Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 11 **Due on Thursday 02/04/2021 at 10.00 am**

Exercise 1 (4 points)**.** For $V \in C(\mathcal{A}^{\mathbb{Z}})$ is real-valued, consider the Hamiltonian H over $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z})$ is given by

$$(H_\omega \psi)(n) = \psi(n+1) + \psi(n-1) + V(n^{-1}\omega)\psi(n), \qquad \omega \in \mathcal{A}^{\mathbb{Z}}.$$

Specifically, $\mathcal{R} = \{0, 1\}$ with $t_1 \equiv 1$ and $t_0 := \frac{1}{2}V$.

Let $p \in \mathcal{A}^{N+1}$ be a finite word and $\omega_p := p^\infty$. Prove that

$$\sigma(H_\Omega) = \sigma(H_{\omega_p}) = \overline{\bigcup_{x \in [0, \frac{2\pi}{N+1})} \sigma\big(A(\omega_p, x)\big)}$$

holds where $A(\omega_p, x)$ is an $(N+1) \times (N+1)$ matrix given by

$$A(\omega_p, x) = \begin{pmatrix} V(\omega_p) & 1 & 0 & \dots & 0 & e^{i(N+1)x} \\ 1 & V(1^{-1}\omega_p) & 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \dots & & & V((N-1)^{-1}\omega_p) & 1 \\ e^{-i(N+1)x} & 0 & \dots & 0 & 1 & V(N^{-1}\omega_p) \end{pmatrix}.$$

Exercise 2 (4 points)**.** Let $p \in \mathcal{A}^{N+1}$ be a finite word and $\omega_p := p^\infty$. Consider the matrices $A(\omega_p, \theta)$ for $\theta \in [0, 2\pi)$ defined by

$$A(\omega_p, \theta) = \begin{pmatrix} V(\omega_p) & 1 & 0 & \dots & 0 & e^{i\theta} \\ 1 & V(1^{-1}\omega_p) & 1 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \dots & & & V((N-1)^{-1}\omega_p) & 1 \\ e^{-i\theta} & 0 & \dots & 0 & 1 & V(N^{-1}\omega_p) \end{pmatrix}.$$

Prove that there is a polynomial $P(\lambda) = \sum_{j=0}^{N+1} a_j \lambda^j$ such that each a_j is independent of θ and the following holds.

(a) The characteristic polynomial $\chi_\theta(\lambda) := det\big(\lambda - A(\omega_p, \theta)\big)$ satisfies

$$\chi_\theta(\lambda) = P(\lambda) - 2\cos(\theta).$$

(b) The equality $\sigma(H_{\omega_p}) = \big\{\lambda \in \mathbb{R} \,\big|\, |P(\lambda)| \leq 2\big\}$ holds.

Exercise 3 (4 points)**.** Let $p \in \mathcal{A}^{N+1}$ be a finite word and $\omega_p := p^\infty$ with $p \in \mathcal{A}^{N+1}$. Consider the matrices $A(\omega_p, \theta)$ for $\theta \in [0, 2\pi)$ defined in the previous Exercise 2. Prove the following statements.

(a) For $\theta \in [0, \pi]$, let $\lambda_0^\theta \leq \lambda_1^\theta \leq \ldots \leq \lambda_N^\theta$ be the eigenvalues of $A(\omega_p, \theta)$. Prove that
  - if N is even, then

$$\lambda_N^0 > \lambda_N^\theta > \lambda_N^\pi \geq \lambda_{N-1}^\pi > \lambda_{N-1}^\theta > \lambda_{N-1}^0 \geq \lambda_{N-2}^0 > \lambda_{N-1}^\theta > \ldots \geq \lambda_0^0 > \lambda_0^\theta > \lambda_0^\pi,$$

  - if N is odd, then

$$\lambda_N^0 > \lambda_N^\theta > \lambda_N^\pi \geq \lambda_{N-1}^\pi > \lambda_{N-1}^\theta > \lambda_{N-1}^0 \geq \lambda_{N-2}^0 > \lambda_{N-1}^\theta > \ldots \geq \lambda_0^\pi > \lambda_0^\theta > \lambda_0^0.$$

- We have

$$\sigma(H_{\omega_p}) = \bigcup_{j=0}^{N} I_j$$

where the intervals $I_j := [\lambda_j^0, \lambda_j^\pi]$ (where we use the convention that $[a, b] = [b, a]$ if $b < a$) can touch at most at their boundaries.

