Random walks on Ramanujan digraphs and complexes

Ori Parzanchevski
Hebrew University of Jerusalem

Analysis and Geometry on Graphs and Manifolds, Potsdam 2017

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.

Ramanujan Graphs

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.
- G is a good expander if all eigenvalues except $\pm k$ are close to zero.

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.
- G is a good expander if all eigenvalues except $\pm k$ are close to zero.
- G is a **Ramanujan graph** if

$$Spec(A) \subseteq \{-k\} \cup \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right] \cup \{k\}.$$

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.
- G is a good expander if all eigenvalues except $\pm k$ are close to zero.
- G is a **Ramanujan graph** if

$$Spec(A) \subseteq \{-k\} \cup \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right] \cup \{k\}.$$

- Optimal?

- Adjacency operator of a graph: $(Af)(v) = \sum_{w \sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.
- G is a good expander if all eigenvalues except $\pm k$ are close to zero.
- G is a **Ramanujan graph** if

$$Spec(A) \subseteq \{-k\} \cup \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right] \cup \{k\}.$$

- Optimal? For $G = \mathbb{P}^2\mathbb{F}_p$ (lines against planes in $\mathbb{F}_p^3$),

$$Spec(A) = \left\{-k, -\sqrt{k-1}, \sqrt{k-1}, k\right\}.$$

- Adjacency operator of a graph: $(Af)(v) = \sum_{w\sim v} f(w)$.
- For a k-regular graph G,

$$Spec(A) = -k \leq \lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = k.$$

- k is an eigenvalue of the constant eigenfunction.
- $-k \in Spec(A)$ iff G is bipartite.
- G is a good expander if all eigenvalues except $\pm k$ are close to zero.
- G is a **Ramanujan graph** if

$$Spec(A) \subseteq \{-k\} \cup \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right] \cup \{k\}.$$

- Optimal? For $G = \mathbb{P}^2\mathbb{F}_p$ (lines against planes in $\mathbb{F}_p^3$),

$$Spec(A) = \left\{-k, -\sqrt{k-1}, \sqrt{k-1}, k\right\}.$$

- Alon-Boppana: For $\varepsilon > 0$, there are only finitely many k-regular graphs such that

$$Spec(A) \subseteq \{-k\} \cup \left[-2\sqrt{k-1} + \varepsilon, 2\sqrt{k-1} - \varepsilon\right] \cup \{k\}.$$

- Why $2\sqrt{k-1}$?

- Why $2\sqrt{k-1}$?
- Every k-regular graph is a quotient of the k-regular tree T_k.

- Why $2\sqrt{k-1}$?
- Every k-regular graph is a quotient of the k-regular tree T_k.
- Kesten ('59):

$$Spec\left(A_{T_k}|_{L^2(V)}\right) = \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right].$$

- Why $2\sqrt{k-1}$?
- Every k-regular graph is a quotient of the k-regular tree T_k.
- Kesten ('59):

$$Spec\left(A_{T_k}|_{L^2(V)}\right) = \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right].$$

- T_k is the Cayley graph of $\mathbf{F}_{k/2}$: the free group on $k/2$ generators.

- Why $2\sqrt{k-1}$?
- Every k-regular graph is a quotient of the k-regular tree T_k.
- Kesten ('59):

$$Spec\left(A_{T_k}|_{L^2(V)}\right) = \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right].$$

- T_k is the Cayley graph of $\mathbf{F}_{k/2}$: the free group on $k/2$ generators.
- Moreover (Kesten): if $G = \langle S \rangle$ and $S = S^{-1}$ then

$$Spec\left(A_{Cay(G,S)}\right) \subseteq \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$$

only if S freely generates G.

- Why $2\sqrt{k-1}$?
- Every k-regular graph is a quotient of the k-regular tree T_k.
- Kesten ('59):

$$Spec\left(A_{T_k}|_{L^2(V)}\right) = \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right].$$

- T_k is the Cayley graph of $\mathbf{F}_{k/2}$: the free group on $k/2$ generators.
- Moreover (Kesten): if $G = \langle S \rangle$ and $S = S^{-1}$ then

$$Spec\left(A_{Cay(G,S)}\right) \subseteq \left[-2\sqrt{k-1}, 2\sqrt{k-1}\right]$$

  only if S freely generates G.
- General Alon-Boppana Theorem (Serre, Grinberg, Grigorchuk-Żuk): if G_n is an infinite family of quotients of $\widetilde{G}$ then

$$\overline{\lim_n}\, \lambda_2(G_n) \geq \lambda_2\left(\widetilde{G}\right).$$

- Existence?

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.

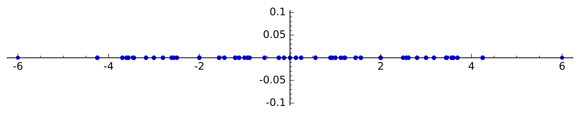
Ramanujan graphs

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.
- LPS: explicit construction as Cayley graphs of $PGL_2(\mathbb{F}_q)$, $PSL_2(\mathbb{F}_q)$.

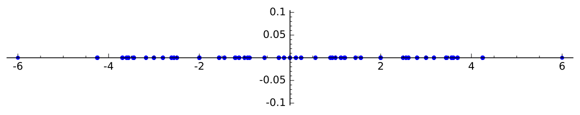
- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.
- LPS: explicit construction as Cayley graphs of $PGL_2(\mathbb{F}_q)$, $PSL_2(\mathbb{F}_q)$.



Adjacency spectrum of $PGL_2(\mathbb{F}_{13})$ with respect to $\left(\begin{smallmatrix} 4 & 0 \\ 0 & 11 \end{smallmatrix}\right)^{\pm 1}, \left(\begin{smallmatrix} 1 & 2 \\ 11 & 1 \end{smallmatrix}\right)^{\pm 1}, \left(\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right)^{\pm 1}$

$$2\sqrt{6-1} \approx 4.4721$$

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.
- LPS: explicit construction as Cayley graphs of $PGL_2(\mathbb{F}_q)$, $PSL_2(\mathbb{F}_q)$.

