Non self-adjoint Laplacians on a directed graph

Marwa BALTI

Carthage University (FSB) & Nantes University (LMJL)

31 July 2017:
International Conference at the University of Potsdam, Germany

Abstract

We consider a non self-adjoint Laplacian on a directed graph with non symmetric edge weights. We analyse spectral properties of this Laplacian under a Kirchhoff's assumption. Moreover we establish isoperimetric inequalities in terms of the numerical range to show the absence of the essential spectrum of our Laplacian.

T. Kato
Perturbation theory for linear operators, (1976).

A. Grigor'yan
Analysis on graphs. Lecture Notes, (2011/2012) .

W. D. Evans, R. T. Lewis and A. Zettl
Non self-adjoint operators and their essential spectra, (1983) ...

Plan

Notion of graphs

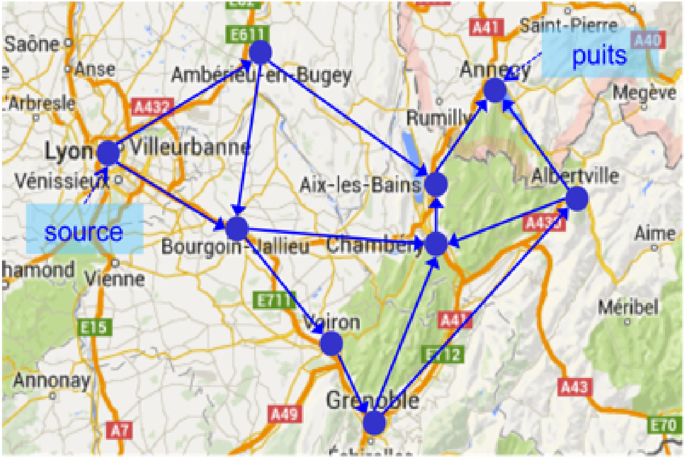
We call oriented or directed graph, the couple $G = (V, \vec{E})$, where V is a countable set of vertices, and $\vec{E} \subset V \times V$ is a set of directed edges. For two vertices x, y of V, we denote by (x, y) the edge that connects x to y, we also say that x and y are *neighbors*.
For all $x \in V$, we set:

- The edge (x, x) is called a loop.
- $V_x^+ = \left\{ y \in V, \quad (x, y) \in \vec{E} \right\}$
- $V_x^- = \left\{ y \in V, \quad (y, x) \in \vec{E} \right\}$
- $V_x = V_x^+ \cup V_x^-$. G is locally finite if for all $x \in V$,

$$\# V_x < \infty.$$

Example of a directed graph

Transport network



A path between two vertices x and y in V is a finite set of directed edges $(x_1, y_1);\ (x_2, y_2); ..; (x_n, y_n),\ n \geq 2$ such that

$$x_1 = x,\ y_n = y \text{ and } x_i = y_{i-1}\ \ \forall\, 2 \leq i \leq n.$$

- $G = (V, \vec{E})$ is called connected if two vertices are always related by a path.
- $G = (V, \vec{E})$ is called strongly connected if for all vertices x, y there is a path from x to y and one from y to x.

In this work, we suppose that G is without loops, locally finite, connected and satisfy:

$$\forall x \in V, \quad \# V_x^+ \neq 0 \text{ et } \# V_x^- \neq 0.$$

We define a weighted graph

Definition

Weighted graph: *A directed weighted graph is a triple* (G, m, b) *, where* G *is a directed graph,* $m : V \to \mathbb{R}_+^*$ *is a weight on* V *and* $b : V \times V \to [0, \infty)$ *is a weight satisfying the following conditions:*

