# Quotients of finite-dimensional operators by symmetry representations

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(joint work with R. Band, G. Berkolaiko and W. Liu)

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The Leverhulme Trust

# Outline

- Simple example - Part I
- Simple example - Part II
- Motivations/background - Isospectrality, spectral computation
- Results - New definition, alignment with previous notions, generalisations
- Other interesting examples and discussion

# Simple example - Part I

- **Line graph:** Laplacian on 5 vertices, $G = \{e, r\}$, $L\pi(r) = \pi(r)L$

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \qquad \pi(r) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- **Reflection symmetry:** Eigenfunctions are either even or odd under reflection, $[\pi(r)f](x) = f(-x) = \pm f(x)$

$$\sigma(L) = \left\{0,\ \frac{1}{2}(3-\sqrt{5}),\ \frac{1}{2}(5-\sqrt{5}),\ \frac{1}{2}(3+\sqrt{5}),\ \frac{1}{2}(5+\sqrt{5})\right\}$$

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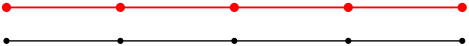

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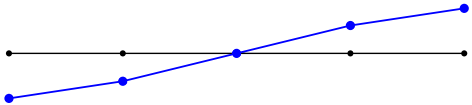

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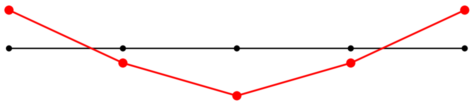

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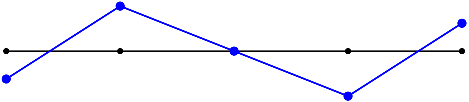

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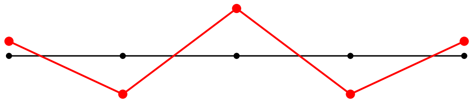

# Simple example - Part I

- **Isolate even functions:** Modify eigenvalue equation with *information* of trivial representation $f(x) = f(-x)$

$$\begin{aligned}
\lambda f(0) &= [Lf](0) = 2f(0) - f(1) - f(-1) = 2f(0) - 2f(1) \\
\lambda f(1) &= [Lf](1) = 2f(1) - f(2) - f(0) \\
\lambda f(2) &= [Lf](2) = f(2) - f(1)
\end{aligned}$$

- **Matrix form:** Reading off coefficients gives

$$\lambda f = \widetilde{L}_+ f, \qquad \widetilde{L}_+ = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

- **Normalisation:** If we choose $g(0) = f(0), g(x) = \sqrt{2}f(x), x = 1, 2$, then

$$\sum_{x=0}^{2} g(x)^2 = \sum_{x=-2}^{2} f(x)^2 = 1$$

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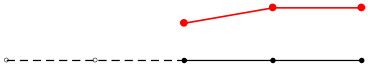

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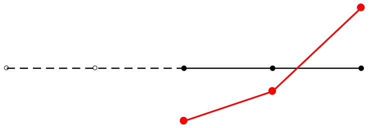

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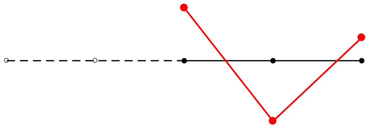

# Simple example - Part II

- **Alternative viewpoint:** Take a basis of even/odd vectors

$$\Theta_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Theta_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

- **Conjugate with original operator:** Then we have $L_\pm = \Theta_\pm^* L \Theta_\pm$, i.e.

$$\Theta_+^* L \Theta_+ = \begin{pmatrix} 2 & -\sqrt{2} & 0 \\ -\sqrt{2} & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad \Theta_- L \Theta_- = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

- **Block diagonalisation:** $Z = (\Theta_+ \ \Theta_-)$, then

$$Z^* \pi(r) Z = \mathrm{diag}(1, 1, 1, -1, -1), \qquad Z^* L Z = L_+ \oplus L_-$$

# Simple example - Part II

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- **Unitary equivalence:** Take any unitary matrix, then following also valid

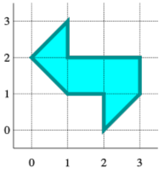
$$\Theta_\pm \mapsto \Theta_\pm U \qquad , L_\pm \mapsto U^* L_\pm U$$

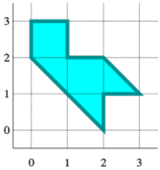
- **The question:** Given unitary invariance, for arbitrary symmetries,

How do we consistently choose $\Theta$?

