Kac regular sets in geometry and probability

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Analysis and Geometry on Graphs and Manifolds

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Joint work with Francesco Bei (University of Lyon).

Let $\Omega \subset M$ be an open subset of a Riemannian manifold M.
Under which assumption on Ω does one have

$$W_0^{1,2}(\Omega) = \{f|_\Omega : f \in W_0^{1,2}(M),\ f = 0 \text{ a.e. in } M \setminus \Omega\}\textbf{?} \tag{1}$$

Ω's having this property have been called **Kac regular** by Herbst/Zhao (1987) and Stroock (1967) for $M = \mathbb{R}^m$.

• We will see in a moment that Kac regularity is equivalent to realizing Dirichlet boundary conditions by the potential $\infty \cdot 1_{M\setminus\Omega}$, and also to a property of Brownian motion.

• $W_0^{1,2}(\Omega)$ is the form domain of the Dirichlet realization of $-\Delta$ in $L^2(\Omega)$ $\rightsquigarrow$ we can replace $-\Delta$ with $-\Delta + V$ and ask the same question again.

• More generally, we can also consider covariant Schrödinger operators (magnetic Schrödinger operators, squares of Dirac operators,...)

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On the literature:

• Herbst/Zhao have shown that for $M = \mathbb{R}^m$ and $V = 0$ that Ω's with $\partial\Omega$ having the segment property are Kac regular, showing that natural Kac regularity results should depend on the local regularity of $\partial\Omega$.

• Hedberg & Stollmann: There exists a Kac irregular set $\Omega \subset \mathbb{R}^m$ with $\mathbb{R}^m \setminus \Omega$ compact and equal to the closure of its interior (disproving a conjecture by Simon).

• If one replaces 'a.e.' with 'q.e.', the equality of Sobolev spaces is true for arbitrary Ω's (Adams/Hedberg for $M = \mathbb{R}^m$) and thus does not see anything from the local regularity of $\partial\Omega$.

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Definition

A *semibounded Schrödinger bundle over M* is a datum $(E, \nabla, V) \to M$ with

- ∇ a metric covariant derivative on the complex metric vector bundle $E \to M$,
- $V : M \to \mathrm{End}(E)$ is an L^2_{loc} potential,

such that (for simplicity)

$$\langle \nabla f, \nabla f \rangle + \langle Vf, f \rangle \geq 0 \quad \text{for all } f \in C_c^\infty(M, E).$$

• Given an open subset $\Omega \subset M$ let $H_\Omega(\nabla, V)$ denote the Friedrichs realization of $\nabla^\dagger \nabla + V$ in $L^2(\Omega, E)$.
• $H_\Omega(\nabla, V)$ is $\nabla^\dagger \nabla + V$ with Dirichlet boundary conditions on Ω.
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- Define $W_0^{1,2}(\Omega, E; \nabla, V) \subset L^2(\Omega, E)$ to be the closure of $C_c^\infty(\Omega, E)$ w.r.t. 'Sobolev norm'

$$\|f\|_{\nabla,V}^2 := \|f\|^2 + \|\nabla f\|^2 + \langle Vf, f\rangle .$$

- $W_0^{1,2}(\Omega, E; \nabla, V)$ is the form domain of $H_\Omega(\nabla, V)$.
- Which properties of Ω imply in the validity of

$$W_0^{1,2}(\Omega, E; \nabla, V) = \left\{ f|_\Omega : f \in W_0^{1,2}(M, E; \nabla, V), f|_{M\setminus\Omega} = 0 \text{ a.e.} \right\}? \tag{2}$$

It is elementary to see that '$\subset$' in (2) is true for all Ω's.

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Examples:

• Magnetic Schrödinger operators: $\nabla = d + \sqrt{-1}\alpha$ with α a 1-form on M, $V : M \to \mathbb{R}$ in L^2_{loc}, on the trivial line bundle over M, identifying sections with functions M$\to \mathbb{C}$. For $\alpha = 0$ we recover Schrödinger operators:

Notation: $H_\Omega(V) := H_\Omega(d + \sqrt{i}\alpha, V)|_{\alpha=0}$ with form domain $W^{1,2}_0(\Omega; V) \subset L^2(\Omega)$; $H_\Omega := H_\Omega(V)|_{V=0}$ with form domain $W^{1,2}_0(\Omega) \subset L^2(\Omega)$.

• Given a Dirac bundle $(E, c, \nabla) \to M$ with its Dirac operator $D(c, \nabla)$. Lichnerowicz formula:

$$V(c, \nabla) := D(c, \nabla)^2 - \nabla^\dagger \nabla : M \to \mathrm{End}(E)$$

is a (smooth) potential $\rightsquigarrow$ the semibounded Schrödinger bundle $(E, \nabla, V(c, \nabla)) \to M$.

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$W(M)$: Wiener space of continuous paths $\omega : [0,\infty) \to \hat{M} = M \cup \infty_M$,
$\mathbb{X} : [0,\infty) \times W(M) \longrightarrow M, \quad \mathbb{X}_t(\omega) := \omega(t)$,
$\mathbb{P}^x$: Brownian motion measure on $W(M)$ with $\mathbb{P}^x\{\mathbb{X}_0 = x\} = 1$.