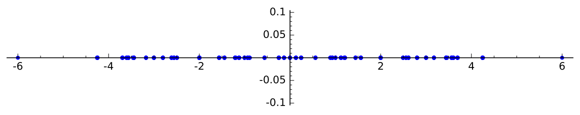


Adjacency spectrum of $PGL_2(\mathbb{F}_{13})$ with respect to $\begin{pmatrix} 4 & 0 \\ 0 & 11 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 2 \\ 11 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^{\pm 1}$

$$2\sqrt{6-1} \approx 4.4721$$

- Morgenstern: $p^k + 1$

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.
- LPS: explicit construction as Cayley graphs of $PGL_2(\mathbb{F}_q)$, $PSL_2(\mathbb{F}_q)$.



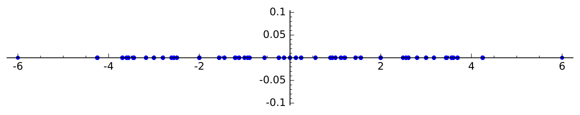
Adjacency spectrum of $PGL_2(\mathbb{F}_{13})$ with respect to $\begin{pmatrix} 4 & 0 \\ 0 & 11 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 2 \\ 11 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^{\pm 1}$

$$2\sqrt{6-1} \approx 4.4721$$

- Morgenstern: $p^k + 1$
- Marcus-Spielman-Srivastava '13: Bipartite Ramanujan graphs for all k.

Ramanujan graphs

- Existence? K_n is Ramanujan, but $k = |Verts| - 1$.
- $\mathbb{P}^2\mathbb{F}_p$ is Ramanujan, but $k \approx \sqrt{|V|}$.
- Lubotzky-Philips-Sarnak, Margulis: for $k = prime + 1$, there exist k-regular Ramanujan graphs with $|V| \to \infty$.
- LPS: explicit construction as Cayley graphs of $PGL_2(\mathbb{F}_q)$, $PSL_2(\mathbb{F}_q)$.



Adjacency spectrum of $PGL_2(\mathbb{F}_{13})$ with respect to $\begin{pmatrix} 4 & 0 \\ 0 & 11 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 2 \\ 11 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}^{\pm 1}$

$$2\sqrt{6-1} \approx 4.4721$$

- Morgenstern: $p^k + 1$
- Marcus-Spielman-Srivastava '13: Bipartite Ramanujan graphs for all k.
- Non-bipartite is still open!

Ramanujan graphs

- Well known: Random graphs are good expanders.

Ramanujan graphs

- Well known: Random graphs are good expanders.
- Alon's conjecture (Friedman's theorem): Random regular graphs are almost Ramanujan

- Well known: Random graphs are good expanders.
- Alon's conjecture (Friedman's theorem): Random regular graphs are almost Ramanujan
  - For any $\varepsilon > 0$, the probability that

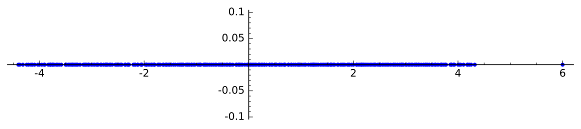
$$Spec\,(A) \subseteq \left[-2\sqrt{k-1} - \varepsilon, 2\sqrt{k-1} + \varepsilon\right] \cup \{k\}$$

goes to 1 as $|V| \to \infty$.

- Well known: Random graphs are good expanders.
- Alon's conjecture (Friedman's theorem): Random regular graphs are almost Ramanujan
  - For any $\varepsilon > 0$, the probability that

$$Spec\,(A) \subseteq \left[-2\sqrt{k-1}-\varepsilon, 2\sqrt{k-1}+\varepsilon\right] \cup \{k\}$$

goes to 1 as $|V| \to \infty$.

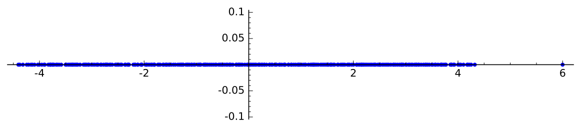


Random regular graph with $k = 6$, $|V| = 300$

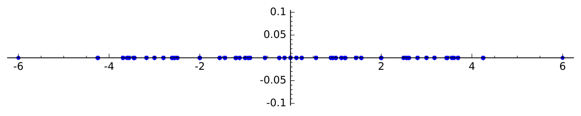
- Well known: Random graphs are good expanders.
- Alon's conjecture (Friedman's theorem): Random regular graphs are almost Ramanujan
  - For any $\varepsilon > 0$, the probability that

$$Spec\,(A) \subseteq \left[-2\sqrt{k-1}-\varepsilon, 2\sqrt{k-1}+\varepsilon\right] \cup \{k\}$$

goes to 1 as $|V| \to \infty$.



Random regular graph with $k = 6$, $|V| = 300$



Digraphs

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.

Digraphs

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.

Digraphs

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.
- $-k \in Spec\,(A)$ if G is bipartite

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.
- $-k \in Spec\,(A)$ if G is bipartite
- $e^{\frac{2\pi i}{3}} k \in Spec\,(A)$ if G is 3-periodic

Digraphs

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.
- $-k \in Spec\,(A)$ if G is bipartite
- $e^{\frac{2\pi i}{3}} k \in Spec\,(A)$ if G is 3-periodic, ...
- We call all of these trivial.

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.
- $-k \in Spec\,(A)$ if G is bipartite
- $e^{\frac{2\pi i}{3}} k \in Spec\,(A)$ if G is 3-periodic, ...
- We call all of these trivial.
- A might be non-diagonalizable, or not normal (=unitarily diagonalizable).

- For a digraph G, $Spec\,(A_G) \subseteq \mathbb{C}$ in general.
- For a k-regular digraph, $k \in Spec\,(A)$.
- $-k \in Spec\,(A)$ if G is bipartite
- $e^{\frac{2\pi i}{3}} k \in Spec\,(A)$ if G is 3-periodic, ...
- We call all of these trivial.
- A might be non-diagonalizable, or not normal (=unitarily diagonalizable). Arguments using inner product break down.

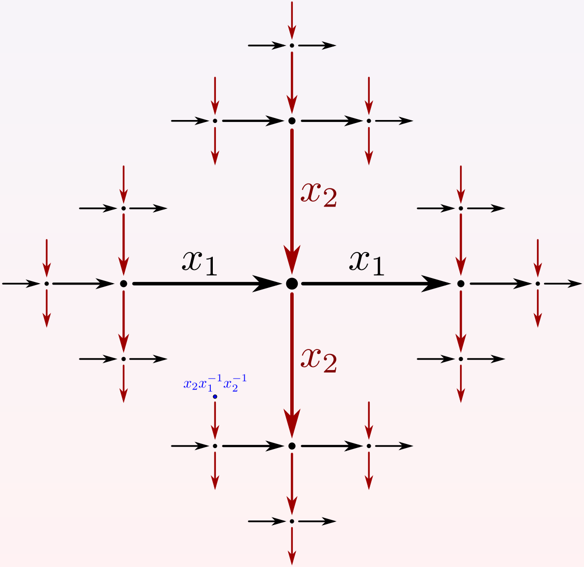
What is the universal object?

