GEOMETRIE IN POTSDAM

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Wolfgang Pauli and Niels Bohr watching a spinning top (1954)

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Preface

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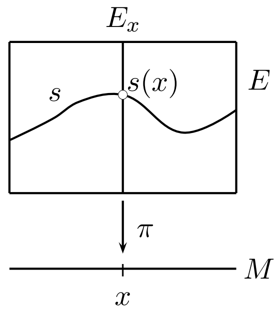
1.1. Differential operators

We start by looking at linear differential operators on manifolds. Later we will specialize to Laplace and Dirac-type operators.

Let M be an n-dimensional differentiable manifold and let $\pi : E \to M$ be a vector bundle. Recall that a **section** of E is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.
We define

$$C^\infty(M, E) := \{\text{smooth sections of } E\}.$$



Definition 1.1.1. Let E and F be $\mathbb{K}$-vector bundles over M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
A **differential operator of order (at most) *k*** is a linear mapping $P : C^\infty(M, E) \to C^\infty(M, F)$ such that for any local coordinate system $x^1, \ldots, x^n$ on $U \subset M$ and any local trivializations $E|_U \xrightarrow{\approx} U \times \mathbb{K}^p$ and $F|_U \xrightarrow{\approx} U \times \mathbb{K}^q$ there exist smooth maps $A^\alpha : U \to \mathrm{Mat}(q \times p, \mathbb{K})$ such that

$$Pv|_U = \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x^1 \ldots \partial^{\alpha_n} x^n}$$

for all $v \in C^\infty(M, E)$. Here $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Notation 1.1.2. We define

$$\mathscr{D}\!\mathit{iff}_k(E, F) := \{P : C^\infty(M, E) \to C^\infty(M, F) \mid P \text{ differential operator of order} \le k\}.$$

The vector spaces $\mathscr{D}\!\mathit{iff}_k(E, F)$ form a filtration,

$$\cdots \supset \mathscr{D}\!\mathit{iff}_{k+1}(E, F) \supset \mathscr{D}\!\mathit{iff}_k(E, F) \supset \cdots \supset \mathscr{D}\!\mathit{iff}_0(E, F) = C^\infty(M, \mathrm{Hom}(E, F)).$$

Example 1.1.3. Let M be a Riemannian manifold, let $E = M \times \mathbb{R}$ be the trivial real line bundle and $F = TM$ be the tangent bundle of M. The **gradient** is a differential operator of order 1 from E to F, grad $\in \mathscr{D}\!\mathit{iff}_1(M \times \mathbb{R}, TM)$. In local coordinates, we have:

$$\operatorname{grad} v = \sum_i g^{ij}(x) \frac{\partial v}{\partial x^i} \frac{\partial}{\partial x^j}.$$

Comparing the coefficients in this formula with the coefficients A^α in Definition 1.1.1, we find:

$$A^{(0,\dots,\overset{i}{\overset{\downarrow}{1}},\dots,0)} = \left(g^{1i}, \dots, g^{ni}\right)^\top, \qquad A^{(0\dots,0)} = (0, \dots, 0)^\top.$$

Example 1.1.4. Let M be a Riemannian manifold, let $E = TM$ be the tangent bundle of M and let $F = M \times \mathbb{R}$ be the trivial real line bundle. The **divergence** is a differential operator of order 1 from E to F, div $\in \mathscr{D}\!\mathit{iff}_1(TM, M \times \mathbb{R})$. In local coordinates, we have for $Y = \sum_i y^i \frac{\partial}{\partial x^i}$:

$$\operatorname{div}(Y) = \sum_i \frac{\partial y^i}{\partial x^i} + \sum_{ij} \Gamma^i_{ij} y^j.$$

The coefficients are

$$A^{(0,\dots,\overset{i}{\overset{\downarrow}{1}},\dots,0)} = (0, \dots, \overset{i}{\overset{\downarrow}{1}}, \dots, 0), \qquad A^{(0\dots,0)} = \Big(\sum_i \Gamma^i_{i1}, \dots, \sum_i \Gamma^i_{in}\Big).$$

Here Γ^k_{ij} denote the Christoffel symbols of the Riemannian metric with respect to the coordinates $x^1, \dots, x^n$.

Example 1.1.5. Let M be a Riemannian manifold and consider $E = \Lambda^m T^*M$ and $F = \Lambda^{m+1} T^*M$. The **exterior derivative** d is a differential operator of order 1 from E to F, $d \in \mathscr{D}\!\mathit{iff}_1(\Lambda^m T^*M, \Lambda^{m+1} T^*M)$.

Example 1.1.6. Let E be an arbitrary vector bundle over M with connection ∇ and let $F = T^*M \otimes E$. Then ∇ is a differential operator of first order from E to F.

Example 1.1.7. Consider $M = \mathbb{C}$ and the trivial complex line bundle $E = F := M \times \mathbb{C}$. Then

$$\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right), \quad \text{where } z = x + iy,$$

is a differential operator of order 1 from E to F.

Remark 1.1.8. Let $E, F, G \to M$ be vector bundles over a smooth manifold M. If $P \in \mathscr{D}\!\mathit{iff}_k(E, F)$ and $Q \in \mathscr{D}\!\mathit{iff}_l(F, G)$ then $Q \circ P \in \mathscr{D}\!\mathit{iff}_{k+l}(E, G)$.

Example 1.1.9. Let M be a Riemannian manifold and consider $E = G = M \times \mathbb{R}$ and $F = TM$. Then $\Delta = -\mathrm{div} \circ \mathrm{grad} \in \mathscr{Diff}_2(E, G)$, where Δ denotes the **Laplace-Beltrami operator**.

For a given differential operator $P \in \mathscr{Diff}_k(E, F)$ and a covector $\xi \in T_x^*M$ we construct a linear mapping $\sigma_k(P, \xi) : E_x \to F_x$ as follows: We choose a smooth function $f : M \to \mathbb{R}$ such that $f(x) = 0$ and $df(x) = \xi$. We then set for $e \in E_x$:

$$\sigma_k(P, \xi) \cdot e := \frac{1}{k!} P\big(f^k \tilde{e}\big)|_x, \tag{1.1}$$

where $\tilde{e} \in C^\infty(M, E)$ is any extension of e, i.e. $\tilde{e}(x) = e$. As we shall see, this definition is independent of the choice of $\tilde{e}$ and f. In local coordinates and local trivializations, we compute:

$$\begin{aligned} \sigma_k(P, \xi) \cdot e &= \frac{1}{k!} \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial^{|\alpha|} \left(f^k \tilde{e}\right)}{\partial^{\alpha_1} x^1 \cdots \partial^{\alpha_n} x^n}(x) \\ &= \frac{1}{k!} \sum_{|\alpha| = k} A^\alpha(x) \frac{\partial^{|\alpha|} \left(f^k\right)}{\partial^{\alpha_1} x^1 \cdots \partial^{\alpha_n} x^n}(x) \cdot \tilde{e}(x) \\ &= \sum_{|\alpha| = k} A^\alpha(x) \cdot \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \cdot e\,. \end{aligned} \tag{1.2}$$

The second equality holds because by assumption $f(x) = 0$, so that all terms vanish in which f^k is differentiated less than k times. The last equality holds by a similar argument: If one of the factors in f^k is differentiated more than once, there is another factor which remains without differentiation and hence vanishes at x.
Since the right hand side of (1.2) is independent of the choice of $\tilde{e}$ and f, so is the left hand side. This shows that $\sigma_k(P, \xi)$ is well defined by (1.1).
For any $\xi \in T_x^*M$, we have constructed a homomorphism $\sigma_k(P, \xi) : E_x \to F_x$. Thus we have $\sigma_k(P, \cdot) \in \mathrm{Hom}(\pi^*E, \pi^*F)$, where $\pi : T^*M \to M$ is the projection to the foot point.

Definition 1.1.10. Let $E, F \to M$ be vector bundles over a smooth manifold M and let $P \in \mathscr{Diff}_k(E, F)$. Then $\sigma_k(P, \cdot) \in \mathrm{Hom}(\pi^*E, \pi^*F)$ is called the **principal symbol** of the operator P.

Remark 1.1.11. The principal symbol $\sigma_k(P, \cdot)$ contains the coefficients of the highest order derivatives of $P \in \mathscr{Diff}_k(E, F)$. In particular, we have

$$\sigma_k(P, \xi) = 0 \text{ for all } \xi \in T^*M \quad \Leftrightarrow \quad A^\alpha = 0 \text{ for all } |\alpha| = k \quad \Leftrightarrow \quad P \in \mathscr{Diff}_{k-1}(E, F).$$

In other words: The sequence

$$0 \to \mathscr{D}\!\mathit{iff}_{k-1}(E,F) \longrightarrow \mathscr{D}\!\mathit{iff}_{k}(E,F) \xrightarrow{\sigma_k(P,\cdot)} \mathrm{Hom}\,(\pi^*E, \pi^*F)$$

is exact.

