Riesz transform without Gaussian heat kernel bound

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joint work with T. Coulhon, J. Feneuil and E. Russ

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$$e^{-t\Delta} f(x) = \int_M h_t(x, y) f(y) d\mu(y), \quad \forall f \in L^2(M), a.e.\, x \in M.$$

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Let $B(x,r)$ open ball with center $x \in M$ and radius $r > 0$. Denote $V(x,r) := \mu(B(x,r))$. M satisfies the **doubling volume property** if

$$V(x,2r) \leq CV(x,r), \quad \forall r > 0, x \in M. \tag{D}$$

Background of Riesz transform on Riemannian manifolds

Strichartz (1983): For which kind of non-compact Riemannian manifold M and for which $p \in (1, \infty)$, there holds $\| |\nabla f| \|_p \simeq \|\Delta^{1/2} f\|_p$?

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The Riesz transform (formally $\nabla \Delta^{-1/2}$) is L^p bounded on M if

$$\| |\nabla f| \|_p \leq C \|\Delta^{1/2} f\|_p, \quad \forall f \in \mathcal{C}_0^\infty(M), \tag{R_p}$$

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Well-known results: The Riesz transform is L^p bounded for $1 < p < \infty$ on

- Euclidean spaces $\mathbb{R}^n$
- Riemannian manifolds with non-negative Ricci curvature (Bakry, Littlewood-Paley theory)
- Lie groups with polynomial growth endowed with a sublaplacian (Alexopoulos)

Gaussian heat kernel estimates and Riesz transform

Theorem (Coulhon, Duong 1999)

Let M be a complete non-compact Riemannian manifold satisfying (D) and

$$h_t(x,y) \le \frac{C}{V(x,\sqrt{t})} \exp\Big(-c\frac{d^2(x,y)}{t}\Big), \forall x,y \in M, t>0. \qquad (UE)$$

Then the Riesz transform is of weak type $(1,1)$ and $L^p(M)$ bounded for $1 < p \le 2$.

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Remarks Under (D) and (UE), (R_p) may not hold for $p > 2$.
Examples: manifolds consisting of two copies of $\mathbb{R}^n \backslash \{B(0,1)\}$ $(n \geq 2)$ (see [Coulhon-Duong 1999], [Carron-Coulhon-Hassell 2006] etc).

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Examples: Riemannian manifolds with a spectral gap.

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Question: If (D) is assumed, can we replace (UE) by some other natural heat kernel estimates like the sub-Gaussian estimates?

Sub-Gaussian heat kernel estimate

Sub-Gaussian heat kernel upper estimate (UE_m):

$$h_t(x,y) \le \frac{C}{V(x,\rho^{-1}(t))} \exp\left(-cG(d(x,y),t)\right),$$

where $\rho(t) = \begin{cases} t^2, & 0 < t < 1, \\ t^m, & t \ge 1; \end{cases}$ and $G(r,t) = \begin{cases} r^2/t, & t \le r, \\ (r^m/t)^{1/(m-1)}, & t \ge r. \end{cases}$

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Given any $\alpha, m \in \mathbb{R}_+$ such that $\alpha > 1$ and $2 < m \le \alpha + 1$, there always exist manifolds satisfying $V(x,r) \simeq r^\alpha$ for $r \ge 1$ and (UE_m). See for example [Barlow, Coulhon, Grigor'yan 2001], [Hebisch, Saloff-Coste 2001], [Barlow 2004] etc.

Viscek manifolds

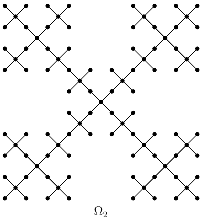

Figure: **Vicsek graph in** $\mathbb{R}^2$

M Vicsek manifold built in $\mathbb{R}^N$ with $N \in \mathbb{N}$, $N \geq 2$. Then $V(x,r) \simeq r^D$, $r \geq 1$, where $D = \log_3(2^N + 1)$; and

$$h_t(x,y) \leq \frac{C}{t^{\frac{D}{D+1}}} \exp\Big(- c\Big(\frac{d^{D+1}(x,y)}{t}\Big)^{1/D}\Big), \quad t \geq 1.$$

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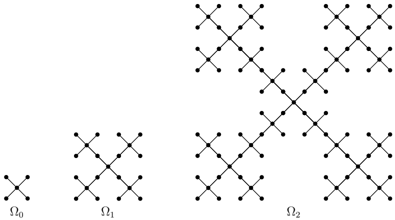

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It also satisfies the non-standard Poincaré inequality:

$$\int_B |f - f_B|^2 d\mu \leq C r_B^{D+1} \int_B |\nabla f|^2 d\mu, \quad \forall r_B \geq 1, \forall f \in \mathcal{C}_0^\infty(M).$$

Riesz transform for $1 \leq p \leq 2$ without Gaussian estimate

Theorem (C., Coulhon, Feneuil, Russ 2017)

Let M be a complete non-compact Riemannian manifold satisfying (D) and (UE_m), then the Riesz transform is weak $(1,1)$ bounded and L^p bounded for $1 < p \leq 2$.