What is the universal object?

(a) $Cay\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$

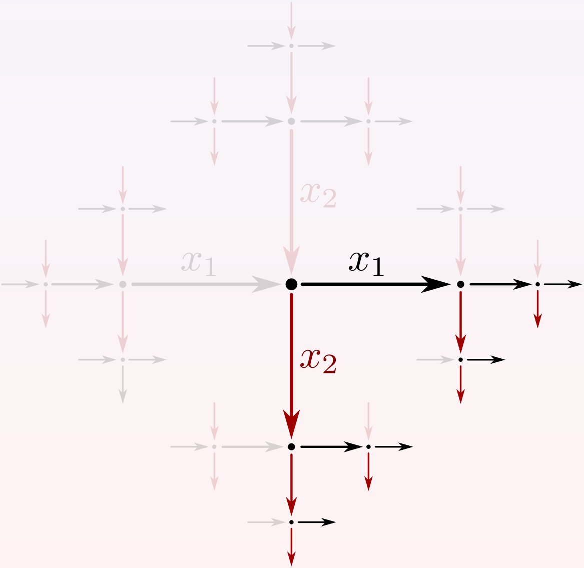
What is the universal object?

(a) $\mathit{Cay}\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$



(b) $Cay\left(\mathbf{FSG}_k, \{x_1, \ldots, x_k\}\right)$ (Free semigroup - only positive letters)

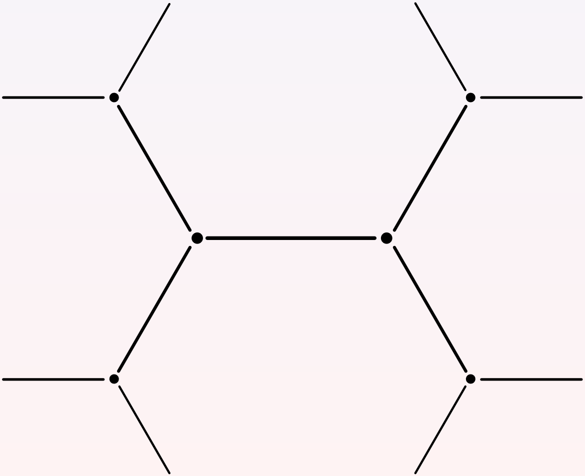
(b) $Cay\,(\mathbf{FSG}_k, \{x_1, \ldots, x_k\})$ (Free semigroup - only positive letters)



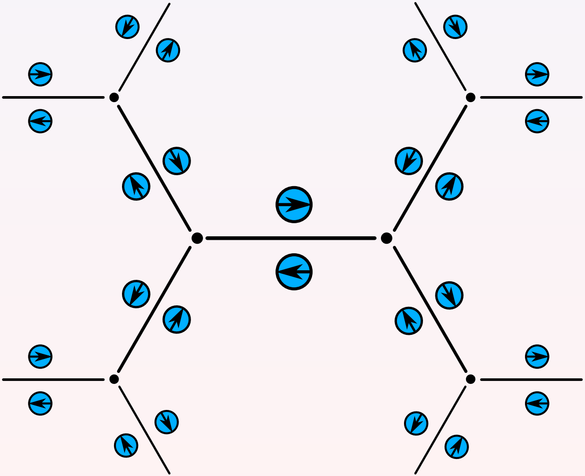
(c) Line-digraph of the $(k+1)$-regular tree:

(c) Line-digraph of the $(k+1)$-regular tree:
Vertices: **directed edges** of T_k
Edges: $e_1 \to e_2$ iff e_1, e_2 is a non-backtracking path

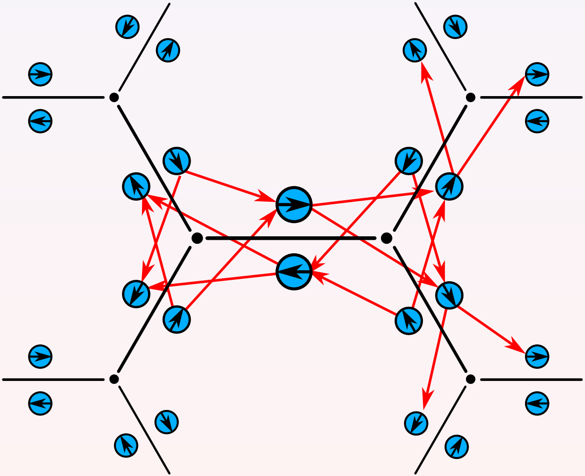
(c) Line-digraph of the $(k+1)$-regular tree:
Vertices: **directed edges** of T_k
Edges: $e_1 \to e_2$ iff e_1, e_2 is a non-backtracking path



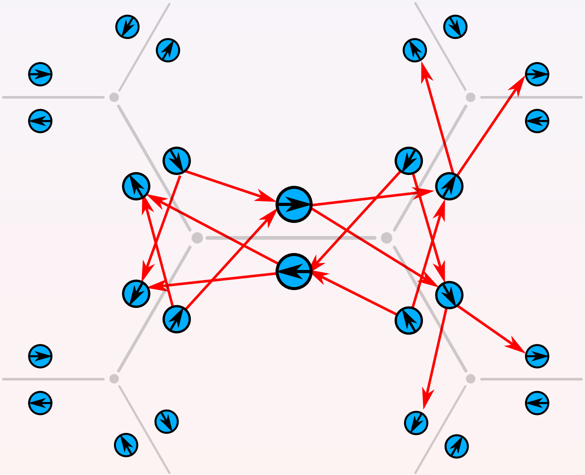
(c) Line-digraph of the $(k+1)$-regular tree:
Vertices: **directed edges** of T_k
Edges: $e_1 \to e_2$ iff e_1, e_2 is a non-backtracking path



(c) Line-digraph of the $(k+1)$-regular tree:
Vertices: **directed edges** of T_k
Edges: $e_1 \to e_2$ iff e_1, e_2 is a non-backtracking path



