Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 6 **Due on Thursday 12/17/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let $\mathcal{A} := \{a, b\}$ and $\Omega := \overline{Orb(\omega)} \subseteq \mathcal{A}^{\mathbb{Z}}$ be the orbit closure of $\omega \in \mathcal{A}^{\mathbb{Z}}$ defined by

$$\omega(n) := \begin{cases} a, & n \leq 0, \\ b, & n \geq 1, \end{cases} \qquad n \in \mathbb{Z}.$$

(a) Prove that Ω is an isolated point in $\mathcal{J}$.
(b) Show that there exists a sequence periodic $\omega_n \in \mathcal{A}^{\mathbb{Z}}$ such that (ω_n) converges to ω in $\mathcal{A}^{\mathbb{Z}}$.
(c) Is Ω weakly aperiodic or strongly aperiodic?

Exercise 2 (4 points)**.** Let (X, G) be a dynamical system. Prove that if $E \subseteq C(X)$ is dense and $\mu_n(f) \to \mu(f)$ for all $f \in E$, then $\mu_n \rightharpoonup \mu$.

Exercise 3 (4 points)**.** Consider the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ over a finite alphabet $\mathcal{A}$. For $k \in \mathbb{N}$ and $u \in \mathcal{A}^{2k+1}$, define

$$O(u) := \left\{ \omega \in \mathcal{A}^{\mathbb{Z}} \,\middle|\, \omega|_{\{-k,\dots,k\}} = u \right\},$$

which is a basis for the topology on $\mathcal{A}^{\mathbb{Z}}$. Let $\mu_n, \mu \in \mathcal{M}^1(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$, $n \in \mathbb{N}$. Prove that the following assertions are equivalent.

(i) $\mu_n \rightharpoonup \mu$ in the vague topology,
(ii) $\mu_n(O(u)) \to \mu(O(u))$ for all $u \in \mathcal{A}^{2k+1}$ and $k \in \mathbb{N}$.

Hint: Use the Theorem of Stone-Weierstrass.

Exercise 4 (4 points)**.** Let $\mathcal{A} := \{a, b\}$ and $\omega, \rho \in \mathcal{A}^{\mathbb{Z}}$ be defined by

$$\omega(n) := \begin{cases} a, & n \leq 0, \\ b, & n \geq 1, \end{cases} \qquad \rho(n) := \begin{cases} b, & n \leq 0, \\ a, & n \geq 1, \end{cases} \qquad n \in \mathbb{Z},$$

Consider the subshift $\Omega := \overline{Orb(\omega)} \cup \overline{Orb(\rho)}$.

(a) Determine the set $\mathcal{M}^1(\Omega, \mathbb{Z})$.
(b) Let $\mu \in \mathcal{M}^1(\Omega, \mathbb{Z})$. Construct a sequence of periodic $\Omega_n \in \mathcal{J}$ with unique $\mathbb{Z}$-invariant probability measure μ_n such that
  - $\Omega_n \to \Omega$ in $\mathcal{J}$,
  - $\mu_n \rightharpoonup \mu$.

Hint: (a) Figure out on which set any $\mu \in \mathcal{M}^1(\Omega, \mathbb{Z})$ can be supported. (b) You can use the following statement: For every $\lambda \in [0, 1]$, there are $p_n \leq q_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ such that

- $\frac{p_n}{q_n} \to \lambda$,
- $p_n \to \infty$,
- $q_n - p_n \to \infty$,

where $\frac{p_n}{q_n}$ is a reduced fractions.