Exercise 4 (4 points)**.** Let $\mathcal{A} := \{a, b\}$, $\omega_n := (ba^{2n+1})^\infty \in \mathcal{A}^\mathbb{Z}$ and

$$\omega(n) := \begin{cases} a, & n \neq 0, \\ b, & n = 0, \end{cases} \qquad n \in \mathbb{Z}.$$

Let H be the Hamiltonian defined by

$$(H_\rho \psi)(n) = \psi(n+1) + \psi(n-1) + V(n^{-1}\rho)\psi(n), \qquad \rho \in \mathcal{A}^\mathbb{Z},$$

where

$$V(\omega) := \begin{cases} 0, & \omega(0) = a, \\ 2, & \omega(0) = b. \end{cases}$$

(a) Compute the distance of the dynamical systems $\overline{Orb(\omega)}$ and $Orb(\omega_n)$ with respect to the Hausdorff metric δ_H defined in the lecture.
(b) Prove that $d_H(\sigma(H_{\omega_n}), \sigma(H_\omega))$ tends to zero if $n \to \infty$ by providing a suitable upper bound for the Hausdorff distance of the spectra.

Hint: According to Sheet 7, Exercise 3, we have $Orb(\omega_n) \to \overline{Orb(\omega)}$.

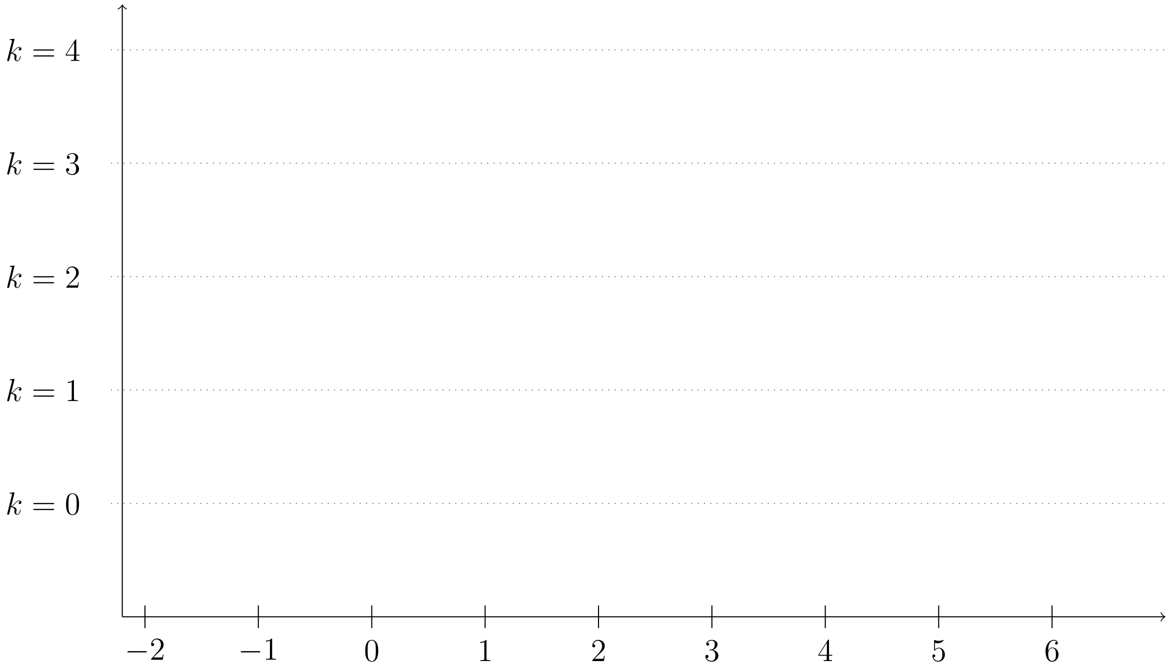
Bonus exercise 1 (2 points)**.** Let S be the Fibonacci substitution defined by $S(a) := ab$ and $S(b) := a$ for $\mathcal{A} := \{a, b\}$. Let $V : \mathcal{A}^\mathbb{Z} \to \mathbb{R}$ be defined by

$$V(\omega) := \begin{cases} 0, & \omega(0) = b, \\ 4, & \omega(0) = a. \end{cases}$$

Hamiltonian H over $(\mathcal{A}^\mathbb{Z}, \mathbb{Z})$ is given by

$$(H_\omega \psi)(n) = \psi(n+1) + \psi(n-1) + V(n^{-1}\omega)\psi(n), \qquad \omega \in \mathcal{A}^\mathbb{Z}.$$

Compute numerically (with your favorite computer tool) the spectrum of H_{ω_i} for $i = 0, 1, 2, 3, 4$ where $\omega_i := S^i(b^\infty)$ and draw the spectra into the following plot:



$k=4$
$k=3$
$k=2$
$k=1$
$k=0$
-2
-1
0
1
2
3
4
5
6