Existence of minimizers for weighted L^p-Hardy inequalities

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Abstract

Let $p \in (1,\infty)$, $\alpha \in \mathbb{R}$, and $\Omega \subsetneq \mathbb{R}^N$ be a $C^{1,\gamma}$-domain with a compact boundary $\partial\Omega$, where $\gamma \in (0,1]$. Denote by $\delta_\Omega(x)$ the distance of a point $x \in \Omega$ to $\partial\Omega$. Let $\widetilde{W}_0^{1,p;\alpha}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_{\widetilde{W}_0^{1,p;\alpha}(\Omega)} := \left(\| \, |\nabla \varphi| \, \|^p_{L^p(\Omega,\delta_\Omega^{-\alpha})} + \|\varphi\|^p_{L^p(\Omega,\delta_\Omega^{-(\alpha+p)})} \right)^{1/p}.$$

We study the following two variational constants: the *weighted Hardy constant*

$$H_{\alpha,p}(\Omega) := \inf \left\{ \int_\Omega |\nabla\varphi|^p \delta_\Omega^{-\alpha} \mathrm{d}x \,\middle|\, \int_\Omega |\varphi|^p \delta_\Omega^{-(\alpha+p)} \mathrm{d}x = 1, \varphi \in \widetilde{W}_0^{1,p;\alpha}(\Omega) \right\},$$

and the *weighted Hardy constant at infinity*

$$\lambda^\infty_{\alpha,p}(\Omega) := \sup_{K \Subset \Omega} \inf_{W_c^{1,p}(\Omega\setminus K)} \left\{ \int_{\Omega\setminus K} |\nabla\varphi|^p \delta_\Omega^{-\alpha} \mathrm{d}x \,\middle|\, \int_{\Omega\setminus K} |\varphi|^p \delta_\Omega^{-(\alpha+p)} \mathrm{d}x = 1 \right\}.$$

We show that $H_{\alpha,p}(\Omega)$ is attained if and only if the spectral gap $\Gamma_{\alpha,p}(\Omega) := \lambda^\infty_{\alpha,p}(\Omega) - H_{\alpha,p}(\Omega)$ is strictly positive. Moreover, we obtain tight decay estimates for the corresponding minimizers.