,g) is vacuum at infinity, we have the following result.

Theorem 33 ([18, Theorem 1])**.** *Suppose that (M,g) is asymptotic to Schwarzschild with mass $m>0$ and center of mass C and suppose that $R=0$ outside a compact set. Then*

$$\lim_{\kappa\to 0}|\Sigma(\kappa)|^{-1}\int_{\Sigma(\kappa)}x\,\mathrm{d}\mu=C.$$

4.2. **Uniqueness of large area-constrained Willmore spheres.** Let $\{\Sigma_\ell\}_{\ell=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_\ell\subset M$ enclosing B_1 such that $\rho(\Sigma_\ell)\to\infty$. Suppose that

$$\int_{\Sigma_\ell}H^2\,\mathrm{d}\mu\leq 16\,\pi \tag{22}$$

for every ℓ. Note that, equivalently, $m_H(\Sigma_\ell)\geq 0$.

Remark 34. *For ease of exposition, we only consider area-constrained Willmore spheres that enclose B_1. This assumption is not necessary for the following uniqueness results.*

4.2.1. *Curvature estimates.*

Proposition 35. *There holds, uniformly for all $x\in\Sigma_\ell$,*

$$\begin{aligned}|h-\lambda(\Sigma_\ell)^{-1}\,g|_{\Sigma_\ell}|^4=&\,O(|x|_{\bar g}^{-4})\left(\int_{\Sigma_\ell\cap B_{1/4\,|x|_{\bar g}}(x)}|h-\lambda(\Sigma_\ell)^{-1}\,g|_{\Sigma_\ell}|^2\,\mathrm{d}\mu\right)^2\\&+O(|x|_{\bar g}^{-8})+O(\kappa(\Sigma_\ell)^2)\int_{\Sigma_\ell\cap B_{1/4\,|x|_{\bar g}}(x)}|h-\lambda(\Sigma_\ell)^{-1}\,g|_{\Sigma_\ell}|^2\,\mathrm{d}\mu.\end{aligned}$$

Proof. This follows from an adaptation of the integral curvature estimates proved in [21]. □

It follows from (22) that

$$\int_{\Sigma_\ell\cap B_{1/4\,|x|_{\bar g}}(x)}|h-\lambda(\Sigma_\ell)^{-1}\,g|_{\Sigma_\ell}|^2\,\mathrm{d}\mu=O(\lambda(\Sigma_\ell)^{-1}+\rho(\Sigma_\ell)^{-2}). \tag{23}$$

Corollary 36. *There holds*

$$|x|_{\bar g}\,|h-\lambda(\Sigma_\ell)^{-1}\,g|_{\Sigma_\ell}|=O(\lambda(\Sigma_\ell)^{-1/2}+\rho(\Sigma_\ell)^{-1}).$$

4.2.2. *A general convexity criterion.*

Lemma 37. *Let $f\in C^1(\mathbb{R}^3)$ be a non-negative function satisfying*

$$\sum_{i=1}^{3}x^i\,\partial_i(|x|^2_{\bar g}\,f)\le 0. \tag{24}$$

For every $\xi_1,\xi_2\in\mathbb{R}^3$ with $|\xi_1|_{\bar g}$, $|\xi_2|_{\bar g}<1$ and $\lambda>0$ there holds

$$\int_{S_{\xi_1,\lambda}}\bar g(\bar\nu,\xi_2-\xi_1)\,f\,\mathrm{d}\bar\mu\ge\int_{S_{\xi_2,\lambda}}\bar g(\bar\nu,\xi_2-\xi_1)\,f\,\mathrm{d}\bar\mu.$$