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The strategy for the proof is the same as the Gaussian case in [Coulhon, Duong 1999].

- Singular integral techniques developed by Duong and McIntosh;
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It suffices to prove that for all $\lambda > 0$,

$$\mu\{x : |\nabla \Delta^{-1/2} f| > \lambda\} \leq \frac{C}{\lambda} \|f\|_1.$$

Formally one can write

$$\nabla \Delta^{-1/2} f = \int_0^\infty \nabla e^{-t\Delta} f \frac{dt}{\sqrt{t}}.$$

Using the Calderón-Zygmund decomposition, one can deduce the weak $(1,1)$ boundedness to the following estimate: for all $y \in M$, all $r, s > 0$,

$$\int_{d(x,y)\geq r} |\nabla_x h_s(x,y)|\, d\mu(x) \lesssim \begin{cases} \dfrac{1}{\sqrt{s}} \exp\Big(- c\dfrac{r^2}{s}\Big), & 0 < s < 1, \\ \dfrac{1}{\sqrt{s}} \exp\Big(- c\Big(\dfrac{r^m}{s}\Big)^{1/(m-1)}\Big), & s \geq 1. \end{cases}$$

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Key ingredients

- Chain rule for $u(x,t) = h_t(x,y)$:

$$\Delta u^p(x,t) = pu^{p-1}(x,t)\Delta u(x,t) - p(p-1)u^{p-2}(x,t)|\nabla_x u(x,t)|^2.$$

- Estimate $\Big\| |\nabla h_t(\cdot,y)| \exp\Big(c\Big(\frac{d^m(x,y)}{t}\Big)^{1/(m-1)}\Big)\Big\|_p$ for $1 < p < 2$, $t \geq 1$.

Riesz transform for $p > 2$ on Vicsek manifolds

Theorem (C. 2014; C., Coulhon, Feneuil, Russ 2017)

For any Vicsek manifold, (RR_p) does not hold for all $p \in (1, 2)$. Consequently, (R_p) does not hold for all $2 < p < \infty$.

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- This is an improvement of the result in [Coulhon, Duong 2003], where (RR_p) was shown to be false for $1 < p < \frac{2D}{D+1}$.
- This result shows that the conjunction of (D) and the non-standard Poincaré inequality does not imply the existence of $\varepsilon > 0$ such that (R_p) holds for $p \in (2, 2+\varepsilon)$.

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Corollary (C., Coulhon, Feneuil, Russ 2017)

For any Vicsek manifold. Let $1 < p < \infty$. Then (R_p) holds if and only if $1 < p \leq 2$ and (RR_p) holds if and only if $2 \leq p < \infty$.

All our results have their counterparts in the graph setting.

Idea for the proof

Let M be the Vicsek manifold with the volume growth $V(x,r) \simeq r^D$, $r \geq 1$. Denote $D' = \frac{2D}{D+1}$. Then M satisfies

$$\|f\|_p^{1+\frac{p}{(p-1)D'}} \leq C_p \|f\|_1^{\frac{p}{(p-1)D'}} \|\Delta^{1/2} f\|_p, \ \forall f \in \mathcal{C}_0^\infty(M) \text{ such that } \frac{\|f\|_p}{\|f\|_1} \leq 1.$$

Assume that (RR_p) is true, hence $\|f\|_p^{1+\frac{p}{(p-1)D'}} \leq C_p \|f\|_1^{\frac{p}{(p-1)D'}} \||\nabla f|\|_p$.
Choose $\{g_n\}_{n\in\mathbb{N}}$ to contradict the above inequality (from [Barlow, Coulhon, Grigor'yan 2001]):

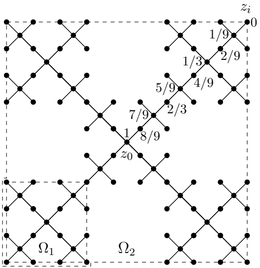

Figure: **The function g_2 on the diagonal $z_0 z_i$**

Thanks very much!