- $b(x, x) = 0$ *for all* $x \in V$
- $b(x, y) > 0$ *iff* $(x, y) \in \vec{E}$

The graph (G, m, b) is called *symmetric* if for all $x, y \in V$,

$$b(x, y) = b(y, x).$$

- $\mathcal{C}(V) = \{f : V \to \mathbb{C}\}$
- $\mathcal{C}_c(V)$ is its subset of finite supported functions.
- The Hilbert space

$$\ell^2(V,m) = \{f \in \mathcal{C}(V), \quad \sum_{x\in V} m(x)|f(x)|^2 < \infty\}$$

endowed with the following inner product:

$$(f,g)_m = \sum_{x\in V} m(x)f(x)\overline{g(x)}.$$

Laplacian and Dichlet Laplacian:
For a weighted connected directed graph $(V, \vec{E}, b)$, we introduce the combinatorial Laplacians:

- We define the Laplacian Δ on $\mathcal{C}_c(V)$ by:

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \in V_x^+} b(x,y)\big(f(x) - f(y)\big).$$

- The Dirichlet Laplacian Δ_U^D, where U is a subset of V, is defined for $f \in \mathcal{C}_c(U)$ and $g : V \to \mathbb{C}$ the extension of f to V by setting $f = 0$ outside U by:

$$\Delta_U^D(f) = \Delta(g)|_U.$$

Adjoint of operator A with domain $D(A)$

Proposition

A formal adjoint of Δ is the operator Δ' defined on $C_c(V)$ by:

$$\Delta' f(x) = \frac{1}{m(x)} \left(\sum_{y\in V} b(x,y) f(x) - \sum_{y \in V} b(y,x) f(y) \right).$$

The operator Δ' can be expressed as a Schrödinger operator with potential $q(x) = \dfrac{1}{m(x)} \sum_{y\in V} \big(b(x,y) - b(y,x)\big),\ x, y \in V$

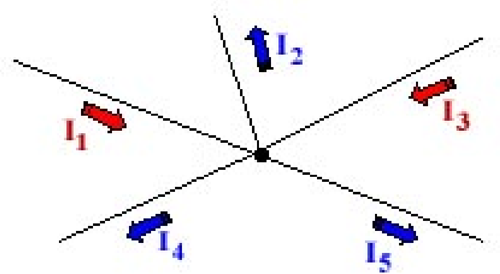
$$\Delta' f(x) = \frac{1}{m(x)} \sum_{y\in V} b(y,x) \big(f(x) - f(y)\big) + q(x) f(x).$$

We introduce the Assumption (β)

The Assumption (β) consists on :

$$\text{for all } x \in V, \quad \beta^+(x) = \beta^-(x).$$

The Assumption (β) is natural, it looks like the Kirchhoff's law in the electrical networks.



Corolary

If the Assumption (β) is satisfied, then

$$\forall f \in \mathcal{C}_c(V), \quad \Delta' f(x) = \frac{1}{m(x)} \sum_{y \in V} b(y,x)\big(f(x) - f(y)\big).$$

Commentaire

The domain of the adjoint Δ^* of Δ is given by:

$$D(\Delta^*) = \left\{ f \in \ell^2(V,m),\ \Delta' f(x) \in \ell^2(V,m) \right\}.$$

Lemma

(Green formula) We suppose that f and g are two founctions of $\mathcal{C}_c(V)$. Then

$$(\Delta f, g)_m + \overline{(\Delta g, f)}_m = \sum_{(x,y)\in\vec{E}} b(x,y)\big(f(x)-f(y)\big)\big(\overline{g(x)-g(y)}\big).$$

Definition

The numerical range of an operator T with domain $D(T)$, denoted by $W(T)$ is the non-empty set

$$W(T) = \{(Tf, f), \quad f \in D(T), \quad \| f \| = 1\}.$$

Proposition

Δ is a closable operator.