Warning. In the literature, the definition of $\sigma_k(P,\xi)$ often contains another factor i^k.

Convention. If k is clear from the context, we will write $\sigma(P,\xi)$ instead of $\sigma_k(P,\xi)$.

Example 1.1.13. We compute the principal symbol of the gradient, see Example 1.1.3. We fix a covector $\xi \in T_x^*M$. Since $E_x = \mathbb{R}$, we have to apply $\sigma(\mathrm{grad}\,,\xi)$ to a real number, say 42. A convenient extension of 42 to a smooth section of E is the constant function $x \mapsto 42$.
Let $f : M \to \mathbb{R}$ be a smooth function such that $f(x) = 0$ and $df(x) = \xi$. By the definition of $\sigma(\mathrm{grad}\,,\xi)$, we have[1]

$$\begin{aligned} \sigma(\mathrm{grad}\,,\xi)\cdot 42 &= \mathrm{grad}\,(f\cdot 42)(x) \\ &= 42\cdot \mathrm{grad}\, f(x) \\ &= 42\cdot df(x)^\sharp \\ &= 42\cdot \xi^\sharp. \end{aligned}$$

In short: $\sigma(\mathrm{grad}\,,\xi) = \xi^\sharp$. Here $\sharp : T^*M \to TM$ denotes the "musical isomorphism" induced by the Riemannian metric.

Example 1.1.14. We compute the principal symbol of the divergence. Here $E_x = T_xM$, so we have to apply $\sigma(\mathrm{div}\,,\xi)$ to a tangent vector $Y \in T_xM$. Let $\tilde{Y}$ be a smooth vector field such that $\tilde{Y}(x) = Y$. Again let $f : M \to \mathbb{R}$ be a smooth function such that $f(x) = 0$ and $df(x) = \xi$. Then we have

$$\begin{aligned} \sigma(\mathrm{div}\,,\xi)Y &= \mathrm{div}\,\big(f\cdot\tilde{Y}\big)(x) \\ &= \underbrace{f(x)}_{=0}\cdot\mathrm{div}\,\big(\tilde{Y}\big)(x) + \big\langle \mathrm{grad}\, f(x), \tilde{Y}(x)\big\rangle \\ &= \big\langle \xi^\sharp, Y\big\rangle \\ &= \xi(Y). \end{aligned}$$

Thus $\sigma(\mathrm{div}\,,\xi) = \xi$.

[1] Here $\xi^\sharp$ is the vector in T_xM dual to $\xi \in T_x^*M$ with respect to the Riemannian metric, i.e., for any $Y \in T_xM$ we have $\langle \xi^\sharp, Y\rangle = \xi(Y)$.

Example 1.1.15. We compute the principal symbol of the exterior derivative d. Let $\omega \in \Lambda^k T_x^* M$ and extend ω to a smooth k-form $\tilde{\omega} \in \Omega^k(M)$ such that $\tilde{\omega}(x) = \omega$. Then we have

$$\begin{aligned}\sigma(d,\xi)\omega &= d\big(f \cdot \tilde{\omega}\big)(x) \\ &= \big(df \wedge \tilde{\omega} + f \cdot d\tilde{\omega}\big)\big|_x \\ &= df(x) \wedge \omega + \underbrace{f(x)}_{=0} \cdot d\tilde{\omega}\big|_x \\ &= \xi \wedge \omega.\end{aligned}$$

Hence $\sigma(d,\xi) = \xi \wedge \cdot$.

Example 1.1.16. We compute the principal symbol of a connection ∇ on a vector bundle E. Let $e \in E_x$ and extend e to a smooth section $\tilde{e} \in C^\infty(M,E)$ such that $\tilde{e}(x) = e$. Then we have

$$\begin{aligned}\sigma(\nabla,\xi)e &= \nabla\big(f\tilde{e}\big)\big|_x \\ 2 &= \big(df \otimes \tilde{e} + f \cdot \nabla\tilde{e}\big)\big|_x \\ &= df(x) \otimes e + \underbrace{f(x)}_{=0} \cdot \big(\nabla\tilde{e}\big)\big|_x \\ &= \xi \otimes e.\end{aligned}$$

Thus $\sigma(\nabla,\xi) = \xi \otimes \cdot$.

Example 1.1.17. We compute the principal symbol of $P = \bar{\partial}$. Let $z \in \mathbb{C}$ and extend z to a section $\tilde{z} \in C^\infty(M,E)$ such that $\tilde{z}(x) = z$. Then we have

$$\begin{aligned}\sigma\big(\bar{\partial},\xi\big)z &= \bar{\partial}\big(f \cdot \tilde{z}\big) \\ &= \frac{1}{2}\Big(\frac{\partial}{\partial x}\big(f \cdot \tilde{z}\big) + i\,\frac{\partial}{\partial y}\big(f \cdot \tilde{z}\big)\Big)\Big|_x \\ &= \frac{1}{2}\Big(\frac{\partial f}{\partial x}(x) \cdot z + i\,\frac{\partial f}{\partial y}(x) \cdot z\Big) \\ &= \frac{1}{2}\Big(\xi\Big(\frac{\partial}{\partial x}\Big) + i\,\xi\Big(\frac{\partial}{\partial y}\Big)\Big) \cdot z \\ &= \xi(\bar{\partial})z.\end{aligned}$$

In the next to last equality, we used $\xi\left(\frac{\partial}{\partial x}\right) = df|_x\left(\frac{\partial}{\partial x}\right) = \frac{\partial f}{\partial x}(x)$. Thus $\sigma(\bar{\partial},\xi) = \xi(\bar{\partial})$.

Remark 1.1.18. Let E, F, G be vector bundles over a smooth manifold M, let $P \in \mathscr{Diff}_k(E,F)$ and $Q \in \mathscr{Diff}_l(F,G)$. Then we have

$$\sigma_{k+l}(Q \circ P, \xi) = \sigma_l(Q,\xi) \circ \sigma_k(P,\xi).$$

Example 1.1.19. We compute the principal symbol of the Laplace-Beltrami operator Δ from the principal symbols of div and grad:

$$\sigma_2(\Delta,\xi) = \sigma_2(-\mathrm{div} \circ \mathrm{grad}\,,\xi) = -\sigma_1(\mathrm{div}\,)\cdot\sigma_1(\mathrm{grad}\,) = -\xi\big(\xi^\sharp\big) = -|\xi^\sharp|^2 = -|\xi|^2.$$

In the following let M be a Riemannian manifold and $E, F \to M$ be Riemannian or Hermitian vector bundles.

Lemma 1.1.20. *For any $P \in \mathscr{D}\!\mathit{iff}_k(E,F)$ there is a unique $P^* \in \mathscr{D}\!\mathit{iff}_k(F,E)$ such that*

$$\int_M \langle Pu, v\rangle_F \, dvol = \int_M \big\langle u, P^*v\big\rangle_E \, dvol \tag{1.3}$$

holds for all $u \in C^\infty(M,E)$, $v \in C^\infty(M,F)$ with compact supports.

Definition 1.1.21. The operator $P^* \in \mathscr{D}\!\mathit{iff}_k(F,E)$ satisfying (1.3) is called the **operator formally adjoint** to P.

Proof of Lemma 1.1.20. Uniqueness:
Let $u \in C^\infty(M,E)$ and $v \in C^\infty(M,F)$ be sections with supports in a coordinate neighborhood $U \subset M$. Using local trivializations of E and F over U by orthonormal frames we compute:

$$\begin{aligned}
\int_U \langle Pu, v\rangle_F \, dvol &= \int_U \Big\langle \sum_{|\alpha|\le k} A^\alpha \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}, v\Big\rangle \sqrt{\det g}\, dx \\
&= \sum_{|\alpha|\le k} \int_U \Big\langle \frac{\partial^{|\alpha|}u}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}, \sqrt{\det g}\big(A^\alpha\big)^\top v\Big\rangle\, dx \\
&\overset{\substack{\text{integ.}\\ \text{by parts}}}{=} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \int_U \Big\langle u, \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}\Big(\sqrt{\det g}\big(A^\alpha\big)^\top v\Big)\Big\rangle\, dx \\
&= \int_U \Big\langle u, \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}\big(\sqrt{\det g}(A^\alpha)^\top v\big)}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n} \frac{1}{\sqrt{\det g}}\Big\rangle\, dvol.
\end{aligned}$$

Thus

$$P^*v = \frac{1}{\sqrt{\det g}} \sum_{|\alpha|\le k} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}\big(\sqrt{\det g}(A^\alpha)^\top v\big)}{\partial^{\alpha_1}x_1\cdots\partial^{\alpha_n}x_n}. \tag{1.4}$$

Now let $v \in C^\infty(M,F)$ be an arbitrary section with compact support. We choose an open covering of M with local trivializations and a partition of unity subordinated to it. Then v is a finite sum of sections of the form considered above. Since P^* is required to be linear, it is uniquely determined by the local formula (1.4).