Proof. We may assume that $\lambda=1$. Moreover, we may assume that $\xi_2\neq\xi_1$ and that

$$e_3=\frac{\xi_2-\xi_1}{|\xi_2-\xi_1|_{\bar g}}.$$

We define the hemispheres

$$S^\ell_+=\{x\in S_1(\xi_\ell):\bar g(\bar\nu,\xi_2-\xi_1)\ge 0\}\qquad\text{and}\qquad S^\ell_-=\{x\in S_1(\xi_\ell):\bar g(\bar\nu,\xi_2-\xi_1)\le 0\}$$

where $\ell=1,2$. We parametrize S_2^+ via

$$\Psi:(0,\pi)\times(0,2\,\pi)\to S_2^+\qquad\text{given by}\qquad\Psi(\zeta,\varphi)=\xi_2+(\sin\zeta\,\sin\varphi,\sin\zeta\,\cos\varphi,\cos\zeta).$$

and S_1^+ by

$$(0,\pi)\times(0,2\,\pi)\to S_1^+\qquad\text{where}\qquad(\theta,\varphi)\mapsto\xi_1+(\sin\theta\,\sin\varphi,\sin\theta\,\cos\varphi,\cos\theta).$$

Note that, given ζ, there is $\theta=\theta(\zeta)$ with $\theta\le\zeta$ and $t=t(\zeta)>1$ such that

$$t\,[\xi_1+(\sin\theta\,\sin\varphi,\sin\theta\,\cos\varphi,\cos\theta)]=\xi_2+(\sin\zeta\,\sin\varphi,\sin\zeta,\cos\varphi,\cos\zeta). \tag{25}$$

By a direct computation,

$$\dot\theta\,\sin\theta\,\cos\theta\ge t^{-2}\,\sin\zeta\,\cos\zeta.$$

Using that f is non-negative and (24), it follows that

$$\begin{aligned}&\int_{S^1_+}f\,\bar g(\bar\nu,\xi_2-\xi_1)\,\mathrm{d}\bar\mu-\int_{S^2_+}f\,\bar g(\bar\nu,\xi_2-\xi_1)\,\mathrm{d}\bar\mu\\&\ge|\xi_2-\xi_1|_{\bar g}\int_0^{2\,\pi}\int_0^{\pi}\left[t^{-2}\,f(t^{-1}\,\Psi(\zeta,\varphi))-f(\Psi(\zeta,\varphi))\right]\,\sin\zeta\,\cos\zeta\,\mathrm{d}\zeta\,\mathrm{d}\varphi\\&\ge 0.\end{aligned}$$

The same argument shows that

$$\int_{S^1_-}f\,\bar g(\bar\nu,\xi_2-\xi_1)\,\mathrm{d}\bar\mu-\int_{S^2_-}f\,\bar g(\bar\nu,\xi_2-\xi_1)\,\mathrm{d}\bar\mu\ge 0.$$

□

4.2.3. *Local uniqueness.*

Proposition 38. *Let (M,g) be asymptotic to Schwarzschild and suppose that*

$$\sum_{i=1}^{3} x^i\,\partial_i(|x|_{\bar g}^2\,R(x)) \leq 0. \tag{26}$$

Let $\{\Sigma_\ell\}_{\ell=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_\ell \subset M$ enclosing B_1 such that $\rho(\Sigma_\ell)\to\infty$, $m_H(\Sigma_\ell)\geq 0$, and $\Sigma_\ell \neq \Sigma(\kappa)$ for every $\kappa\in(0,\kappa_0)$. Then $\rho(\Sigma_\ell)=o(\lambda(\Sigma_\ell))$.

Proof. Suppose for a contradiction, that, passing to a subsequence, $\lambda(\Sigma_\ell)=O(\rho(\Sigma_\ell))$. By Corollary 36, $\lambda(\Sigma_\ell)^{-1}\,\Sigma_\ell$ converges smoothly to $S_1(\xi)$ for some $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}<1$. In particular, $\Sigma_\ell=\Sigma_{\xi_\ell,\lambda_\ell}$ for suitable $\xi_\ell\in\mathbb{R}^3$ and $\lambda_\ell>1$ with $\lambda_\ell\to\infty$. By Lemma 29 and Lemma 37, G_{λ_ℓ} is strictly convex. In conjunction with Proposition 28, we see that $\Sigma_\ell=\Sigma(\kappa_\ell)$ for suitable $\kappa_\ell\in(0,\kappa)$. □

Remark 39. *Proposition 38 is in general not true without the assumption* (26) *even when $R\geq 0$.*

4.2.4. *Slowly divergent area-constrained Willmore spheres.* We aim to prove the following improvement on Proposition 38.