(c) Line-digraph of the $(k+1)$-regular tree:
Vertices: **directed edges** of T_k
Edges: $e_1 \to e_2$ iff e_1, e_2 is a non-backtracking path



Ramanujan digraphs

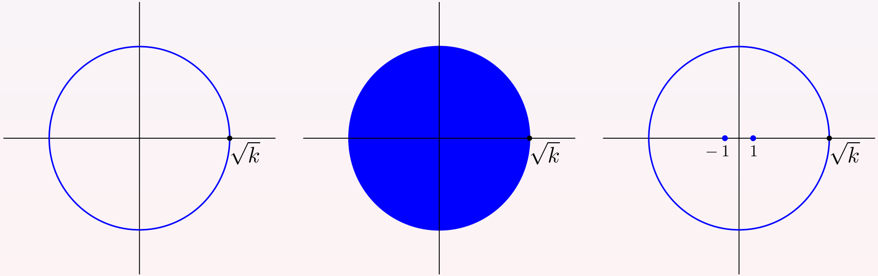
Candidates:

- $\mathit{Cay}\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$
- $\mathit{Cay}\left(\mathbf{FSG}_k, \{x_1, \ldots, x_k\}\right)$ (Free semigroup)
- $\mathrm{LDG}\left(T_{k+1}\right)$ (Line-digraph)

Candidates:

- $Cay\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$
- $Cay\left(\mathbf{FSG}_k, \{x_1, \ldots, x_k\}\right)$ (Free semigroup)
- $\mathrm{LDG}\left(T_{k+1}\right)$ (Line-digraph)

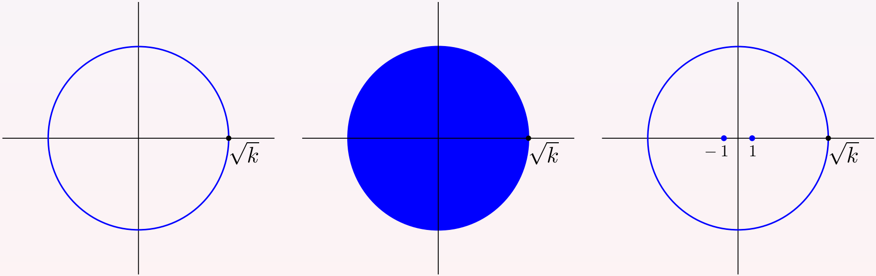
Spectra (I think):



Candidates:

- $Cay\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$
- $Cay\left(\mathbf{FSG}_k, \{x_1, \ldots, x_k\}\right)$ (Free semigroup)
- $\mathrm{LDG}\left(T_{k+1}\right)$ (Line-digraph)

Spectra (I think):

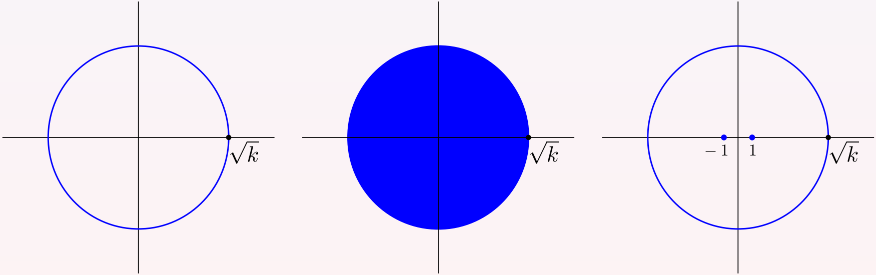


- The spectrum of $\mathrm{LDG}\left(G\right)$ is the spectrum of nonbacktracking random walk on G.

Candidates:

- $Cay\left(\mathbf{F}_k, \{x_1, \ldots, x_k\}\right)$
- $Cay\left(\mathbf{FSG}_k, \{x_1, \ldots, x_k\}\right)$ (Free semigroup)
- $\mathrm{LDG}\left(T_{k+1}\right)$ (Line-digraph)

Spectra (I think):



- The spectrum of $\mathrm{LDG}\left(G\right)$ is the spectrum of nonbacktracking random walk on G.
- $\pm 1 \in Spec\left(\mathrm{LDG}\left(T_{k+1}\right)\right)$ come from paths from ∞ to ∞.

- Definition: a k-regular digraph is a Ramanujan digraph if

$$Spec\,(A) \subseteq \left\{ z \in \mathbb{C} \;\middle|\; |z| \leq \sqrt{k} \text{ or } |z| = k \right\}.$$

- Definition: a k-regular digraph is a Ramanujan digraph if

$$Spec\left(A\right) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} \text{ or } |z| = k \right\}.$$

Example:

- Definition: a k-regular digraph is a Ramanujan digraph if

$$Spec\left(A\right) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} \text{ or } |z| = k \right\}.$$

Example:

- Theorem (Ihara-Hashimoto): for a $(k+1)$-regular graph,

$$Spec\left(A_{\mathrm{LDG}(G)}\right) = \{\pm 1\}^{\beta_1} \cup \left\{ \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \,\middle|\, \lambda \in Spec\left(A_G\right) \right\}$$

- Definition: a k-regular digraph is a Ramanujan digraph if

$$Spec\left(A\right) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} \text{ or } |z| = k \right\}.$$

Example:

- Theorem (Ihara-Hashimoto): for a $(k+1)$-regular graph,

$$Spec\left(A_{\mathrm{LDG}(G)}\right) = \{\pm 1\}^{\beta_1} \cup \left\{ \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \,\middle|\, \lambda \in Spec\left(A_G\right) \right\}$$

- Exercise: $\lambda \in \left[-2\sqrt{k}, 2\sqrt{k}\right] \Leftrightarrow \left|\frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2}\right| \leq \sqrt{k}$.

- Definition: a k-regular digraph is a Ramanujan digraph if

$$Spec\left(A\right) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} \text{ or } |z| = k \right\}.$$

Example:

- Theorem (Ihara-Hashimoto): for a $(k+1)$-regular graph,

$$Spec\left(A_{\mathrm{LDG}(G)}\right) = \{\pm 1\}^{\beta_1} \cup \left\{ \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \,\middle|\, \lambda \in Spec\left(A_G\right) \right\}$$

- Exercise: $\lambda \in \left[-2\sqrt{k}, 2\sqrt{k}\right] \Leftrightarrow \left|\frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2}\right| \leq \sqrt{k}$.
- So, G is $(k+1)$-Ramanujan graph $\Leftrightarrow \mathrm{LDG}\left(G\right)$ is a k-Ramanujan digraph.