Existence: Let $v \in C^\infty(M,F)$ be a smooth section with compact support. We now use formula (1.4) to define P^*v if v has support in U. For general v we use a partition of unity to write it as a sum of sections with supports contained in coordinate patches. It is tedious but straightforward to check that this definition is independent of the choice of coordinates, trivializations, and partition of unity. $\square$

Remark 1.1.22. For any $P \in \mathscr{D}\!\mathit{iff}_k(E,F)$ we have $(P^*)^* = P$. This is obvious from equation (1.3) and the uniqueness of the formal adjoint.

Example 1.1.23. The gradient is a first order operator $\mathrm{grad} : C^\infty(M) \to C^\infty(M,TM)$, so grad^* maps vector fields to functions. By definition, for any function $u \in C^\infty(M)$ and any vector field $Y \in C^\infty(M,TM)$, both with compact support, we have

$$\begin{aligned}
\int_M u(x)\big(\mathrm{grad}^* Y\big)(x)\, dvol(x) &= \int_M \langle \mathrm{grad}\, u(x), Y(x)\rangle\, dvol(x) \\
&= \int_M \big(\mathrm{div}\,(uY) - u\,\mathrm{div}\, Y\big)\, dvol(x) \\
&= -\int_M u\,\mathrm{div}\, Y\, dvol(x).
\end{aligned}$$

In the last step we used the Gauß divergence theorem. Thus $\mathrm{grad}^* = -\mathrm{div}$. By Remark 1.1.22 we then also have $\mathrm{div}^* = -\mathrm{grad}$.

Remark 1.1.24. For differential operators $P \in \mathscr{D}\!\mathit{iff}_k(E,F)$ and $Q \in \mathscr{D}\!\mathit{iff}_l(F,G)$ we have

$$(Q \circ P)^* = P^* \circ Q^*.$$

Definition 1.1.25. Let M be a Riemannian manifold and let E be a Riemannian or Hermitian vector bundle over M.
Then $P \in \mathscr{D}\!\mathit{iff}_k(E,E)$ is called **formally self-adjoint** iff $P = P^*$.

Example 1.1.26. We consider the bundle $E = M \times \mathbb{R}$ and $P = \Delta$. We then have

$$P^* = -(\mathrm{div} \circ \mathrm{grad})^* = -\mathrm{grad}^* \circ \mathrm{div}^* = -\mathrm{div} \circ \mathrm{grad} = P.$$

Thus the Laplace-Beltrami operator is formally self-adjoint.

Lemma 1.1.27. *Let M be a Riemannian manifold. Let E and F be Riemannian or Hermitian vector bundles over M, and let $P \in \mathscr{D}\!\mathit{iff}_k(E,F)$. Then for any $\xi \in T^*M$ we have*

$$\sigma_k\big(P^*,\xi\big) = (-1)^k \sigma_k(P,\xi)^*. \tag{1.5}$$

Proof. Since only the terms of order k contribute to the principal symbol $\sigma_k(P,\cdot)$, we write

$$Pu = \sum_{|\alpha|=k} A^\alpha(x)\, \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n}\, u + \text{l.o.t.}$$

where "l.o.t." stands for "lower order terms". By (1.4) the adjoint of P is given by

$$\begin{aligned} P^*v &= \frac{1}{\sqrt{\det g}} \sum_{|\alpha|=k} (-1)^k \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} \Big(\sqrt{\det g}\, A^\alpha(x)^\top v\Big) + \text{l.o.t.} \\ &= \frac{1}{\sqrt{\det g}} \sum_{|\alpha|=k} (-1)^k \sqrt{\det g}\, A^\alpha(x)^\top \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} + \text{l.o.t.} \\ &= \sum_{|\alpha|=k} (-1)^k\, A^\alpha(x)^\top \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1}x_1 \cdots \partial^{\alpha_n}x_n} + \text{l.o.t.} \end{aligned}$$

Thus, by the local formula (1.2) for the principal symbol, we have

$$\sigma_k\big(P^*,\xi\big) = (-1)^k \sum_{|\alpha|=k} \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} A^\alpha(x)^\top = (-1)^k \sigma_k(P,\xi)^*. \qquad \square$$

1.2. Sobolev spaces

Next we introduce Sobolev spaces which are important function spaces for the analysis of the kind of differential operators which we will be considering later.

Definition 1.2.1. Let M be a Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle. For

$$u,v \in C_c^\infty(M,E) := \big\{ w \in C^\infty(M,E) \,\big|\, \operatorname{supp}(w) \Subset M \big\}$$

we define the ***L^2*-scalar product** by

$$(u,v)_{L^2} := \int_M \big\langle u(x), v(x) \big\rangle\, dvol(x).$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean or Hermitian scalar product in E_x. The L^2-norm is then given by

$$\|u\|_{L^2} = \sqrt{(u,u)_{L^2}} = \Big(\int\limits_M |u(x)|^2 \, dvol(x) \Big)^{1/2}.$$

Remark 1.2.2. The formally adjoint operator P^* of P is therefore characterized by the property

$$(Pu, v)_{L^2} = \big(u, P^* v\big)_{L^2}, \qquad \forall u \in C_c^\infty(M,E), v \in C_c^\infty(M,F).$$

Definition 1.2.3. We define $L^2(M,E)$ as the completion of $C_c^\infty(M,E)$ with respect to the L^2-norm:

$$L^2(M,E) := \overline{C_c^\infty(M,E)}^{\|\cdot\|_{L^2}},$$

i.e. elements of $L^2(M,E)$ are equivalence classes of Cauchy sequences in the vector space $(C_c^\infty(M,E), \|\cdot\|_{L^2})$.

Remark 1.2.4. $L^2(M,E)$ is a Hilbert space with scalar product $(\cdot,\cdot)_{L^2}$.

Let $M = T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ be the n-dimensional torus. Let $E = M \times \mathbb{C}$ be the trivial complex line bundle. Sections in E are complex functions on the torus. We may also consider them as $2\pi\mathbb{Z}^n$-periodic functions on $\mathbb{R}^n$. For any $k \in \mathbb{Z}^n$ put

$$u_k(x) := (2\pi)^{-\frac{n}{2}} e^{i\langle k,x\rangle}.$$

Then $u_k : \mathbb{R}^n \to \mathbb{C}$ is smooth and $u_k(x + 2\pi p) = u_k(x)$ for any $p \in \mathbb{Z}^n$. Hence u_k descends to a (smooth) function $u_k : T^n \to \mathbb{C}$ on the torus.

Fact. The family $(u_k)_{k\in\mathbb{Z}^n}$ is an orthonormal Hilbert space basis of $L^2(T^n, T^n \times \mathbb{C})$. In particular, any $v \in L^2(T^n, T^n \times \mathbb{C})$ can be uniquely written as

$$v = \sum_{k\in\mathbb{Z}^n} \underbrace{\hat{v}(k)}_{\in\mathbb{C}} \cdot u_k,$$

where

$$\hat{v}(k) = (v, u_k)_{L^2} = \int\limits_{T^n} v(x)\overline{u_k(x)} \, dx = (2\pi)^{-\frac{n}{2}} \int\limits_{T^n} v(x) e^{-i\langle k,x\rangle} \, dx$$

is the k-th Fourier coefficient of v. Moreover,

$$\begin{aligned}
\|v\|_{L^2}^2 = (v,v)_{L^2} &= \left(\sum_k \hat{v}(k)u_k, \sum_l \hat{v}(l)u_l\right) \\
&= \sum_{k,l} \hat{v}(k)\overline{\hat{v}(l)}\underbrace{(u_k,u_l)}_{=\delta_{kl}} \\
&= \sum_k \hat{v}(k)\overline{\hat{v}(k)} \\
&= \sum_k \left|\hat{v}(k)\right|^2. \qquad (1.6)
\end{aligned}$$

The equation (1.6) is known as **Parseval's theorem**.

For any $v \in C^1(T^n)$ we find, using integration by parts:

$$\widehat{\frac{\partial v}{\partial x_j}}(k) = \left(\frac{\partial v}{\partial x_j}, u_k\right)_{L^2} = -\left(v, \frac{\partial u_k}{\partial x_j}\right)_{L^2} = -\left(v, ik_j\, u_k\right)_{L^2} = i\,k_j(v,u_k)_{L^2} = i\,k_j\,\hat{v}(k).$$

Hence the Fourier transform turns derivatives into multiplications by polynomials.

