Schrödinger operators over dynamical systems

Winter semester 2020 **Dr. S. Beckus**

Sheet 1 **Due on Thursday 11/12/2020 at 10.00 am**

Exercise 1 (4 points)**.** Let $\mathcal{A}$ be a finite set equipped with the discrete topology and G be a countable group. Let $K_n \subseteq G, n \in \mathbb{N}$ be finite such that

- $K_n \subsetneq K_{n+1}$ for all $n \in \mathbb{N}$ and
- $\bigcup_{n\in\mathbb{N}} K_n = G$.

Prove the following statements.

(a) The map $d : \mathcal{A}^G \times \mathcal{A}^G \to [0,\infty)$ defined by

$$d(\omega,\rho) := \min\left\{1, \inf\left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N},\, \omega|_{K_n} = \rho|_{K_n} \right\}\right\}$$

is an ultra metric on $\mathcal{A}^G$.

(b) The metric space $(\mathcal{A}^G, d)$ is a totally bounded and complete metric space (and hence compact).

Exercise 2 (4 points)**.** Let $\mathcal{A}$ be a finite set equipped with the discrete topology and G be a countable group. Show that $(\mathcal{A}^G, G)$ is a (topological) dynamical system.

Exercise 3 (4 points)**.** Let $G := \mathbb{C} \times \mathbb{R}$ be equipped with the composition and inverse defined by

$$(v,s)(w,t) := (v + w, s + t - \Im(v \cdot \overline{w}))\,,$$
$$(v,s)^{-1} := (-v,-s),$$

for $(v,s),(w,t) \in G$. Furthermore, define $\|\cdot\| : G \to [0,\infty)$ by

$$\|(v,s)\| := \left(|v|^4 + 4s^2\right)^{\frac{1}{4}}\,.$$

Prove the following assertions

(a) The set G is a non-abelian group if equipped with the composition and inverse defined before.

(b) The map $\|\cdot\| : G \to [0,\infty)$ satisfies the following.
- $\|\cdot\|$ is definite, namely $\|(v,s)\| > 0$ if and only if $(v,s) \neq (0,0)$
- For all $g \in G$, we have $\|g\| = \|g^{-1}\|$.
- $\|\cdot\|$ satisfies the triangle inequality, namely

$$\|gh\| \le \|g\| + \|h\|, \qquad g,h \in G.$$

Exercise 4 (4 points)**.** Let $G := \mathbb{C} \times \mathbb{R}$ be equipped with the Euclidean topology (i.e. $|(v_1,v_2,s)| := \sqrt{v_1^2 + v_2^2 + s^2}$) composition and inverse defined by

$$(v,s)(w,t) := (v + w, s + t - \Im(v \cdot \overline{w}))\,,$$
$$(v,s)^{-1} := (-v,-s),$$

for $(v,s),(w,t) \in G$. Furthermore, define $\|\cdot\| : G \to [0,\infty)$ by

$$\|(v,s)\| := \left(|v|^4 + 4s^2\right)^{\frac{1}{4}}.$$

Prove the following assertions.

(a) The map $d : G \times G \to [0,\infty)$ defined by

$$d(g,h) := \|g^{-1}h\|, \qquad g,h \in G,$$

is a left-invariant metric on G, namely d is a metric and $d(hg_1, hg_2) = d(g_1, g_2)$ holds for all $g_1, g_2, h \in G$.

(b) For $g \in G$ with $|g|^2 \leq \frac{1}{2}$, we have

$$|g| \leq \|g\| \leq \sqrt{2|g|}$$

where $|\cdot| : G \to [0,\infty)$ is the absolute value on $G = \mathbb{C} \times \mathbb{R}$. Is topology induced on G by d equals to the Euclidean topology on $G = \mathbb{C} \times \mathbb{R}$?

(c) The group G with the induced topology of d is a topological group.

Bonus exercise 1 (1 point)**.** Let T be a topological space. A function $f : T \to \mathbb{R}$ is called *lower semi-continuous* at $t_0 \in T$ if for every $r < f(t_0)$, there is a neighborhood U of t_0 such that $r < f(t)$ for all $t \in U$.

Let $\{f_i\}_{i\in I}$ be a family of lower semi-continuous functions $f_i : T \to \mathbb{R}$. If $\sup_{i\in I} |f_i(t)| < \infty$ for every $t \in T$, then $f(t) := \sup_{i\in I} f_i(t)$ is lower semi-continuous.

Bonus exercise 2 (1 point)**.** Let $\mathcal{A}$ be a finite set equipped with the discrete topology and G be a countable group. Show that the balls in $\mathcal{A}^G$ are clopen (i.e. closed and open).