# Motivations and background

- **Isospectrality:** [Sunada '85, Gordon, Webb & Wolpert '92]

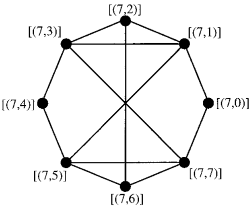


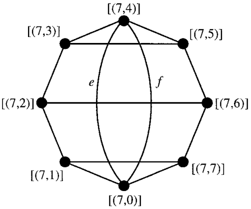




Quotient graphs are isospectral if

$$\mathrm{Ind}_{H_1}^{G}(\mathrm{triv}) \cong \mathrm{Ind}_{H_2}^{G}(\mathrm{triv})$$

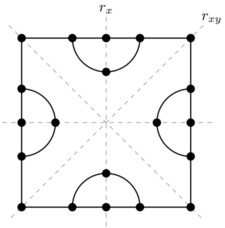
- **Isospectrality in discrete graphs:** [Brooks '99, Halbeisen & Hungerbühler '99]

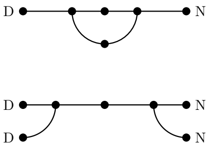


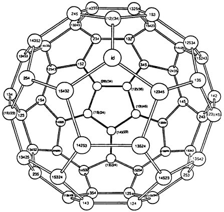
# Motivations and background

- **Obtaining quotients with different representations:** Generalisation - quotients isospectral if $\mathrm{Ind}_{H_1}^{G}(\rho_1) \cong \mathrm{Ind}_{H_2}^{G}(\rho_2)$ [Band, Parzanchevski & Ben-Shach '09]




- **Spectral computation in transitive graphs:** [Chung & Sternberg '92]



# Questions arising

• **Spectral interpretation:** [Band, Parzanchevski & Ben-Shach '09] - A quotient operator is any operator $\widetilde{\text{Op}}$ such that

$$\mathbf{E}_{\lambda}^{\widetilde{\text{Op}}} \cong \text{Hom}_G(V_\rho, \mathbf{E}_{\lambda}^{\text{Op}})$$

Works for self-adjoint operators - what about other types?

• **Fixed points:** [Halbeisen & Hungerbühler '99] - Extend to other representations, remove fixed point conditions

• **Transitive graphs:** [Chung & Sternberg '92] - What about non-transitive graphs?

# Representation theory

- **Symmetry:** We say that a finite-dimensional operator Op is $\pi$-symmetric if

$$\pi(g)\,\mathrm{Op} = \mathrm{Op}\,\pi(g) \qquad \forall g \in G$$

Assume $\pi(g)$ to be permutation matrices.

- **Representations:** Vectors transform in following manner

$$[\pi(g)\phi_k](x) = \phi_k(g^{-1}x) = \sum_{l=1}^{r} \phi_l(x)\rho(g)_{lk} \qquad \forall g \in G, k = 1, \ldots, r$$

$\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$.

- **Hom space:** We denote by $\mathrm{Hom}_G(V_\rho, V_\pi)$ the space of all $\phi : V_\rho \to V_\pi$ such that

$$\pi(g)\phi = \phi\rho(g) \qquad \forall g \in G$$

# Constructing a new definition

- **Vectorisation:** Introduce procedure vec : $M_{n\times m}(\mathbb{C}) \to \mathbb{C}^{nm}$, e.g.

$$\text{vec}\left(\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\right) = \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix}$$

- **Kernel space:** Applying vec gives $(\mathbf{I}_r \otimes \pi(g))\,\text{vec}(\phi) = (\rho(g)^T \otimes \mathbf{I}_p)\,\text{vec}(\phi)$

$$\mathcal{K}_G(\rho,\pi) = \bigcap_{g\in G} \ker\left[\mathbf{I}_r \otimes \pi(g) - \rho(g)^T \otimes \mathbf{I}_p\right].$$

Thus $\psi = \text{vec}(\phi) \in \mathcal{K}_G(\rho,\pi) \iff \phi \in \text{Hom}_G(V_\rho, V_\pi)$.

- **Quotient operator:** Let Op be $\pi$-symmetric and $\Theta$ a basis for $\mathcal{K}_G(\rho,\pi)$, then

$$\text{Op}_\rho := \Theta^*[\mathbf{I}_r \otimes \text{Op}]\Theta$$

# Properties of quotient operator

$$\mathrm{Op}_\rho := \Theta^*[\mathbf{I}_r \otimes \mathrm{Op}]\Theta$$

- **Operator type:** One may choose any operator Op, only requirement is $\pi$-symmetric.