Example 1.2.6. For $v \in C^2(T^n)$ we have:

$$\widehat{\Delta v}(k) = -\sum_j \widehat{\frac{\partial^2 v}{\partial x_j^2}} = -\sum_j (i\,k_j)^2\hat{v}(k) = \sum_j k_j^2\,\hat{v}(k) = |k|^2\hat{v}(k).$$

Definition 1.2.7. Let $s \in \mathbb{R}$, and let $v, w \in C^\infty(T^n)$. We set

$$(v,w)_{H^s} := \sum_{k\in\mathbb{Z}^n} \hat{v}(k)\cdot\overline{\hat{w}(k)}\cdot\left(1+|k|^2\right)^s.$$

The norm induced by $(v,w)_{H^s}$ is given by

$$\|v\|_{H^s} = \sqrt{(v,v)_{H^s}} = \Big(\sum_{k\in\mathbb{Z}^n} \left|\hat{v}(k)\right|^2\cdot\left(1+|k|^2\right)^s\Big)^{1/2}.$$

Furthermore we define the Sobolev space

$$H^s(T^n) := \overline{C^\infty(T^n)}^{\|\cdot\|_{H^s}}.$$

The Hilbert space $(H^s(T^n), (\cdot,\cdot)_{H^s})$ is called the **Sobolev space** of degree s.

Remark 1.2.8. For $s = 0$ we have $\|\cdot\|_{H^0} = \|\cdot\|_{L^2}$ by Parseval's theorem, and hence $H^0(T^n) = L^2(T^n)$.

Remark 1.2.9. If $s_1 \leq s_2$ then $\left(1 + |k|^2\right)^{s_1} \leq \left(1 + |k|^2\right)^{s_2}$, hence $\|v\|_{H^{s_1}} \leq \|v\|_{H^{s_2}}$ for all $v \in C^\infty(T^n)$. Thus the identity on $C^\infty(T^n)$ extends uniquely to a continuous embedding $H^{s_2}(T^n) \hookrightarrow H^{s_1}(T^n)$.

Example 1.2.10. Let $v \in C^\infty(T^n)$. We compute:

$$\begin{aligned}
\|v\|_{H^1}^2 &= \sum_k \left|\hat{v}(k)\right|^2 \left(1 + |k|^2\right) \\
&= \sum_k \left|\hat{v}(k)\right|^2 + \sum_k |k|^2 \hat{v}(k)\overline{\hat{v}(k)} \\
&\overset{1.2.6}{=} \|v\|_{L^2}^2 + \sum_k \widehat{\Delta v}(k)\overline{\hat{v}(k)} \\
&= \|v\|_{L^2}^2 + (\Delta v, v)_{L^2} \\
&= \|v\|_{L^2}^2 + \|\operatorname{grad} v\|_{L^2}^2. \qquad (1.7)
\end{aligned}$$

In the last equation, we used $\operatorname{div}^* = -\operatorname{grad}$. Hence the Sobolev norm $\|\cdot\|_{H^1}$ controls the derivatives up to first order in the square mean.

More generally, the Sobolev norm $\|\cdot\|_{H^s}$ controls the derivatives up to order s in the square mean. To have pointwise control on the derivatives, we need another norm:

Definition 1.2.11. For $s \in \mathbb{N}_0$ and $v \in C^s(T^n)$ put

$$\|v\|_{C^s} := \max_{|\alpha| \leq s} \max_{x \in T^n} \left| \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}(x) \right|.$$

By definition, the C^s-norm $\|\cdot\|_{C^s}$ controls the first s derivatives pointwise.

Remark 1.2.12. The C^s-norm $\|\cdot\|_{C^s}$ turns the space $C^s(T^n)$ of s-times continuously differentiable functions on T^n into a Banach space.

Let us look again at Example 1.2.10. Here we have:

$$\begin{aligned}
|v(x)|^2 &\leq \|v\|_{C^1}^2, \\
|\operatorname{grad} v(x)|^2 &= \sum_{j=1}^n \left| \frac{\partial v}{\partial x_j}(x) \right|^2 \leq n \cdot \|v\|_{C^1}^2.
\end{aligned}$$

We thus have

$$\begin{aligned}\|v\|_{H^1}^2 &\stackrel{(1.7)}{=} \int\limits_{T^n} \left(|v(x)|^2 + |\operatorname{grad} v(x)|^2\right)\, dx\\ &\le (n+1)\cdot \|v\|_{C^1}^2 \int\limits_{T^n} dx\\ &= (2\pi)^n (n+1)\cdot \|v\|_{C^1}^2\end{aligned}$$

and hence

$$\|\cdot\|_{H^1} \le (2\pi)^{\frac{n}{2}} \sqrt{n+1}\cdot \|\cdot\|_{C^1}.$$

Similarly, for $s \in \mathbb{N}$ we find constants $C(n,s)$ such that

$$\|\cdot\|_{H^s} \le C(n,s)\cdot \|\cdot\|_{C^s}.$$

These estimates yield continuous embeddings of the C^s-spaces into Sobolev spaces,

$$C^s(T^n) \hookrightarrow H^s(T^n).$$

The following theorem yields embeddings of Sobolev spaces into C^k-spaces:

Theorem 1.2.13 (Sobolev embedding theorem). *Let $l \in \mathbb{N}$ and $s > l + \frac{n}{2}$. Then for each $u \in H^s(T^n)$ the Fourier series*

$$\sum_{k\in\mathbb{Z}^n} \hat{u}(k)\cdot u_k$$

converges absolutely in the C^l-norm and therefore defines a function $u \in C^l(T^n)$. Moreover, there is a constant $C = C(n,l,s) > 0$ such that

$$\|u\|_{C^l} \le C\cdot \|u\|_{H^s}, \qquad \forall\, u \in H^s(T^n).$$

Hence the above Fourier expansion defines a continuous embedding

$$H^s(T^n) \hookrightarrow C^l(T^n).$$

Proof. A direct computation gives

$$\begin{aligned}
\|u\|_{C^l}^2 &= \Big\| \sum_k \hat{u}(k) \cdot u_k \Big\|_{C^l}^2 \\
&\leq \left(\sum_k |\hat{u}(k)| \cdot \|u_k\|_{C^l} \right)^2 \\
&\leq C_1 \left(\sum_k |\hat{u}(k)| \cdot \left(1+|k|^2\right)^{\frac{l}{2}} \right)^2 \\
&= C_1 \left(\sum_k |\hat{u}(k)| \cdot \left(1+|k|^2\right)^{\frac{s}{2}} \cdot \left(1+|k|^2\right)^{\frac{l-s}{2}} \right)^2 \\
&\overset{\text{CS}}{\leq} C_1 \underbrace{\left(\sum_k |\hat{u}(k)|^2 \cdot \left(1+|k|^2\right)^{s} \right)}_{=\|u\|_{H^s}^2} \cdot \sum_k \left(1+|k|^2\right)^{l-s}
\end{aligned}$$

For $s - l > \frac{n}{2}$, we have $\sum\limits_k \left(1+|k|^2\right)^{l-s} \leq c(n) \int_{\mathbb{R}^n} (1+|k|^2)^{-(s-l)}\, dk < \infty$, hence there exists a constant $C(n,l,s)$ such that

$$\|u\|_{C^l}^2 \leq C(n,l,s) \cdot \|u\|_{H^s}^2. \qquad \square$$

Theorem 1.2.14 (Rellich embedding theorem). *Let $s_1 < s_2$. Then the embedding*

$$H^{s_2}(T^n) \hookrightarrow H^{s_1}(T^n)$$

is a compact operator, i.e., every bounded sequence in $H^{s_2}(T^n)$ has a convergent subsequence in $H^{s_1}(T^n)$.

Proof. Let $(u_m)_{m\in\mathbb{N}}$ be a bounded sequence in $H^{s_2}(T^n)$. Then we have for all $m \in \mathbb{N}$:

$$\|u_m\|_{H^{s_1}}^2 = \sum_{k\in\mathbb{Z}^n} \left(1+|k|^2\right)^{s_1} |\hat{u}_m(k)|^2 \leq \sum_{k\in\mathbb{Z}^n} \left(1+|k|^2\right)^{s_2} |\hat{u}_m(k)|^2 = \|u_m\|_{H^{s_2}}^2 \leq C.$$

In particular, for all $k \in \mathbb{Z}^n$ and all $m \in \mathbb{N}$

$$\left(1+|k|^2\right)^{s_1} |\hat{u}_m(k)|^2 \leq C$$

and thus for all $k \in \mathbb{Z}^n$ and all $m \in \mathbb{N}$

$$\left(1+|k|^2\right)^{\frac{s_1}{2}} |\hat{u}_m(k)| \leq \sqrt{C}.$$

Hence there exists a subsequence $(u_{m_i})_{m_i}$ such that the sequence $(\left(1+|k|^2\right)^{\frac{s_1}{2}} \cdot \hat{u}_{m_i}(k))_i$ of complex numbers converges for a fixed $k \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is countable we can successively take such a subsequence for any $k \in \mathbb{Z}^n$. After taking a diagonal subsequence, again denoted by $(u_m)_m$ we get that the sequence $(\left(1+|k|^2\right)^{\frac{s_1}{2}} \cdot \hat{u}_m(k))_m$ converges as $m \to \infty$ for every $k \in \mathbb{Z}^n$.