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0\\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}^{\pm 1}$$

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0 \\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i \\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2 \\ -2 & 1\end{pmatrix}^{\pm 1}$$

generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^{\pm 1}$$

generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$, which acts simply transitively on $Verts\,(T_6)$.

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0\\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}^{\pm 1}$$

  generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$, which acts simply transitively on $Verts\,(T_6)$.
- (T_6 is the the symmetric space of the p-adic Lie group $U_2\,(\mathbb{Q}_5)$).

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0\\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}^{\pm 1}$$

  generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$, which acts simply transitively on $Verts\,(T_6)$.
- (T_6 is the the symmetric space of the p-adic Lie group $U_2\left(\mathbb{Q}_5\right)$).
- From deep arithmetic theorems (Ramanujan, Eichler, Igusa, Ihara, Jacquet-Langlands, Deligne...) it follows that they generate finite graphs with the same spectrum when projected modulo q.

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0\\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}^{\pm 1}$$

  generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$, which acts simply transitively on $Verts\,(T_6)$.
- (T_6 is the the symmetric space of the p-adic Lie group $U_2\,(\mathbb{Q}_5)$).
- From deep arithmetic theorems (Ramanujan, Eichler, Igusa, Ihara, Jacquet-Langlands, Deligne...) it follows that they generate finite graphs with the same spectrum when projected modulo q. E.g.:

$$\left(\begin{smallmatrix}1+2i & \\ & 1-2i\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}1+2\cdot 8 & \\ & 1-2\cdot 8\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}4 & \\ & 11\end{smallmatrix}\right) \pmod{13}.$$

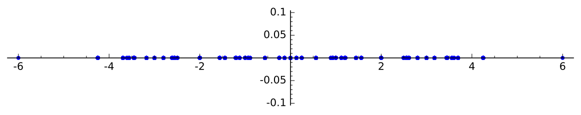
Ramanujan Cayley graphs

- Lubotzky-Philips-Sarnak: The matrices

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1+2i & 0\\ 0 & 1-2i\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2i\\ 2i & 1\end{pmatrix}^{\pm 1}, \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}^{\pm 1}$$

  generate a free group $\Gamma \leq U\left(\mathbb{Z}\left[\frac{1}{\sqrt{5}}\right]\right)$, which acts simply transitively on $Verts\,(T_6)$.
- (T_6 is the the symmetric space of the p-adic Lie group $U_2\,(\mathbb{Q}_5)$).
- From deep arithmetic theorems (Ramanujan, Eichler, Igusa, Ihara, Jacquet-Langlands, Deligne...) it follows that they generate finite graphs with the same spectrum when projected modulo q. E.g.:

$$\left(\begin{smallmatrix}1+2i & \\ & 1-2i\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}1+2\cdot 8 & \\ & 1-2\cdot 8\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}4 & \\ & 11\end{smallmatrix}\right) \pmod{13}.$$



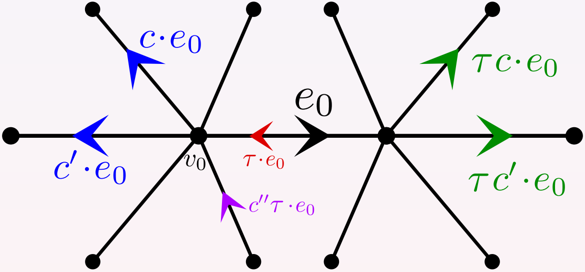
(Adjacency spectrum of $PGL_2\,(\mathbb{F}_{13})$ with respect to $\left(\begin{smallmatrix}4 & 0\\ 0 & 11\end{smallmatrix}\right)^{\pm 1}, \left(\begin{smallmatrix}1 & 2\\ 11 & 1\end{smallmatrix}\right)^{\pm 1}, \left(\begin{smallmatrix}1 & 3\\ 3 & 1\end{smallmatrix}\right)^{\pm 1}$)

- P-Sarnak: Construction of $C \leq U(2), \tau \in U(2)$, acting on T_k, so that

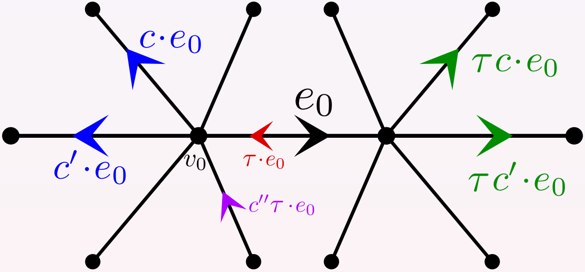
- P-Sarnak: Construction of $C \leq U(2), \tau \in U(2)$, acting on T_k, so that
  - C fixes $v_0 \in V(T_k)$ and acts simply-transitively on its neighbors.

- P-Sarnak: Construction of $C \leq U(2), \tau \in U(2)$, acting on T_k, so that
  - C fixes $v_0 \in V(T_k)$ and acts simply-transitively on its neighbors.
  - τ is an involution which flips an edge e_0 touching the origin.

- P-Sarnak: Construction of $C \leq U(2), \tau \in U(2)$, acting on T_k, so that
  - C fixes $v_0 \in V(T_k)$ and acts simply-transitively on its neighbors.
  - τ is an involution which flips an edge e_0 touching the origin.

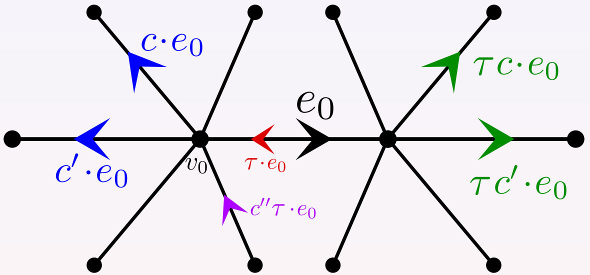


- P-Sarnak: Construction of $C \leq U(2), \tau \in U(2)$, acting on T_k, so that
  - C fixes $v_0 \in V(T_k)$ and acts simply-transitively on its neighbors.
  - τ is an involution which flips an edge e_0 touching the origin.