- **Decomposition:** If $R \cong \bigoplus \rho$ then $\mathrm{Op}_R \cong \bigoplus \mathrm{Op}_\rho$. In particular, if $\rho$ are irreps of $G$, then
$$\mathrm{Op} \cong \bigoplus_\rho [\mathbf{I}_{\deg \rho} \otimes \mathrm{Op}_\rho],$$

- **Normality:** If $\mathrm{Op}^* \mathrm{Op} = \mathrm{Op}\,\mathrm{Op}^*$ then $\mathrm{Op}_\rho^* \mathrm{Op}_\rho = \mathrm{Op}_\rho \mathrm{Op}_\rho^*$.

- **Spectral property:** For any Op we now have $\mathbf{E}_\lambda^{\mathrm{Op}_\rho} \cong \mathrm{Hom}_G(V_\rho, \mathbf{E}_\lambda^{\mathrm{Op}})$

- **Unitary equivalence:** Choosing a different basis $\widetilde{\Theta} = \Theta U$ leads to equivalent operator.

# Choosing a basis

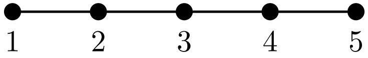
- **Orbits:** Let $\mathcal{P} = \{1, \ldots, p\}$ be set of points. Then *orbit* of $i$ is

$$O_i := \{j \in \mathcal{P} : \exists g \in G \text{ s.t. } i = gj\}.$$

Then *fundamental domain* a set of one representative from each orbit, $\mathcal{D} = \{1, \ldots, |\mathcal{D}|\}$.

- **Example:** Line graph on 5 vertices

$$O_1 = \{1, 5\}, \qquad O_2 = \{2, 4\}, \qquad O_3 = \{3\}$$

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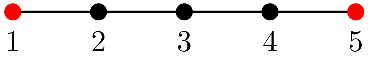


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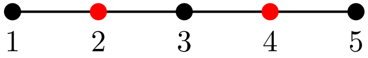


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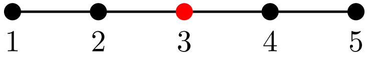


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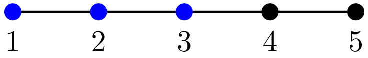


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# Choosing a basis

**Theorem (J., Band, Berkolaiko, Liu)**

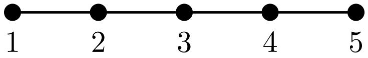
*Let columns of matrix $\Theta_i$ be an orthonormal basis for*

$$\mathcal{K}_G^i(\rho,\pi) := K_G(\rho,\pi) \cap [V_\rho \otimes X_i], \qquad X_i := \mathrm{span}\{\mathbf{e}_j : j \in O_i\}.$$

*Then columns of $\Theta = (\Theta_1 \ \dots \ \Theta_{|\mathcal{D}|})$ form orthonormal basis for $\mathcal{K}_G(\pi,\rho)$.*

- **Example:** Line graph on 5 vertices, $\mathcal{K}(\pm,\pi) = \ker[\pi(r) \mp \mathbf{I}_5]$

$$\Theta_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Theta_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix},$$



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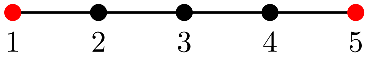
*Then columns of $\Theta = (\Theta_1 \ \dots \ \Theta_{|\mathcal{D}|})$ form orthonormal basis for $\mathcal{K}_G(\pi,\rho)$.*

- **Example:** Line graph on 5 vertices, $\mathcal{K}^1(\pm,\pi) = \ker[\pi(r) \mp \mathbf{I}_5] \cap X_1$

$$\Theta_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Theta_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix},$$



# Choosing a basis

**Theorem (J., Band, Berkolaiko, Liu)**

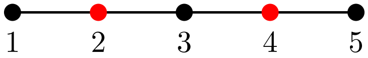
*Let columns of matrix* $\Theta_i$ *be an orthonormal basis for*

$$\mathcal{K}_G^i(\rho,\pi) := K_G(\rho,\pi) \cap [V_\rho \otimes X_i], \qquad X_i := \mathrm{span}\{\mathbf{e}_j : j \in O_i\}.$$

*Then columns of* $\Theta = (\Theta_1 \;\ldots\; \Theta_{|\mathcal{D}|})$ *form orthonormal basis for* $\mathcal{K}_G(\pi,\rho)$.