We show that $(u_m)_m$ is a Cauchy sequence in $H^{s_1}(T^n)$: Let $\varepsilon > 0$. Choose $R > 0$ sufficiently large such that

$$\left(1+R^2\right)^{s_1-s_2} < \frac{\varepsilon}{8C}.$$

Choose $N \in \mathbb{N}$ sufficiently large such that for all $m, l > N$ we have

$$\sum_{\substack{k\in\mathbb{Z}^n \\ |k|\le R}} \left(1+|k|^2\right)^{s_1} |\hat{u}_m(k) - \hat{u}_l(k)|^2 < \frac{\varepsilon}{2}. \tag{1.8}$$

This is possible because the (finitely many) summands come from the Cauchy sequences of complex numbers constructed above. We decompose $\|u_m - u_l\|^2_{H^{s_1}}$ into

$$\|u_m - u_l\|^2_{H^{s_1}} = \sum_{\substack{k\in\mathbb{Z}^n \\ |k|\le R}} \left(1+|k|^2\right)^{s_1} |\hat{u}_m(k) - \hat{u}_l(k)|^2 + \sum_{\substack{k\in\mathbb{Z}^n \\ |k|> R}} \left(1+|k|^2\right)^{s_1} |\hat{u}_m(k) - \hat{u}_l(k)|^2 .$$

According to (1.8), for $m, l \ge N$, the first sum can be estimated by $\frac{\varepsilon}{2}$. In the second sum we have $|k| > R$. Using $s_1 < s_2$, we get for the summands of the second sum:

$$(1+|k|^2)^{s_1} = (1+|k|^2)^{s_1-s_2}(1+|k|^2)^{s_2} < (1+R^2)^{s_1-s_2}(1+|k|^2)^{s_2} < \frac{\varepsilon}{8C}(1+|k|^2)^{s_2}.$$

Thus, for $m, l \ge N$, we have:

$$\begin{aligned}
\|u_m - u_l\|^2_{H^{s_1}} &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{8C} \sum_{\substack{k\in\mathbb{Z}^n \\ |k|> R}} \left(1+|k|^2\right)^{s_2} |\hat{u}_m(k) - \hat{u}_l(k)|^2 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{8C} \|u_m - u_l\|^2_{H^{s_2}} \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{8C} \Big(\underbrace{\|u_m\|_{H^{s_2}}}_{\le\sqrt{C}} + \|u_l\|_{H^{s_2}} \Big)^2 \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{8C}\, 4C = \varepsilon.
\end{aligned}$$

Thus $(u_m)_m$ is a Cauchy sequence in the Banach space $H^{s_1}(T^n)$ and hence converges. □

Definition 1.2.15. A sequence $(u_m)_{m\in\mathbb{N}}$ in a Hilbert space H **converges weakly** to $u \in H$ iff

$$(u_m, v) \to (u, v) \quad \text{as } m \to \infty \quad \forall\, v \in H.$$

One then writes $u_m \rightharpoonup u$.

Lemma 1.2.16. *a) Weak limits are unique.*

b) If $u_m \to u$ then $u_m \rightharpoonup u$, i.e., convergence implies weak convergence.

c) If $\dim(H) = \infty$ then weak convergence does not imply convergence.

d) If $A : H_1 \to H_2$ is a bounded linear operator and $u_m \rightharpoonup u$ then $Au_m \rightharpoonup Au$.

e) In a Hilbert space each bounded sequence has a weakly convergent subsequence.

Proof. a) Suppose $u_m \rightharpoonup u$ and $u_m \rightharpoonup u'$. Now for every $v \in H$ we have

$$(u - u', v) = \lim_{m\to\infty} (u_m, v) - \lim_{m\to\infty} (u_m, v) = 0,$$

hence $u - u' = 0$.

b) Suppose $u_m \to u$ in H. For any $v \in H$, the Cauchy-Schwarz inequality yields $|(u_m, v) - (u, v)| = |(u_m - u, v)| \leq \|u_m - u\| \cdot \|v\| \to 0$.

c) Let H be infinite dimensional and let $(e_m)_{m\in\mathbb{N}}$ be an orthonormal basis of H. For any $v \in H$, we have $\sum_m |(e_m, v)|^2 \leq |v|^2 < \infty$. Thus $(e_m, v) \to 0$ for all $v \in H$ and hence $e_m \rightharpoonup 0$. But since $\|e_m\| = 1$, we have $\|e_m\| \to 1$ so that $e_m \nrightarrow 0$ in H. If $(e_m)_m$ converged in H then, by a) and b), the limit would have to be 0. Thus $(e_m)_m$ does not converge at all in H.

d) We have $(Au_m, v)_{H_2} = (u_m, A^*v)_{H_1} \to (u, A^*v)_{H_1} = (Au, v)_{H_2}$ for all $v \in H_2$.

e) For a proof of this part see e.g. Section 14 in [6]. □

Corollary 1.2.17. *Let $s_1 < s_2$. For each bounded sequence in $H^{s_2}(T^n)$ there exist a subsequence $(u_m)_{m\in\mathbb{N}}$ and an element $u \in H^{s_2}(T^n)$ such that*

$$\begin{aligned} u_m &\rightharpoonup u \quad \text{in } H^{s_2}(T^n), \\ u_m &\to u \quad \text{in } H^{s_1}(T^n). \end{aligned}$$

Proof. We use Lemma 1.2.16. Let $(u_m)_m$ be a bounded sequence in $H^{s_2}(T^n)$. Then, by e), after passing to a subsequence, $u_m \rightharpoonup u$ in $H^{s_2}(T^n)$ for some $u \in H^{s_2}(T^n)$. Passing again to a subsequence, Rellich's theorem 1.2.14 yields $u_m \to v$ in $H^{s_1}(T^n)$ for some $v \in H^{s_1}(T^n)$. By b), $u_m \rightharpoonup v$ in $H^{s_1}(T^n)$. By d) we also have $u_m \rightharpoonup u$ in $H^{s_1}(T^n)$. Since by a) weak limits are unique, it follows that $u = v$. In particular, $u \in H^{s_1}(T^n)$ and $u_m \to u$ in $H^{s_1}(T^n)$. □

Remark 1.2.18. For vector-valued functions $u = (u_1, \dots, u_l) \in C^\infty(T^n, \mathbb{C}^l)$ we put

$$\|u\|_{H^s}^2 := \sum_{j=1}^{l} \|u_j\|_{H^s}^2.$$

Then for the corresponding Sobolev space $H^s(T^n, \mathbb{C}^l)$ the embedding theorems of Sobolev 1.2.13 and Rellich 1.2.14 still hold.

Obviously, $C^\infty(T^n, \mathbb{R}^l) \subset C^\infty(T^n, \mathbb{C}^l)$. Since $H^s(T^n, \mathbb{R}^l) \subset H^s(T^n, \mathbb{C}^l)$ is a closed (real) subspace, the theorems of Sobolev and Rellich also hold for $H^s(T^n, \mathbb{R}^l)$.

In the following, let M be a compact manifold and $E \to M$ be a $\mathbb{K}$-vector bundle, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $x : M \supset U \to U' \subset \mathbb{R}^n$ be a chart such that $\overline{U'} \subset (0, 2\pi) \times \cdots \times (0, 2\pi)$. Then $\pi : \mathbb{R}^n \to T^n = \mathbb{R}^n/_{2\pi\mathbb{Z}^n}$ maps U' diffeomorphically onto some $U'' \subset T^n$.

By restricting U if necessary, we can assume that $E|_U$ is trivial, i.e., there exists a diffeomorphism $\phi : E|_U \to U \times \mathbb{K}^l$ such that

$$\begin{tikzcd} E|_U \arrow[rr, "\phi"] \arrow[dr] & & U \times \mathbb{K}^l \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes and ϕ is a linear map in each fiber.
Put $\tilde{x} := \pi \circ x$. For $v \in C^\infty(M, E)$ with $\mathrm{supp}(v) \subset U$ define $v_{\phi,x} \in C^\infty(T^n, \mathbb{K}^l)$ by

$$v_{\phi,x} := \begin{cases} \mathrm{pr}_2 \circ \phi \circ v \circ \tilde{x}^{-1} & \text{on } U'', \\ 0 & \text{on } T^n - U''. \end{cases}$$

Since M is compact, we can cover M by finitely many open sets U_j such that for each U_j we have a chart $x_{(j)}$ and a local trivialization $\phi_{(j)} : E|_{U_j} \to U_j \times \mathbb{K}^l$ as above.
Choose a partition of unity $\chi_j \in C^\infty(M, \mathbb{R})$ subordinate to the covering (U_j), i.e.,

$$0 \le \chi_j \le 1, \quad \mathrm{supp}(\chi_j) \subset U_j, \quad \sum_j \chi_j = 1.$$

For $u \in C^\infty(M, E)$ put

$$\|u\|_{H^s}^2 := \sum_j \left\| (\chi_j\, u)_{\phi_{(j)}, x_{(j)}} \right\|_{H^s(T^n)}^2, \tag{1.9}$$

$$H^s(M, E) := \overline{C^\infty(M, E)}^{\|\cdot\|_{H^s}}. \tag{1.10}$$

The definition of $\|u\|_{H^s}^2$ depends on the choice of the U_j, $x_{(j)}$, $\phi_{(j)}$ and χ_j. But one can check (not difficult but technical) that different choices of these lead to equivalent

H^s-norms. Thus the Sobolev space $H^s(M,E)$ does not depend upon the various choices in its construction.