If $\Omega \subset \hat{M}$ is an open set, then one calls

$$\alpha_\Omega := \inf\left\{t > 0 : \mathbb{X}_t \in M \setminus \Omega\right\} : W(M) \longrightarrow [0,\infty]$$

$$\beta_\Omega := \inf\left\{t > 0 : \int_0^t 1_{M\setminus\Omega}(\mathbb{X}_s)ds > 0\right\} : W(M) \longrightarrow [0,\infty],$$

the **first exit time of $\mathbb{X}$ from Ω**, resp. the **first penetration time of $\mathbb{X}$ to $M \setminus \Omega$.** Elementary:

$$\alpha_\Omega \leq \beta_\Omega, \tag{3}$$

$$\{t < \alpha_\Omega\} = \left\{\mathbb{X}_s \in \Omega \text{ for all } s \in [0,t]\right\}, \tag{4}$$

$$\{t < \beta_\Omega\} = \left\{\mathbb{X}_s \in \Omega \text{ for a.e. } s \in [0,t]\right\} \tag{5}$$

Proposition (I.Herbst/Z.Zhao for $\mathbb{R}^m$; F. Bei/B.G.)

Let $\Omega \subset M$ be an arbitrary open subset. The following properties are equivalent:

- *Ω is Kac regular, that is,*

$$W_0^{1,2}(\Omega) = \left\{ f|_\Omega : f \in W_0^{1,2}(M), f|_{M\setminus\Omega} = 0 \text{ a.e. } \right\}, \tag{6}$$

- *for all $x \in \Omega$ one has $\mathbb{P}^x\{\alpha_\Omega = \min(\beta_\Omega, \alpha_M)\} = 1$,*
- *for all $t \geq 0$ one has*

$$\mathrm{s-}\lim_{n\to\infty} \exp\big(- tH_M(n1_{M\setminus\Omega})\big) = \exp\big(- tH_\Omega\big) P_\Omega. \tag{7}$$

The last condition means that Dirichlet boundary conditions on Ω are realized by means of the potential $V(x) := \infty \cdot 1_{M\setminus\Omega}$.

Theorem (F.Bei/B.G.)

Let Ω be an arbitrary open subset of M.
a) *If Ω is Kac regular, then for every regular Schrödinger bundle $(E,\nabla,V)\to M$ one has*

$$W^{1,2}_0(\Omega,E;\nabla,V)=\big\{f|_\Omega : f\in W^{1,2}_0(M,E;\nabla,V), f|_{M\setminus\Omega}=0\,\mathrm{a.e.}\big\},$$

and

$$\mathrm{s-}\lim_{n\to\infty}\exp\big(-tH_M(\nabla,V+n1_{M\setminus\Omega})\big)=\exp\big(-tH_\Omega(\nabla,V)\big)P_\Omega.$$

b) *If Ω is Lipschitz exhaustable, in the sense that there exist $\Omega_n\subset\Omega$ relatively compact and open with Lipschitz boundary, such that $\Omega_n\nearrow\Omega$, $\overline{\Omega_n}\nearrow\overline{\Omega}$, then Ω is Kac regular.*

Sketch of proof in the scalar case (= usual Schrödinger operators):

• For arbitrary Ω, the generator $\tilde{H}_\Omega(V)$ of the semigroup

$$P_t f(x) = \int_{\{t<\min(\beta_\Omega,\alpha_M)\}} e^{-\int_0^t V(\mathbb{X}_s)} f(\mathbb{X}_t) dP^x$$

is induced by the restriction of form of $H_M(V)$ to the domain $\left\{ f|_\Omega : f \in W_0^{1,2}(M;V), f|_{M\setminus\Omega} = 0 \,\mathrm{a.e.} \right\}$

• For arbitrary Ω one has

$$\mathrm{s-}\lim_{n\to\infty} \exp\big(- tH_M(n1_{M\setminus\Omega})\big) = \exp\big(- tH_\Omega\big) P_\Omega,$$

$$e^{-tH_\Omega(V)}(x) = \int_{\{t<\alpha_\Omega\}} f(\mathbb{X}_t) e^{-\int_0^t V(\mathbb{X}_s) ds} dP^x.$$

• In the covariant case, one has to use the covariant Feynman-Kac formula; rather technical (no monotonicity).

• Lipschitz exhaustable sets are Kac regular: locally, use the existence of a Sobolev extension operator for bounded Lipschitz sets, so that

$$\mathbb{P}^x\{\alpha_{\Omega_n} = \min(\beta_{\Omega_n}, \alpha_M)\} = 1 \quad \text{for all } n. \tag{8}$$

Then using $\alpha_{\Omega_n} \to \alpha_\Omega$ (requires $\Omega_n \to \Omega$) and $\beta_{\Omega_n} \to \beta_\Omega$ (requires $\Omega_n \to \Omega$ (uses $\overline{\Omega_n} \to \overline{\Omega}$), we can take $n \to \infty$ in (8), showing Kac regularity.

Remarks and outlook:

- Typical applications of Kac regularity: Uniqueness results for Laplace type equations.
- Ω is Lipschitz exhaustable, if $\partial\Omega$ is smooth.
- **Conjecture:** Ω is Lipschitz exhaustable, if $\partial\Omega$ has a locally Lipschitz boundary.
- Possibly nonlocal Dirichlet forms? $\rightsquigarrow$ Adams/Hedberg

Thank you for listening!