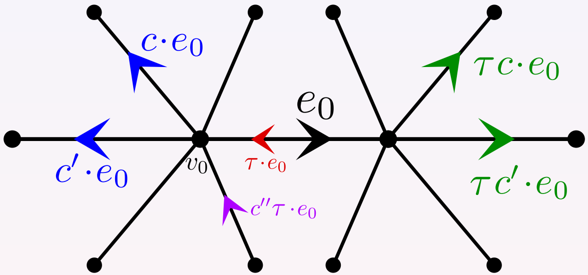


- It follows that $\Gamma = \langle C, \tau \rangle$ acts simply-transitively on the directed edges of the tree.
- E.g.:

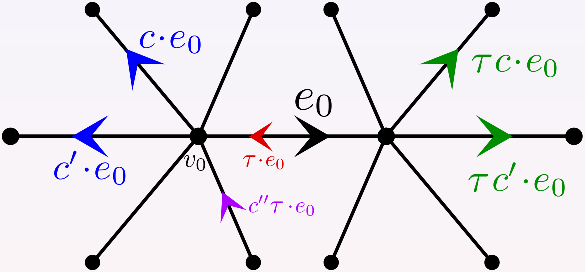
$$\Gamma = \Bigg\langle \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{C}, \underbrace{\tfrac{1}{\sqrt{3}} \left(\begin{smallmatrix} 1 & 1-i \\ 1+i & -1 \end{smallmatrix}\right)}_{\tau} \Bigg\rangle \curvearrowright Edges^{\pm}(T_4).$$



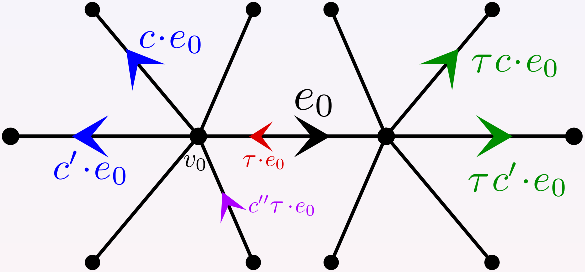
$c{\cdot}e_0$
$\tau c{\cdot}e_0$
e_0
$c'\cdot e_0$
v_0
$\tau\cdot e_0$
$\tau c'\cdot e_0$
$c''\tau\cdot e_0$



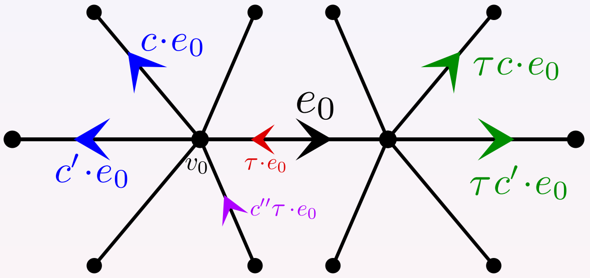
- Observe $S = \{\tau c \mid 1 \neq c \in C\} \quad (|S| = k - 1)$.



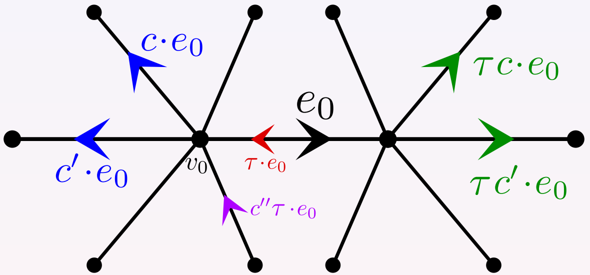
- Observe $S = \{\tau c \mid 1 \neq c \in C\}$ $\quad (|S| = k - 1)$.
- $S \cdot \ldots \cdot S \cdot e_0$ - non-backtracking random walk starting from e_0.



- Observe $S = \{\tau c \,|\, 1 \neq c \in C\}$ $\quad (|S| = k - 1)$.
- $S \cdot \ldots \cdot S \cdot e_0$ - non-backtracking random walk starting from e_0.
- Random walk with S as generators corresponds to NBRW on the tree!
- (Corollary: S generates a free semigroup.)

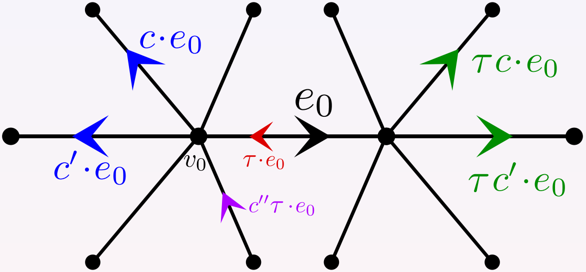


- Observe $S = \{\tau c \mid 1 \neq c \in C\}$ $\quad(|S| = k - 1)$.
- $S \cdot \ldots \cdot S \cdot e_0$ - non-backtracking random walk starting from e_0.
- Random walk with S as generators corresponds to NBRW on the tree!
- (Corollary: S generates a free semigroup.)
- The Ramanujan conjectures (Deligne's theorem) imply that if we project S modulo q we get a finite graph with the same spectrum as before



- Observe $S = \{\tau c \mid 1 \neq c \in C\}$ $(|S| = k - 1)$.
- $S \cdot \ldots \cdot S \cdot e_0$ - non-backtracking random walk starting from e_0.
- Random walk with S as generators corresponds to NBRW on the tree!
- (Corollary: S generates a free semigroup.)
- The Ramanujan conjectures (Deligne's theorem) imply that if we project S modulo q we get a finite graph with the same spectrum as before:

$$\mathrm{Spec}\left(Cay\left(\langle S \pmod q\rangle, S \pmod q\right)\right) \subseteq \mathrm{Spec}\left(\mathrm{LDG}\left(T_{k+1}\right)\right) \cup \{\pm k\},$$



- Observe $S = \{\tau c \,|\, 1 \neq c \in C\}$ $\quad (|S| = k - 1)$.
- $S \cdot \ldots \cdot S \cdot e_0$ - non-backtracking random walk starting from e_0.
- Random walk with S as generators corresponds to NBRW on the tree!
- (Corollary: S generates a free semigroup.)
- The Ramanujan conjectures (Deligne's theorem) imply that if we project S modulo q we get a finite graph with the same spectrum as before:

$$\mathrm{Spec}\left(Cay\left(\langle S \pmod q\rangle, S \pmod q\right)\right) \subseteq \mathrm{Spec}\left(\mathrm{LDG}\left(T_{k+1}\right)\right) \cup \{\pm k\},$$

so we get a Ramanujan Cayley digraph.