Similarly, one can define C^l-norms $\|\cdot\|_{C^l}$ on $C^\infty(M,E)$ and put

$$C^l(M,E) := \overline{C^\infty(M,E)}^{\|\cdot\|_{C^l}} \tag{1.11}$$

Then we get continuous embeddings $C^l(M,E) \hookrightarrow H^l(M,E)$ and the theorems of Sobolev 1.2.13 and Rellich 1.2.14 still hold for $H^s(M,E)$ and $C^l(M,E)$, i.e.,

$$\begin{aligned} H^{s_2}(M,E) \hookrightarrow H^{s_1}(M,E) &\text{ is compact,} && \text{if } s_1 < s_2, \\ H^s(M,E) \hookrightarrow C^l(M,E) &\text{ is continuous,} && \text{if } s > l + \frac{n}{2}. \end{aligned}$$

Remark 1.2.19. The embedding

$$H^s(M,E) \hookrightarrow C^l(M,E), \quad (s > l + \frac{n}{2})$$

is compact.

Proof. Choose s' with $s > s' > l + \frac{n}{2}$. Then we have the embedding

$$H^s(M,E) \xrightarrow{1.2.14} H^{s'}(M,E) \xrightarrow{1.2.13} C^l(M,E)\,.$$

The first embedding is compact by Theorem 1.2.14 whereas the second is continuous by Theorem 1.2.13. Since both embeddings are extensions of the identity, the composition coincides with the embedding $H^s(M,E) \hookrightarrow C^l(M,E)$ in Theorem 1.2.13. Obviously, the composition of a compact map with a continuous map is again compact. $\square$

Let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle with connection $\nabla : C^\infty(M,E) \to C^\infty(M, T^*M \otimes E)$. The connection ∇ and the Levi-Civita connection on T^*M induce a connection

$$\nabla : C^\infty(M, T^*M \otimes E) \to C^\infty(M, T^*M \otimes T^*M \otimes E)$$

on $T^*M \otimes E$.
We define

$$\nabla^2 := \nabla \circ \nabla : C^\infty(M,E) \to C^\infty\left(M, T^*M^{\otimes 2} \otimes E\right)\,.$$

Iterating this construction, we obtain differential operators ∇^k of order k:

$$\nabla^k : C^\infty(M,E) \to C^\infty\left(M, T^*M^{\otimes k} \otimes E\right).$$

The C^l-norm defined above is equivalent to the norm

$$\|u\|_{C^l} = \max_{k=0,\ldots,l} \max_{x \in M} \left|\nabla^k u(x)\right|$$

on $C^\infty(M,E)$. With this new definition of C^l-norms we obtain the same spaces $C^l(M,E)$ as defined in (1.11). Similarly, we can define the H^s-norms for nonnegative integers s by

$$\|u\|_{H^s}^2 = \sum_{k=0}^{s} \left\|\nabla^k u\right\|_{L^2}^2 = \sum_{k=0}^{s} \int_M \left|\nabla^k u(x)\right|^2 dvol(x) \tag{1.12}$$

and we obtain the same spaces $H^s(M,E)$ as defined in (1.10)

Lemma 1.2.20. *Let M be a compact manifold, let $E,F\to M$ be $\mathbb{K}$-vector bundles, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Then every $P\in \mathscr{Diff}_k(E,F)$ extends uniquely to bounded linear maps*

$$C^{l+k}(M,E)\to C^l(M,F), \quad l\in\mathbb{N}_0,$$

and

$$H^{s+k}(M,E)\to H^s(M,F), \quad s\in\mathbb{R}.$$

Proof. 1) Choose a Riemannian metric on M and connections ∇ on E and F. For each $P\in \mathscr{Diff}_k(E,F)$ there exist $A_j\in C^\infty(M,\mathrm{Hom}(T^*M^{\otimes j}\otimes E,F))$ such that

$$P=\sum_{j=0}^{k} A_j\circ\nabla^j.$$

Then we have

$$\|Pu\|_{C^l} = \Big\|\sum_{j=0}^{k} A_j\circ\nabla^j u\Big\|_{C^l} \le \sum_{j=0}^{k}\left\|A_j\circ\nabla^j u\right\|_{C^l}.$$

Now we estimate $\left\|A_j\circ\nabla^j u\right\|_{C^l}$ by $\|u\|_{C^{k+l}}$. For $\nu\in\{0,\dots,l\}$ and any $j\in\{0,\dots,k\}$, we have:

$$\begin{aligned}\left|\nabla^\nu\left(A_j\nabla^j u\right)\right| &\le C_1\sum_{\mu=0}^{\nu}\left|\nabla^\mu A_j\nabla^{\nu-\mu+j}u\right|\\ &\le C_2\sum_{\mu=0}^{\nu}\left|\nabla^{\nu-\mu+j}u\right|\\ &\le C_3\sum_{\mu=0}^{\nu}\|u\|_{C^{\nu-\mu+j}}\\ &\le C_4\|u\|_{C^{k+l}}.\end{aligned}$$

Thus $\|Pu\|_{C^l} \le C\|u\|_{C^{k+l}}$. Hence P extends to a bounded linear map

$$P : C^{k+l}(M,E) \to C^l(M,F).$$

The extension is unique because $C^\infty(M,E)$ is dense in $C^{k+l}(M,E)$.

2) The proof that P extends uniquely to a bounded linear map

$$P : H^{s+k}(M,E) \to H^s(M,F)$$

is similar. $\square$

1.3. Laplace-type and Dirac-type operators

Let M be a Riemannian manifold and let $E \to M$ be a Riemannian or Hermitian vector bundle.

Definition 1.3.1. A differential operator $P \in \mathscr{Diff}_2(E,E)$ is called to be of **Laplace-type** iff

$$\sigma_2(P,\xi) = -|\xi|^2 \cdot \mathrm{id}_{E_x} \qquad \text{for all } x \in M \text{ and } \xi \in T_x^*M.$$

Example 1.3.2. By Example 1.1.19, the Laplace-Beltrami operator Δ is a Laplace-type operator, acting function, i.e., on sections of the trivial line bundle $E = M \times \mathbb{R}$.

Example 1.3.3. Let E be any Riemannian or Hermitian vector bundle and let ∇ be a connection on E. We put $P := \nabla^* \circ \nabla$. Then by Remark 1.1.18, Lemma 1.1.27 and Example 1.1.16, for any covector $\xi \in T_x^*M$ and any $e, e' \in E_x$ we have:

$$\begin{aligned}
\big\langle \sigma_2(P,\xi)e, e'\big\rangle_{E_x} &= \big\langle \sigma_1(\nabla^*,\xi) \circ \sigma_1(\nabla,\xi)e, e'\big\rangle_{E_x} \\
&= -\big\langle \sigma_1(\nabla,\xi)^* \circ \sigma_1(\nabla,\xi)e, e'\big\rangle_{E_x} \\
&= -\big\langle \sigma_1(\nabla,\xi)e, \sigma_1(\nabla,\xi)e'\big\rangle_{T_x^*M\otimes E_x} \\
&= -\langle \xi \otimes e, \xi \otimes e'\rangle_{T_x^*M\otimes E_x} \\
&= -\langle \xi,\xi\rangle_{T_x^*M} \cdot \langle e, e'\rangle_{E_x} \\
&= -|\xi|^2 \cdot \langle e, e'\rangle_{E_x}.
\end{aligned}$$

Since this holds for all $e, e' \in E_x$, we conclude that $\sigma_2(P,\xi) = -|\xi|^2 \cdot \mathrm{id}_E$. Thus the operator P is of Laplace-type. It is called the **connection Laplacian**.

Remark 1.3.4. For any $D \in \mathscr{Diff}_1(E,E)$ the operator $\widetilde{P} := \nabla^*\nabla + D$ is also of Laplace-type. This is obvious, since the first order operator D does not contribute to the principal symbol $\sigma_2(\widetilde{P},\cdot)$.

Lemma 1.3.5. *For every formally self-adjoint Laplace-type operator $P \in \mathscr{D}\!\mathit{iff}_2(E,E)$ there exists a unique metric connection ∇ on E such that*

$$P = \nabla^*\nabla + K,$$

where $K \in C^\infty(M, \mathrm{symEnd}(E))$.

