Neumann domains II
Ground state property of Neumann domains on the torus

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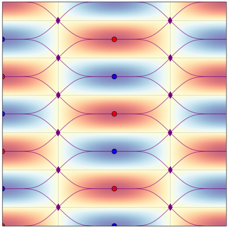
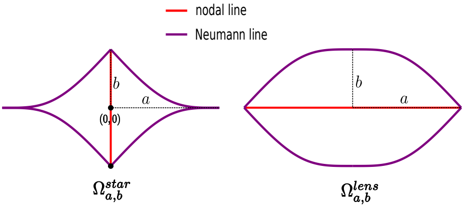
Neumann domains on torus

Laplacian eigenfunctions on $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$: $\quad -\Delta\psi_{a,b} = \lambda_{a,b}\psi_{a,b}$

Eigenvalues: $\quad \lambda_{a,b} = \frac{\pi^2}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right), \quad a = \frac{1}{4m}, \; b = \frac{1}{4n}, \; m, n \in \mathbb{N}_0$

Neumann domains for seperable eigenfunctions:





Origin at center of $\Omega_{a,b}^{star}$: $\quad \psi_{a,b}(x_1, x_2) = \sin\left(\frac{\pi}{2a}x_1\right)\cos\left(\frac{\pi}{2b}x_2\right)$

The result

Definition: ground state property for $\Omega_{a,b}^{star/lens}$ $:\Leftrightarrow$

- lowest non-zero eigenvalue of Neumann Laplacian is simple

  $\Rightarrow$ eigenfunction ψ (mod constant) is called ground state
- $\psi_{a,b}\big|_{\Omega_{a,b}^{star/lens}}$ is ground state on $\Omega_{a,b}^{star/lens}$

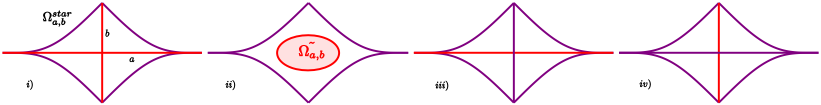
Ground state property depends on $\frac{a}{b}$ $\Rightarrow$ fix $a \in \mathbb{R}_+$ and vary $b \in \mathbb{R}_+$

R. Band, D. Fajman: for every fixed $a > 0$ a $b_0 > 0$ exists such that: $b < b_0 \Rightarrow \Omega_{a,b}^{lens}$ does not satisfy ground state property

R. Band, E., A. Taylor: for every fixed $a > 0$ a $b_0 > 0$ exists such that: $b < b_0 \Rightarrow \Omega_{a,b}^{star}$ does satisfy ground state property

The idea for proving

Symmetry of $\Omega_{a,b}^{star}$ $\Rightarrow$ ground state is symmetric or antisymetric $i)$-$iv)$



Exclude shape

$i)$: Courant's nodal domain theorem

$$\#\{\text{nodal domains}\} \leq n, \quad \lambda_n \quad n-\text{th eigenvalue}$$

$\Rightarrow$ four nodal domains ↯ to λ_2 i.e. $n = 2$

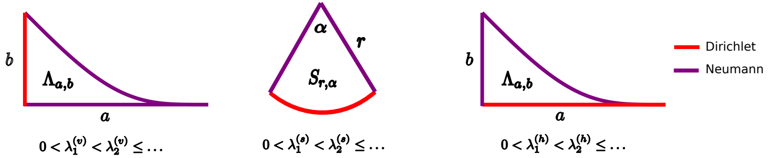
$ii)$:

$$\lambda_2^{(N)}(\Omega_{a,b}^{star}) = \tilde{\lambda}_1^{(D)}(\tilde{\Omega_{a,b}}) \underset{monotonicity}{>} \lambda_1^{(D)}(\Omega_{a,b}^{star}) \underset{Polya}{\geq} \lambda_1^{(N)}(\Omega_{a,b}^{star})$$

Exclusion of shape iii)

Equivalent problem:

For every fixed $a > 0$ a $b_0 > 0$ exists: for all $b < b_0$: $\lambda_1^{(v)} < \lambda_1^{(h)}$



Idea: modified rearrangement with sector $S_{r,\alpha}$ as reference system

Find $S_{r,\alpha}$ such that: for all $b < b_0$:

$$\lambda_1^{(v)} < \lambda_1^{(s)} < \lambda_1^{(h)}$$

$\Rightarrow$ α and r depend on b

The rearrangement

Area of the sector: $|\Lambda_{a,b}|_2 \stackrel{!}{=} |S_{r,\alpha}|_2$

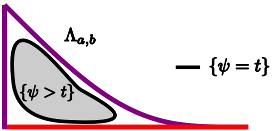
Rearrangement in two steps:

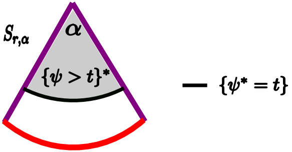
1. rearrangement of sets: $\Lambda_{a,b} \supset \Omega \to \Omega^* \subset S_{r,\alpha}$ such that:

$$|\Omega^*|_2 \stackrel{!}{=} |\Omega|_2 \quad \text{and} \quad \Omega^* = S_{\tilde{r},\alpha}$$