Theorem 40. *Let (M,g) be asymptotic to Schwarzschild and suppose that*

$$\sum_{i=1}^{3} x^i\,\partial_i(|x|_{\bar g}^2\,R(x)) \leq 0.$$

Let $\{\Sigma_\ell\}_{\ell=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_\ell \subset M$ enclosing B_1 such that $\rho(\Sigma_\ell)\to\infty$, $m_H(\Sigma_\ell)\geq 0$, and $\Sigma_\ell \neq \Sigma(\kappa)$ for every $\kappa\in(0,\kappa_0)$. Then $\rho(\Sigma_\ell)=O(\log\lambda(\Sigma_\ell))$.

Let $\{\Sigma_\ell\}_{\ell=1}^\infty$ be a sequence of area-constrained Willmore spheres $\Sigma_\ell\subset M$ enclosing B_1 such that $\rho(\Sigma_\ell)\to\infty$, $m_H(\Sigma_\ell)\geq 0$, $\rho(\Sigma_\ell)=o(\lambda(\Sigma_\ell))$, and $\log(\lambda(\Sigma_\ell))=o(\rho(\Sigma_\ell))$

Lemma 41. *The surfaces $\lambda(\Sigma_\ell)^{-1}\,\Sigma_\ell$ converge to $S_1(\xi)$ in C^1 in $\mathbb{R}^3$ for some $\xi\in\mathbb{R}^3$ with $|\xi|_{\bar g}=1$.*

It follows that, for every ℓ sufficiently large, Σ_ℓ is the Euclidean graph over a nearby coordinate sphere $S_\ell=S_{\lambda_\ell}(\lambda_\ell\,\xi_\ell)$.

Proposition 42. *There holds, as $\ell\to\infty$,*

$$H(\Sigma_\ell)=(2+o(1))\,\lambda(\Sigma_\ell)^{-1}-4\,\lambda(\Sigma_\ell)^{-1}\,|x|_{\bar g}^{-1}+o(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)^{-1})$$

and

$$\kappa(\Sigma_\ell)=o(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)^{-2}).$$

We need the following lemma.

Lemma 43. *There is a constant $c>0$ with the following property. Let $\xi\in\mathbb{R}^3$ and $\lambda>0$. Suppose that u, $f\in\Lambda_0(S_\lambda(\lambda\,\xi))^\perp$ are such that $\bar\Delta u=f$. Then*

$$\sup_{x\in S_\lambda(\lambda\,\xi)}|x|_{\bar g}\,|\bar\nabla u(x)|_{\bar g}\leq c\left(\int_{S_\lambda(\lambda\,\xi)}|f|\,\mathrm{d}\bar\mu+\sup_{x\in S_\lambda(\lambda\,\xi)}|x|_{\bar g}^2\,|f|\right).$$

Proof of Proposition 42. We only sketch the argument. For ease of exposition, we assume that $\kappa(\Sigma_\ell)=0$.

Recall the potential function $N:\mathbb{R}^3\setminus\{0\}\to\mathbb{R}$ of spatial Schwarzschild given by

$$N(x)=(1+|x|_{\bar g}^{-1})^{-1}\,(1-|x|_{\bar g}^{-1})$$

and that $\tilde{D}^2 N = N\,\tilde{Rc}$. Let $F_\ell = N^{-1}\,H(\Sigma_\ell)$. By a direct computation,

$$\Delta F_\ell = (|\mathring{h}|^2 + \kappa + O(|x|_{\bar g}^{-4}) + O(|x|_{\bar g}^{-2}\,|F_\ell|))\,F_\ell + O(|x|_{\bar g}^{-3})\,|x|_{\bar g}\,|\bar\nabla F_\ell|_{\bar g}.$$

By the curvature estimates, we have $|F_\ell| = O(\lambda(\Sigma_\ell)^{-1}) + O(|x|_{\bar g}^{-1}\,(\lambda(\Sigma_\ell)^{-1/2} + \rho(\Sigma_\ell)^{-1}))$. Moreover, using Lemma 41,

$$F_\ell = \operatorname{proj}_{\Lambda_0(S_\ell)} F_\ell + \operatorname{proj}_{\Lambda_0(S_\ell)^\perp} F_\ell = O(\lambda(\Sigma_\ell)^{-1}) + O(\log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)))\sup_{x\in S_\ell}|x|_{\bar g}\,|\bar\nabla F_\ell|_{\bar g}.$$