Ramanujan Cayley digraphs

Example: Using $\Bigg\langle \underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}}_{C}, \underbrace{\tfrac{1}{\sqrt{3}}\left(\begin{smallmatrix}1 & 1-i\\ 1+i & -1\end{smallmatrix}\right)}_{\tau}\Bigg\rangle$, Example: Using $\Big\langle \underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}}_{C}, \underbrace{\tfrac{1}{\sqrt{3}}\left(\begin{smallmatrix}1 & 1-i\\ 1+i & -1\end{smallmatrix}\right)}_{\tau} \Big\rangle$,

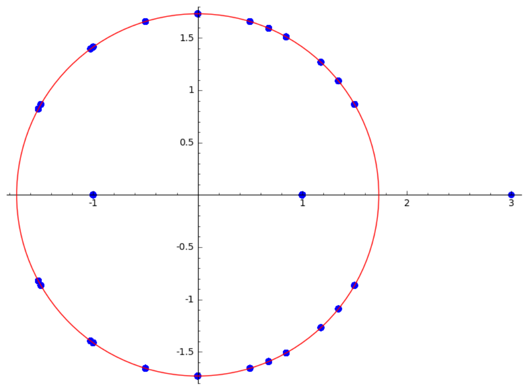
$$S = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & i-1 \\ i+1 & -1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} i-1 & -I \\ -I & -i-1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -i-1 & 1 \\ -1 & i-1 \end{pmatrix} \right\}$$

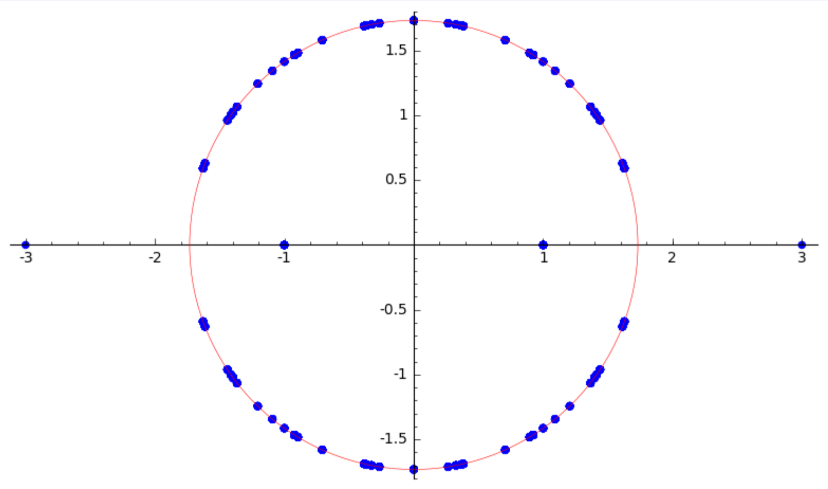
Example: Using
$$\left\langle \underbrace{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}}_{C}, \underbrace{\tfrac{1}{\sqrt{3}}\left(\begin{smallmatrix}1 & 1-i\\ 1+i & -1\end{smallmatrix}\right)}_{\tau} \right\rangle,$$

$$S = \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & i-1\\ i+1 & -1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} i-1 & -I\\ -I & -i-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -i-1 & 1\\ -1 & i-1\end{pmatrix} \right\}$$

and projecting S modulo 13 we obtain

Adjacency spectrum of $PSL_2(\mathbb{F}_{13})$ with respect to $\left(\begin{smallmatrix}12 & 9\\ 7 & 12\end{smallmatrix}\right), \left(\begin{smallmatrix}6 & 8\\ 8 & 9\end{smallmatrix}\right), \left(\begin{smallmatrix}4 & 12\\ 1 & 7\end{smallmatrix}\right)$





Adjacency spectrum of $PGL_2(\mathbb{F}_{17})$ with respect to $\left(\begin{smallmatrix}16 & 14\\ 12 & 16\end{smallmatrix}\right), \left(\begin{smallmatrix}5 & 13\\ 13 & 14\end{smallmatrix}\right), \left(\begin{smallmatrix}3 & 16\\ 1 & 12\end{smallmatrix}\right)$

- Recall T_{p+1} is a symmetric space for $U_2(\mathbb{Q}_p)$ (and $GL_2(\mathbb{Q}_p)$).

- Recall T_{p+1} is a symmetric space for $U_2(\mathbb{Q}_p)$ (and $GL_2(\mathbb{Q}_p)$).
- Bruhat-Tits buildings are infinite, contractible simplicial complexes, which are symmetric spaces for higher dimensional p-adic Lie groups.

- Recall T_{p+1} is a symmetric space for $U_2(\mathbb{Q}_p)$ (and $GL_2(\mathbb{Q}_p)$).
- Bruhat-Tits buildings are infinite, contractible simplicial complexes, which are symmetric spaces for higher dimensional p-adic Lie groups.
- Ramanujan graphs are graphs with the same spectrum of their covering tree.

- Recall T_{p+1} is a symmetric space for $U_2(\mathbb{Q}_p)$ (and $GL_2(\mathbb{Q}_p)$).
- Bruhat-Tits buildings are infinite, contractible simplicial complexes, which are symmetric spaces for higher dimensional p-adic Lie groups.
- Ramanujan graphs are graphs with the same spectrum of their covering tree.
- Ramanujan complexes are complexes with the "same spectrum" as their covering building.

- Recall T_{p+1} is a symmetric space for $U_2(\mathbb{Q}_p)$ (and $GL_2(\mathbb{Q}_p)$).
- Bruhat-Tits buildings are infinite, contractible simplicial complexes, which are symmetric spaces for higher dimensional p-adic Lie groups.
- Ramanujan graphs are graphs with the same spectrum of their covering tree.
- Ramanujan complexes are complexes with the "same spectrum" as their covering building.
- Defined and constructed by Ballantine, Cartwright-Steger-Żuk, Li, Lubotzky-Samuels-Vishne, Sarveniazi, ...

Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs:

Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs: NBRW on a Ramanujan graph covers almost all vertices after $(1+\varepsilon)\log_{k-1}|V|$ steps.

Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs: NBRW on a Ramanujan graph covers almost all vertices after $(1+\varepsilon)\log_{k-1}|V|$ steps.
- Main ingredient: the NBRW operator is almost normal: unitarily equivalent to 2×2 block-diagonal matrix.

Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs: NBRW on a Ramanujan graph covers almost all vertices after $(1+\varepsilon)\log_{k-1}|V|$ steps.
- Main ingredient: the NBRW operator is almost normal: unitarily equivalent to 2×2 block-diagonal matrix.
- We say that if G is a $(k+1)$-Ramanujan graph, then $Line(G)$ is a k-regular 2-normal Ramanujan digraph.

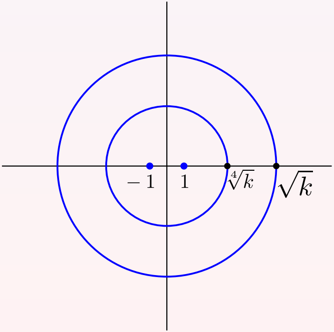
Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs: NBRW on a Ramanujan graph covers almost all vertices after $(1+\varepsilon)\log_{k-1}|V|$ steps.
- Main ingredient: the NBRW operator is almost normal: unitarily equivalent to 2×2 block-diagonal matrix.
- We say that if G is a $(k+1)$-Ramanujan graph, then $Line(G)$ is a k-regular 2-normal Ramanujan digraph.
- [Lubetzky-Lubotzky-P, Kamber]: The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.

Random walks

- Lubetzky-Peres '15: sharp cutoff on Ramanujan graphs: NBRW on a Ramanujan graph covers almost all vertices after $(1+\varepsilon)\log_{k-1}|V|$ steps.
- Main ingredient: the NBRW operator is almost normal: unitarily equivalent to 2×2 block-diagonal matrix.
- We say that if G is a $(k+1)$-Ramanujan graph, then $Line(G)$ is a k-regular 2-normal Ramanujan digraph.
- [Lubetzky-Lubotzky-P, Kamber]: The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- It has more eigenvalues:

Spectrum of 1-geodesic flow on 2-dimensional building / Ramanujan complex



- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.

Ramanujan digraphs

- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- The d-geodesic flow on a Ramanujan complex is a $(d+1)\,!$-normal Ramanujan digraph.

Ramanujan digraphs

- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- The d-geodesic flow on a Ramanujan complex is a $(d+1)\,!$-normal Ramanujan digraph.
- Corollaries: sharp cutoff, optimal diameter, Riemann Hypothesis.

Ramanujan digraphs

- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- The d-geodesic flow on a Ramanujan complex is a $(d+1)\,!$-normal Ramanujan digraph.
- Corollaries: sharp cutoff, optimal diameter, Riemann Hypothesis.
- Theorem (Lubetzky-Lubotzky-P): Any collision-free operator on the Bruhat-Tits building of a semisimple p-adic Lie group induces an r-normal Ramanujan digraph on Ramanujan quotients.

Ramanujan digraphs

- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- The d-geodesic flow on a Ramanujan complex is a $(d+1)\,!$-normal Ramanujan digraph.
- Corollaries: sharp cutoff, optimal diameter, Riemann Hypothesis.
- Theorem (Lubetzky-Lubotzky-P): Any collision-free operator on the Bruhat-Tits building of a semisimple p-adic Lie group induces an r-normal Ramanujan digraph on Ramanujan quotients.
- Collision-free: there is at most one directed path from x to y.

Ramanujan digraphs

- The 1-geodesic flow on a Ramanujan complex of dimension d is a $(d+1)$-normal Ramanujan digraph.
- The d-geodesic flow on a Ramanujan complex is a $(d+1)\,!$-normal Ramanujan digraph.
- Corollaries: sharp cutoff, optimal diameter, Riemann Hypothesis.
- Theorem (Lubetzky-Lubotzky-P): Any collision-free operator on the Bruhat-Tits building of a semisimple p-adic Lie group induces an r-normal Ramanujan digraph on Ramanujan quotients.
- Collision-free: there is at most one directed path from x to y.
- E.g.: NBRW on a tree.

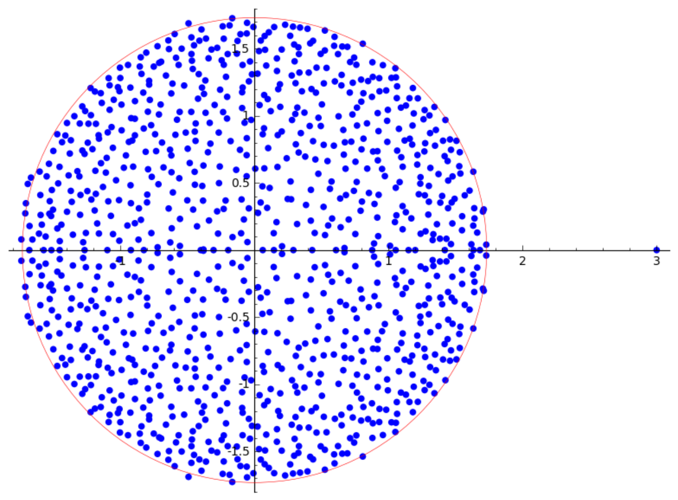
Random digraphs

Random digraphs

- How does the spectrum of a random regular digraph looks like?

Random digraphs

- How does the spectrum of a random regular digraph looks like?



- Conjecture: For any $\varepsilon > 0$, the probability that

$$Spec\,(A) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} + \varepsilon \text{ or } z = k \right\}$$

goes to 1 as $|V| \to \infty$.

Random digraphs

- Conjecture: For any $\varepsilon > 0$, the probability that

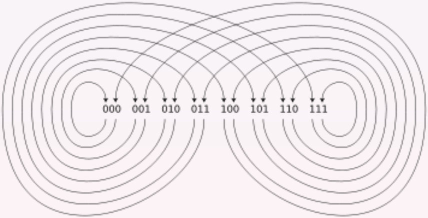
$$Spec\,(A) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{k} + \varepsilon \text{ or } z = k \right\}$$

goes to 1 as $|V| \to \infty$.

- We can show (P-Puder):

$$Prob\left[Spec\,(A) \subseteq \left\{ z \in \mathbb{C} \,\middle|\, |z| \leq \sqrt{2k} \text{ or } z = k \right\} \right] \xrightarrow{|V| \to \infty} 1.$$

- Alon-Boppana theorem?
- DeBruijn digraphs: have $Spec\,(A) = \{0, k\}$ (and arbitrarily large $|V|$).



- Add more assumptions - r-normality, convergence, ...

Thank you!