2. rearrangement of $\psi : \Lambda_{a,b} \to \mathbb{R}_0^+$ to $\psi^* : S_{r,\alpha} \to \mathbb{R}_0^+$

$$\psi^*(\boldsymbol{x}) := \int_0^\infty \chi_{\{\psi>t\}^*}(\boldsymbol{x})\,\mathrm{d}\,t, \quad \{\psi > t\} := \{\boldsymbol{x} \in \Lambda_{a,b};\ \psi(\boldsymbol{x}) > t\}$$



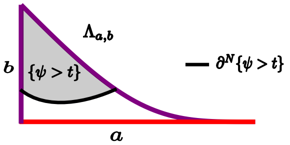


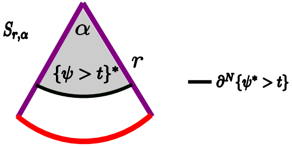
Requirements for rearrangement, *a* fixed, *b* small

i) $\lambda_1^{(v)} < \frac{j_0^2}{r^2} = \lambda_{1,s}$, j_0 first zero of J_0 Bessel function

ii) for almost all $t > 0$: $\partial^N(\cdot)$ non-Neumann part of boundary

$$|\partial^N \{\psi > t\}|_1 > |\partial^N \{\psi^* > t\}|_1,$$





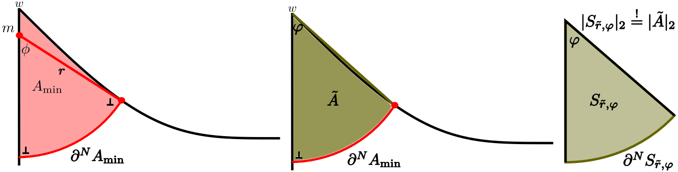
ii)+coarea formula $\Rightarrow$ $\|\nabla\psi^*\| \leq \|\nabla\psi\|$ ($\|\psi^*\| = \|\psi\|$ holds always)

$$\lambda_1^{(h)} = \inf_{\psi \in H^1_{0,D}(\Lambda_{a,b})} \frac{\|\nabla\psi\|^2}{\|\psi\|^2} \geq \inf_{\psi \in H^1_{0,D}(\Lambda_{a,b})} \frac{\|\nabla\psi^*\|^2}{\|\psi^*\|^2} \geq \inf_{\psi \in H^1_{0,D}(S_{r,\alpha})} \frac{\|\nabla\psi\|^2}{\|\psi\|^2} = \lambda_1^{(s)}$$

- *i*) $\Rightarrow$ lower bound $\alpha_{\min}$ for α
- *ii*) $\Rightarrow$ upper bound $\alpha_{\max}$ for α

The upper bound $\alpha_{\max}$

Relative isoperimetric problem in $\Lambda_{a,b}$: determine $\inf_{A\subset\Lambda_{a,b}} \frac{|\partial^N A|_1^2}{2|A|_2}$



First step: find $\forall\eta > 0$ the minimizing set $A_{\min}$ for $\min_{\substack{A\subset\Lambda_{a,b},\\ |A|_2=\eta}} |\partial^N A|_1$

Second step: $|A_{\min}| < |\tilde{A}|,\ |\partial^N \tilde{A}| = |\partial^N A_{\min}| > |\partial^N S_{\tilde{r},\varphi}|,\ \varphi > \frac{\pi}{4}$
implies,

$$\frac{|\partial^N A_{\min}|_1^2}{2|A_{\min}|_2} > \frac{|\partial^N S_{\tilde{r},\varphi}|_1^2}{2|S_{\tilde{r},\varphi}|_2} = \varphi > \frac{\pi}{4} := \alpha_{\max}$$

Third step: $|A_{\min}|_2 \overset{!}{=} |\left\{\psi > t\right\}|_2, \quad (|A_{\min}|_2 = |\left\{\psi > t\right\}^*|_2)$

$$1 \overset{!}{\leq} \frac{|\partial^N\left\{\psi > t\right\}|_1^2}{|\partial^N\left\{\psi^* > t\right\}|_1^2} = \frac{|\partial^N A_{\min}|_1^2}{2|A_{\min}|_2}\frac{2|A_{\min}|_2}{|\partial^N\left\{\psi^* > t\right\}|_1^2} \geq \frac{\alpha_{\max}}{\alpha}$$

The lower bound $\alpha_{\min}$

Asymptotics of $|\Lambda_{a,b}|_2$:

$$|\Lambda_{a,b}|_2 \sim \gamma b^2(1+\mathrm{O}(b)), \quad \gamma \approx 0.6080$$

Ground state of the sector in dependence of $|\Lambda_{a,b}|_2 = |S_{r,\alpha}|_2$:

$$r = \sqrt{\frac{2|\Lambda_{a,b}|_2}{\alpha}}$$

$$\Rightarrow \; \lambda_{1,s} = \frac{j_0^2\alpha}{2|\Lambda_{a,b}|_2} \sim \frac{j_0^2\alpha}{2\gamma b^2}$$

$$\lambda_{a,b} = \frac{\pi^2}{4}(\frac{1}{a^2}+\frac{1}{b^2}) \sim \frac{\pi^2}{4}\frac{1}{b^2}$$

$$\Rightarrow \quad \forall \quad \alpha > \alpha_{\min} := \frac{\gamma\pi^2}{2j_0^2} \approx 0.1652\pi \quad \exists b_0 : \quad \lambda_{a,b} < \lambda_{1,s}$$

$$\Rightarrow \quad \alpha_{\min} < \alpha_{\max} \quad \Rightarrow \quad \text{Claim}$$

Thank you for you attention!