Commentaire

The closure of Δ is the operator $\overline{\Delta}$ whose domain and action are

- $D(\overline{\Delta}) = \Big\{ f \in \ell^2(V, m), \quad \exists\ (f_n)_{n\in\mathbb{N}} \in \mathcal{C}_c(V), \ f_n \to f \text{ and } (\Delta f_n)_n \text{ converge } \Big\}$
- $\Delta f_n \to \overline{\Delta} f, \quad f \in D(\overline{\Delta}).$

Plan

1. Graphs and operators
   - Functional spaces
   - Non self-adjoint Laplacians
   - Assumption (β) and overview on Laplacians
   - Numerical range and applications

2. Cheeger inequality

3. Absence of essential spectrum by Cheeger Theorem

Isoperimetric constants

We define the Cheeger constants on $\Omega \subset V$:

$$h(\Omega) = \inf_{\substack{U \subset \Omega \\ finite}} \frac{b(\partial_E U)}{m(U)} \text{ et } \tilde{h}(\Omega) = \inf_{\substack{U \subset \Omega \\ finite}} \frac{b(\partial_E U)}{\beta^+(U)}$$

we define in addition

$$m_\Omega = \inf \left\{ \frac{\beta^+(x)}{m(x)}, \quad x \in \Omega \right\}$$

$$M_\Omega = \sup \left\{ \frac{\beta^+(x)}{m(x)}, \quad x \in \Omega \right\}$$

The edge boundary

Define for a finite subset U of V, the edge boundary of U

$$\partial_E U = \left\{(x,y) \in \vec{E} : \ (x \in U, \ y \in U^c) \ \text{ or } \ (x \in U^c, \ y \in U)\right\}$$

its measure is given by:

$$b(\partial_E U) = \sum_{(x,y)\in\partial_E U} b(x,y).$$

The Cheeger Theorem

Theorem

Let $\Omega \subset V$, the bottom of the real part of $W(\Delta_\Omega^D)$ satisfies the following control:

$$\frac{h^2(\Omega)}{8} \ \leq \ M_\Omega \nu(\Delta_\Omega^D) \ \leq \ M_\Omega h(\Omega) \tag{1}$$

Proposition

$$m_\Omega \frac{\tilde{h}^2(\Omega)}{8} \ \leq \ \nu(\Delta_\Omega^D). \tag{2}$$

Plan

1. Graphs and operators
   - Functional spaces
   - Non self-adjoint Laplacians
   - Assumption (β) and overview on Laplacians
   - Numerical range and applications
2. Cheeger inequality
3. Absence of essential spectrum by Cheeger Theorem

The essential spectrum σ_{ess} of a closed operator A is: the set of all complex numbers for which the range $R(A-\lambda)$ is not closed or $R(A-\lambda)$ is closed and $\dim\ker(A-\lambda)=\infty$.

Definition

A filtration of $G=(V,\vec{E})$ is a sequence of finite connected subgraphs $\{G_n=(V_n,\vec{E}_n),\quad n\in\mathbb{N}\}$ such that $G_n\subset G_{n+1}$ and:

$$\bigcup_{n\geq 1} V_n = V.$$

Let us denote

$$m_\infty = \lim_{n\to\infty}\ m_{V_n^c}$$

$$M_\infty = \lim_{n\to\infty}\ M_{V_n^c}$$

The Cheeger constant at infinity is defined by:

$$h_\infty = \lim_{n\to\infty} h(V_n^c).$$

$$\eta^{ess}(\overline{\Delta}) = \inf\{\mathcal{R}e\lambda : \quad \lambda \in \sigma_{ess}(\overline{\Delta})\}.$$

Theorem

The essential spectrum of $\overline{\Delta}$ satisfies:

$$\frac{h_\infty^2}{8} \ \leq \ M_\infty \eta^{ess}(\overline{\Delta})$$

and

$$m_\infty \frac{\tilde{h}_\infty^2}{8} \ \leq \ \eta^{ess}(\overline{\Delta}). \tag{3}$$

Definition

G is called with heavy ends if $m_\infty = \infty$.

Theorem

The essential spectrum of $\overline{\Delta}$ on a heavy end graph G with $\tilde{h}_\infty > 0$, is empty.

M. Balti
Non self-adjoint Laplacians on a directed graph.
Filomat, (2017).

Thank you for your attention