- **Example:** Line graph on 5 vertices, $\mathcal{K}^2(\pm,\pi) = \ker[\pi(r) \mp \mathbf{I}_5] \cap X_2$

$$\Theta_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Theta_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix},$$



# Choosing a basis

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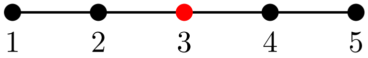
*Then columns of $\Theta = (\Theta_1 \ \ldots \ \Theta_{|\mathcal{D}|})$ form orthonormal basis for $\mathcal{K}_G(\pi,\rho)$.*

- **Example:** Line graph on 5 vertices, $\mathcal{K}^3(\pm,\pi) = \ker[\pi(r) \mp \mathbf{I}_5] \cap X_3$

$$\Theta_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Theta_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{pmatrix},$$



# Form of quotient operator

**Theorem (J., Band, Berkolaiko, Liu)**

*Choose $\Theta$ as in previous theorem, then quotient operator will consist of blocks of the form*

$$[\mathrm{Op}_\rho]_{ij} = \frac{1}{\sqrt{|G_i||G_j|}} \sum_{g \in G} (\Phi_i^* \bar{\rho}(g) \Phi_j)\, \mathrm{Op}_{i,gj},$$

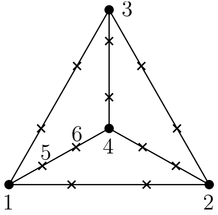
*where $G_i := \{g \in G : gi = i\}$ is fixed point group of $i$ and $\Phi_i$ an orthonormal basis for*

$$\bigcap_{g \in G_i} \ker[\mathbf{I}_r - \rho(g)].$$

- **Structure preservation:** Connections of original operator are preserved in following sense
  - $[\mathrm{Op}_\rho]_{i,j} \neq 0$ only if $\exists$ elements $k \in O_i$, $l \in O_j$ s.t. $Op_{kl} \neq 0$.
  - $[\mathrm{Op}_\rho]_{i,j}$ does not depend on elements from other orbits.

# Example 2 - higher dimensional irreps

- **Reducing points:** Take tetrahedron, invariant under $S_4$. Two orbits given by $O_\bullet = \{1,2,3,4\}$ and $O_\times = \{5,\dots,16\}$.




$$H_{ij} = \begin{cases} a & \bullet \sim \times \\ b & \times \sim \times \\ V_\bullet & \bullet = \bullet \\ V_\times & \times = \times \end{cases}$$

- **Irreducible representation:** 'Standard' representation of $S_4$, generated by

$$R((12)) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; R((23)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; R((34)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Action:** Take $\psi \in \mathcal{K}_G(R,\pi)$, with $\psi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^T$

$$[\pi(g)\psi](x) = R(g)^T \psi(x)$$

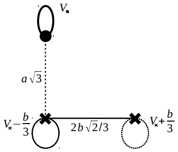
# Example 2 - higher dimensional irreps

- **Fixed points:** Take for example, $G_\bullet = S_3$ (exchange $2,3,4$). Then

$$\psi(\bullet) = \psi(g^{-1}\bullet) = R(g)^T \psi(\bullet) \qquad \forall\ g \in G_\bullet .$$

Therefore $\psi(\bullet) \in \bigcap_{g \in G_\bullet} \ker[\mathbf{I}_r - \bar{R}(g)]$.

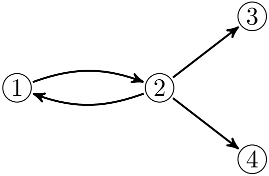
- **Quotient operator:** We have $\dim(\mathcal{K}_G^\bullet(\rho,\pi)) = 1$ and $\dim(\mathcal{K}_G^\times(\rho,\pi)) = 2$, leading to




$$H_R = \begin{pmatrix} V_\bullet & a & a\sqrt{2} \\ a & V_\times + b & 0 \\ a\sqrt{2} & 0 & V_\times - b \end{pmatrix}$$

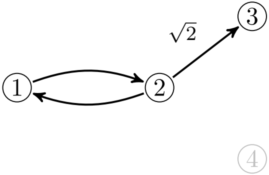
# Example 3 - Non-normal operators

- **Directed graphs:** Take following adjacency matrix




$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- **Form quotient operator:** Fixed point group $G_2 = \mathbb{Z}_2$, so $[A_{\text{triv}}]_{32} = \frac{1}{\sqrt{2}}(A_{32} + A_{42}) = \sqrt{2}$, and in full




$$A_{\text{triv}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

- **Spectrum:** There is (non-unitary) $Q$ such that $Q^{-1}\,\text{diag}(0, 1, -1)Q = A_{\text{triv}}$.

# Conclusions and outlook

- **Summary:**
  - New definition for finite-dimensional quotient operators
  - Generalises previous notions from isospectraility
  - Can show the choice of basis that leads to structure preservation
- **Further research:**
  - Is there an analogous spectral condition?
  - What about antiunitary symmetries?
  - Cellular graphs?