Using Lemma 41, we may apply Lemma 43 and (23) to obtain

$$\begin{aligned}\sup_{x\in\Sigma_\ell}|x|_{\bar g}\,|\nabla \bar F|_{\bar g} = O((\lambda(\Sigma_\ell)^{-1/2} + \rho(\Sigma_\ell)^{-1})^2 + \lambda(\Sigma_\ell)^{-1}\,\log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)))\\ (\lambda(\Sigma_\ell)^{-1} + \log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell))\,\sup_{x\in\Sigma_\ell}|x|_{\bar g}\,|\nabla\bar F|_{\bar g})\\ + \rho(\Sigma_\ell)^{-1}\sup_{x\in\Sigma_\ell}|x|_{\bar g}\,|\nabla\bar F|_{\bar g}.\end{aligned}$$

Absorbing, we obtain

$$\sup_{x\in\Sigma_\ell}|x|_{\bar g}\,|\nabla\bar F|_{\bar g} = O((\lambda(\Sigma_\ell)^{-1/2} + \rho(\Sigma_\ell)^{-1})^2 + \lambda(\Sigma_\ell)^{-1}\,\log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)))\,\lambda(\Sigma_\ell)^{-1}.$$

Using that

$$\operatorname{proj}_{\Lambda_0(S_\ell)^\perp} F_\ell = O(\log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)))\sup_{x\in S_\ell}|x|_{\bar g}\,|\bar\nabla F_\ell|_{\bar g}$$

and that, for instance,

$$\log(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell))\,\rho(\Sigma_\ell)^{-2}\,\lambda(\Sigma_\ell)^{-1} = o(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)^{-1}),$$

it follows that $\operatorname{proj}_{\Lambda_0(S_\ell)^\perp} F_\ell = o(\rho(\Sigma_\ell)^{-1}\,\lambda(\Sigma_\ell)^{-1})$ as claimed. □

Proof of Theorem 40. A lengthy computation using Proposition 42, integration by parts, and the divergence theorem shows that

$$\begin{aligned}0 &= \int_{\Sigma_\ell}(-\Delta H - (|\mathring{h}|^2 + Ric(\nu,\nu))\,H)\,g(\xi_\ell,\nu)\,\mathrm{d}\mu - \kappa\int_\Sigma H\,g(\xi_\ell,\nu)\,\mathrm{d}\mu\\ &= 4\,\pi\,\rho(\Sigma_\ell)^{-2}\,\lambda(\Sigma_\ell)^{-1} - \lambda(\Sigma_\ell)^{-1}\int_{\Sigma_\ell}\bar g(\xi_\ell,\nu)\,R\,\mathrm{d}\bar\mu + o(\rho(\Sigma_\ell)^{-2}\,\lambda(\Sigma_\ell)^{-1}).\end{aligned}$$

By Lemma 41, $\bar g(\xi_\ell,\nu)\geq 0$ implies that $|x|_{\bar g}\geq 1/2\,\lambda(\Sigma_\ell)$. In conjunction with the estimates $R\geq 0$, $R = O(|x|_{\bar g}^{-4})$, and $\rho(\Sigma_\ell) = o(\lambda(\Sigma_\ell))$, we conclude that

$$0 = 4\,\pi\,\rho(\Sigma_\ell)^{-2}\,\lambda(\Sigma_\ell)^{-1} - o(\rho(\Sigma_\ell)^{-2}\,\lambda(\Sigma_\ell)^{-1}),$$

a contradiction. □

Conjecture 1. *Let (M,g) be asymptotic to Schwarzschild and suppose that*

$$\sum_{i=1}^3 x^i\,\partial_i(|x|_{\bar g}^2\,R(x))\leq 0.$$

There exist $r>1$ and $A>1$ with the following property. Let $\Sigma\subset M$ be an area-constrained Willmore sphere with non-negative Hawking mass such that $\Sigma\cap B_r=\emptyset$ and $|\Sigma|>A$. Then $\Sigma=\Sigma(\kappa)$ for some $\kappa\in(0,\kappa_0)$.

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