Proof. Let $\widetilde{\nabla}$ be any metric connection on E. Then $D := P - \widetilde{\nabla}^*\widetilde{\nabla}$ is formally self-adjoint and we have:

$$\sigma_2(D,\xi) = \sigma_2(P,\xi) - \sigma_2\left(\widetilde{\nabla}^*\widetilde{\nabla},\xi\right) = -|\xi|^2 + |\xi|^2 = 0.$$

Therefore D is actually a first order operator. Thus we can decompose P as

$$P = \widetilde{\nabla}^*\widetilde{\nabla} + D, \tag{1.13}$$

where $\widetilde{\nabla}^*\widetilde{\nabla}$ is of second order and D is a first order operator.

Any other metric connection ∇ on E is of the form

$$\nabla = \widetilde{\nabla} + B,$$

where $B \in C^\infty(M, T^*M \otimes \mathrm{asymEnd}(E))$. Inserting this into (1.13) gives

$$P = (\nabla - B)^*(\nabla - B) + D = \nabla^*\nabla \underbrace{-\nabla^*B - B^*\nabla + B^*B + D}_{=:K}$$

We want to choose B in such a way that K is of order zero. Since B^*B is of order zero we have:

$$\begin{aligned} K \text{ is of order } 0 &\Longleftrightarrow D - \nabla^*B - B^*\nabla \text{ is of order } 0 \\ &\Longleftrightarrow \sigma_1\left(D - \nabla^*B - B^*\nabla, \xi\right) = 0 \quad \text{for all } \xi \in T^*M. \end{aligned} \tag{1.14}$$

We compute

$$\begin{aligned} \left\langle \sigma_1\left(\nabla^*B + B^*\nabla,\xi\right)e, e'\right\rangle &= \left\langle \big(\sigma_1\left(\nabla^*,\xi\right)\circ B + B^*\circ\sigma_1(\nabla,\xi)\big)e, e'\right\rangle \\ &= -\left\langle Be, \sigma_1(\nabla,\xi)e'\right\rangle + \left\langle \sigma_1(\nabla,\xi)e, Be'\right\rangle \\ &= -\left\langle Be, \xi\otimes e'\right\rangle + \left\langle \xi\otimes e, Be'\right\rangle. \end{aligned}$$

Here, we used $\sigma_1(\nabla^*\circ B,\xi) = \sigma_1(\nabla^*,\xi)\circ\sigma_0(B,\xi) = \sigma_1(\nabla^*,\xi)\circ B$ and Lemma 1.1.27.

Now let $b_1, \ldots, b_n$ be an orthonormal basis of T_xM and let $b_1^*, \ldots, b_n^*$ be the dual basis of T_x^*M. We write $Be = \sum_i b_i^* \otimes B_{b_i} e$ and thus obtain:

$$\begin{aligned}
\left\langle \sigma_1\left(\nabla^* B + B^* \nabla, \xi\right) e, e' \right\rangle &= -\left\langle \sum_i b_i^* \otimes B_{b_i} e, \xi \otimes e' \right\rangle + \left\langle \xi \otimes e, \sum_i b_i^* \otimes B_{b_i} e' \right\rangle \\
&= -\sum_i \langle b_i^*, \xi\rangle \left\langle B_{b_i} e, e' \right\rangle + \sum_i \langle b_i^*, \xi\rangle \left\langle e, B_{b_i} e' \right\rangle \\
&= -\sum_i \xi(b_i) \left\langle B_{b_i} e, e' \right\rangle + \sum_i \xi(b_i) \left\langle e, B_{b_i} e' \right\rangle \\
&= -\left\langle B_{\sum_i \xi(b_i) b_i} e, e' \right\rangle + \left\langle e, B_{\sum_i \xi(b_i) b_i} e' \right\rangle \\
&= -\left\langle B_{\xi^\sharp} e, e' \right\rangle + \left\langle e, B_{\xi^\sharp} e' \right\rangle \\
&= \left\langle \left(B_{\xi^\sharp}^* - B_{\xi^\sharp}\right) e, e' \right\rangle \\
&= -2 \left\langle B_{\xi^\sharp} e, e' \right\rangle .
\end{aligned}$$

Hence $\sigma_1\left(\nabla^* B + B^* \nabla, \xi\right) = -2B_{\xi^\sharp}$. Thus by (1.14), we have:

$$\begin{aligned}
K \text{ is of order } 0 \iff \sigma_1\left(D, \xi\right) &= \sigma_1(\nabla^* B + B^* \nabla, \xi) \\
&= -2\, B_{\xi^\sharp} \qquad\qquad \text{for all } \xi \in T^*M.
\end{aligned}$$

Therefore there is only one possible choice for $B \in C^\infty(M, T^*M \otimes \mathrm{asymEnd}(E))$, namely

$$B_X = -\frac{1}{2}\sigma_1\left(D, X^b\right) \qquad \text{for all } X \in TM.$$

This show uniqueness. As to existence, we observe that this choice of B is possible, since by the following remark, the principal symbol $\sigma_1(D, \xi)$ is antisymmetric. ☐

Remark 1.3.6. If D is a formally self-adjoint operator of order k then we have:

$$\sigma_k(D, \xi) = \sigma_k(D^*, \xi) \overset{(1.5)}{=} (-1)^k\, \sigma_k(D, \xi)^* .$$

Hence the principal symbol of D is antisymmetric if the order k is odd.

Definition 1.3.7. Let M be a Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian vector bundles. An operator $D \in \mathscr{Diff}_1(E, F)$ is of **Dirac-type** iff

$$D^* D \in \mathscr{Diff}_2(E, E) \quad \text{and} \quad DD^* \in \mathscr{Diff}_2(F, F)$$

are of Laplace-type.

Remark 1.3.8. Let $E = F$ and $D = D^*$. Then D is of Dirac-type if and only if D^2 is of Laplace-type.

Example 1.3.9. We consider the bundles $E = M \times \mathbb{R}$ and $F = TM$ and the operator $D = \mathrm{grad} \in \mathscr{Diff}_1(E, F)$. By Example 1.1.23, we have $D^* = \mathrm{grad}^* = -\mathrm{div}$. Hence $D^*D = -\mathrm{div}\,\mathrm{grad} \in \mathscr{Diff}_2(M \times \mathbb{R}, M \times R)$ is the Laplace-Beltrami operator which is of Laplace-type by Example 1.3.2.
We check whether $DD^* = -\mathrm{grad}\,\mathrm{div} \in \mathscr{Diff}_2(TM, TM)$ is also an operator of Laplace-type: For $M = \mathbb{R}^n$ with the Euclidean metric we write a tangent vector $v = v^j \frac{\partial}{\partial x^j} \in TM$ as $v = (v^1, \ldots, v^n)$. Then we have:

$$\begin{aligned} -\mathrm{grad}\,\mathrm{div}\,(v) &= -\mathrm{grad} \sum_{j=1}^n \frac{\partial v^j}{\partial x^j} \\ &= -\left(\sum_{j=1}^n \frac{\partial^2 v^j}{\partial x^1 \partial x^j}, \ldots, \sum_{j=1}^n \frac{\partial^2 v^j}{\partial x^n \partial x^j} \right)^\top . \end{aligned}$$

For the principal symbol of $-\mathrm{grad}\,\mathrm{div}$, we thus find:

$$\begin{aligned} \sigma_2(-\mathrm{grad}\,\mathrm{div}\,, \xi)v &= -\left(\sum_{j=1}^n \xi_1 \xi_j v^j, \ldots, \sum_{j=1}^n \xi_n \xi_j v^j \right)^\top \\ &= -\begin{pmatrix} \xi_1\xi_1 & \ldots & \xi_1\xi_n \\ \vdots & \ddots & \vdots \\ \xi_n\xi_1 & \ldots & \xi_n\xi_n \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} . \end{aligned}$$

Thus,

$$\sigma_2(-\mathrm{grad}\,\mathrm{div}\,, \xi) = -\begin{pmatrix} \xi_1\xi_1 & \ldots & \xi_1\xi_n \\ \vdots & & \vdots \\ \xi_n\xi_1 & \ldots & \xi_n\xi_n \end{pmatrix} \neq -\begin{pmatrix} |\xi|^2 & & 0 \\ & \ddots & \\ 0 & & |\xi|^2 \end{pmatrix}$$

for general $\xi \in T^*M$. Hence DD^* is not of Laplace-type and thus D not of Dirac-type.
For a general Riemannian manifold M we have by Examples 1.1.13 and 1.1.14: $\sigma_1(\mathrm{grad}\,, \xi) = \xi^\sharp$ and $\sigma_1(\mathrm{div}\,, \xi) = \xi$ and thus for all $v \in TM$ and all $\xi \in T^*M$

$$\sigma_2(DD^*, \xi)v = -\sigma_1(\mathrm{grad}\,, \xi) \circ \sigma_1(\mathrm{div}\,, \xi)v = -\xi(v)\xi^\sharp \neq -|\xi|^2 v$$

if v is not a multiple of $\xi^\sharp$.

