One-norm spectrum of a lattice

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Analysis and geometry on graphs and manifolds,
August 3rd 2017, Potsdam, Germany.

Notation

M a compact connected Riemannian manifold without boundary.

$\Delta : C^\infty(M) \to C^\infty(M)$ the Laplace–Beltrami operator.
Spec(M): λ such that there is $f \in C^\infty(M)$ such that $\Delta f = \lambda f$;

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty,$$

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One cannot hear the shape of a drum.

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Therefore $\mathbb{R}^{16}/E_8 \oplus E_8$ and $\mathbb{R}^{16}/D_{16}^+$ are isospectral.

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Parametrization: for $q \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{Z}$ satisfying $\gcd(q, s_j) = 1$ for all j,

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Equivalently, $L(q; s_1, \dots, s_n) = S^{2n-1}/\sim$ where

$$(z_1, \dots, z_n) \sim (\xi^{s_1} z_1, \dots, \xi^{s_n} z_n)$$

for any ξ root of unity of order q.

Isospectral characterization

We associate to $L(q; s_1, \ldots, s_n)$ the *congruence lattice*

$$\mathcal{L}(q; s_1, \ldots, s_n) = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : a_1 s_1 + \cdots + a_n s_n \equiv 0 \pmod{q}\}.$$

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$\mathcal{L}, \mathcal{L}' \subset \mathbb{Z}^n$ are said to be $\|\cdot\|_1$-*isospectral* if, for all $k \geq 0$,

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Theorem (L., Miatello, Rossetti, 2013)

The lens spaces L and L' are isospectral if and only if their associated congruence lattices $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1$-isospectral.

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We conclude that $F_L(z) = F_{L'}(z)$ if and only if $\Theta_{\mathcal{L}}(z) = \Theta_{\mathcal{L}'}(z)$.

Rational expression for $F_L(z)$

By using Ehrhart's theory on counting integer points in polytopes,

Theorem (L., 2015)

Let $L = L(q; s_1, \ldots, s_n)$ and let $\mathcal{L}$ be the associated congruence lattice. Then, there is a polynomial $P_{\mathcal{L}}(z)$ of degree $\leq q(n+1)$ such that

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In 2016: an explicit description for $P_{\mathcal{L}}(z)$ in terms of $\mathcal{L}$.

Generalizations

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Two lens spaces L and L' are p-isospectral for all p if and only if

$$\#\{\mu \in \mathcal{L} : \|\mu\|_1 = k,\ Z(\mu) = \ell\} = \#\{\mu \in \mathcal{L}' : \|\mu\|_1 = k,\ Z(\mu) = \ell\}$$

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  - A sequence of pairs of 7-dimensional lens spaces.
  - Computational examples.
  - Any example above is strongly isospectral.

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• Explicit p-spectra of lens spaces (2016). I found an explicit description of each p-spectrum of a lens spaces and the following characterization for each p_0: L and L' are p-isospectral for all $0 \leq p \leq p_0$ iff

$$\sum_{\ell=0}^{n} \ell^h \, \Theta_{\mathcal{L}}^{(\ell)}(z) = \sum_{\ell=0}^{n} \ell^h \, \Theta_{\mathcal{L}'}^{(\ell)}(z) \qquad \text{for all } 0 \leq h \leq p_0,$$

where $\Theta_{\mathcal{L}}^{(\ell)} := \sum_{k \geq 0} \#\{\mu \in \mathcal{L} : \|\mu\|_1 = k, \; Z(\mu) = \ell\} \, z^k$.

• A computational study (2017). The previous description let us to make a computational study of p-isospectral lens spaces.