Example 1.3.10. We translate the previous example to differential forms and consider the exterior differential d on $E = \Lambda^0 T^*M$ with values in $F = \Lambda^1 T^*M$. As we have seen, d is not of Dirac-type.
Now we enlarge the bundles to

$$\begin{aligned} E &= \Lambda^{\mathrm{even}} T^*M = \Lambda^0 T^*M \oplus \Lambda^2 T^*M \oplus \Lambda^4 T^*M \oplus \ldots, \qquad \text{and} \\ F &= \Lambda^{\mathrm{odd}} T^*M = \Lambda^1 T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^5 T^*M \oplus \ldots . \end{aligned}$$

On these bundles we consider the **Euler operator**

$$D = d + d^* : C^\infty\big(M, E\big) \to C^\infty\big(M, F\big).$$

We want to show that D is of Dirac-type. We prove that D^*D is of Laplace-type. For each $\xi \in T_x^*M$ and each $\omega \in \Lambda^k T_x^*M$ we have:

$$\begin{aligned}\sigma_2\left(D^*D, \xi\right)\omega &= \sigma_2\left(d^*d + dd^*, \xi\right)\omega \\ &= \big(\sigma_1\left(d^*, \xi\right) \circ \sigma_1(d, \xi) + \sigma_1(d, \xi) \circ \sigma_1\left(d^*, \xi\right)\big)\omega. \end{aligned} \tag{1.15}$$

We have computed earlier in Example 1.1.15 that $\sigma_1(d, \xi) = \xi \wedge (\cdot)$. We also know from (1.5) that $\sigma_1(d^*, \xi) = -\sigma_1(d, \xi)^*$. It remains to compute $\sigma_1(d, \xi)^*$.

For $\xi \in T_x^*M$ and $\omega \in \Lambda^k T_x^*M$ we define $\xi \lrcorner \omega \in \Lambda^{k-1} T_x^*M$ by

$$\xi \lrcorner \omega := \omega\big(\xi^\sharp, \dots\big).$$

<u>Claim:</u> $\xi \lrcorner \cdot$ is the adjoint of $\xi \wedge (\cdot)$.
Let $\xi \in T_x^*M$, $\xi \neq 0$. Let $b_1^* = \xi, b_2^*, \dots, b_n^*$ be an orthogonal basis of T_x^*M. Then we can write each $\omega \in \Lambda^k T_x^*M$ as

$$\begin{aligned}\omega &= \sum_{|I|=k} \omega_I \, b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*, \qquad\qquad I = (1 \le i_1 < \ldots < i_k \le n)\,, \\ &= \sum_{|J|=k-1} \omega_{(1,J)}\, \xi \wedge b_{i_2}^* \wedge \ldots \wedge b_{i_k}^* + \sum_{\substack{|I|=k \\ i_1>1}} \omega_I \, b_{i_1}^* \wedge \ldots \wedge b_{i_k}^* \\ &=: \xi \wedge \omega_\xi + \omega_\xi^\perp.\end{aligned}$$

Thus there is an orthogonal decomposition $\Lambda^k T_x^*M = \xi \wedge \Lambda^{k-1}\big(\xi^\perp\big) \oplus \Lambda^k\big(\xi^\perp\big)$ where $\xi^\perp$ denotes the orthogonal complement of ξ in T_x^*M. Now, on the one hand, we have:

$$\langle \omega, (\xi \wedge \cdot)^* \tau \rangle = \langle \xi \wedge \omega, \tau \rangle = \big\langle \xi \wedge \omega_\xi^\perp, \xi \wedge \tau_\xi \big\rangle = \langle \xi, \xi \rangle \big\langle \omega_\xi^\perp, \tau_\xi \big\rangle.$$

On the other hand, we have:

$$\langle \omega, \xi \lrcorner \tau \rangle = \big\langle \omega, |\xi|^2 \cdot \tau_\xi \big\rangle = \big\langle w_\xi^\perp, |\xi|^2 \tau_\xi \big\rangle.$$

Comparing these two equations yields

$$(\xi \wedge (\cdot))^* = \xi \lrcorner (\cdot)$$

which proves the claim.
For the principal symbol of the codifferential we thus obtain:

$$\sigma_1(d^*, \xi) = -\sigma_1(d, \xi)^* = -(\xi \wedge (\cdot))^* = -\xi \lrcorner (\cdot).$$

Therefore, by (1.15), we have:

$$\begin{aligned}\sigma_2\left(D^*D,\xi\right)\omega &= -\big(\xi\lrcorner\left(\xi\wedge\omega\right)+\xi\wedge\left(\xi\lrcorner\omega\right)\big)\\ &= -\big(\xi\lrcorner\left(\xi\wedge\omega_\xi^\perp\right)+\xi\wedge\left(\xi\lrcorner\omega\right)\big)\\ &= -\big(|\xi|^2\omega_\xi^\perp+|\xi|^2\xi\wedge\omega_\xi\big)\\ &= -|\xi|^2\omega.\end{aligned}$$

Hence $D^*D = d^*d + dd^* =: \Delta_d$ is of Laplace-type. The operator DD^* is also given by $DD^* = d^*d + dd^*$ and the above calculation shows that also DD^* is of Laplace-type. Thus the Euler operator $D = d + d^*$ is of Dirac-type.

Definition 1.3.11. Let $k \in \{0,\dots,n\}$. The operator

$$\Delta_d = D^*D = dd^* + d^*d : C^\infty\big(M,\Lambda^k T^*M\big) \to C^\infty\big(M,\Lambda^k T^*M\big) \tag{1.16}$$

is called the **Hodge Laplacian** in degree k.

For $k = 0$, the Hodge Laplacian Δ_d coincides with the Laplace-Beltrami operator Δ.

Proposition 1.3.12. *Let M be a Riemannian manifold. The Hodge Laplacian Δ_d in degree 1 satisfies the* ***Bochner formula***

$$\Delta_d = \nabla^*\nabla + \mathrm{Ric}\,. \tag{1.17}$$

Here ∇ denotes the Levi-Civita connection of the Riemannian metric and Ric *its Ricci-curvature, considered as an endomorphism field of T^*M.*

Proof. Exercise. □

Remark 1.3.13. If we put $E = F = \Lambda^\bullet T^*M = \Lambda^0T^*M \oplus \Lambda^1T^*M \oplus \Lambda^2T^*M \oplus \dots$ then $D = d + d^*$ is a formally self-adjoint Dirac-type operator on E.

Remark 1.3.14. Let D be a formally self-adjoint Dirac-type operator on a vector bundle E. Then we have:

$$\sigma_1(D,\xi)^2 = \sigma_2\left(D^2,\xi\right) = \sigma_2\left(D^*D,\xi\right) = -|\xi|^2\cdot\mathrm{id}_E.$$

By polarization we obtain:

$$\begin{aligned}
-\big(|\xi|^2 + 2\langle\xi,\eta\rangle + |\eta|^2\big)\,\mathrm{id}_E \\
&= -|\xi+\eta|^2 \cdot \mathrm{id}_E \\
&= \sigma_1(D,\xi+\eta)^2 \\
&= \big(\sigma_1(D,\xi) + \sigma_1(D,\eta)\big)^2 \\
&= \sigma_1(D,\xi)^2 + \sigma_1(D,\xi)\sigma_1(D,\eta) + \sigma_1(D,\eta)\sigma_1(D,\xi) + \sigma_1(D,\eta)^2 \\
&= -|\xi|^2 \cdot \mathrm{id}_E + \sigma_1(D,\xi)\sigma_1(D,\eta) + \sigma_1(D,\eta)\sigma_1(D,\xi) - |\eta|^2 \cdot \mathrm{id}_E.
\end{aligned}$$

This yields the **Clifford relations**

$$\sigma_1(D,\xi)\sigma_1(D,\eta) + \sigma_1(D,\eta)\sigma_1(D,\xi) = -2\langle\xi,\eta\rangle \cdot \mathrm{id}_E. \tag{1.18}$$

Since these relations impose strong restrictions on the bundle E, it is much more difficult to construct Dirac-type operators than Laplace-type operators.

Let V be an n-dimensional oriented Euclidean vector space (later we will choose $V = T_x^*M$) and let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of V. Then

$$\omega := e_1 \wedge \ldots \wedge e_n \in \Lambda^n V$$

is called the **volume element** and does not depend on the particular choice of orthonormal basis.

Lemma 1.3.15. *Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional oriented Euclidean vector space. For each $k \in \{0, \ldots, n\}$ there is a unique isomorphism $* : \Lambda^k V \to \Lambda^{n-k} V$ such that*

$$\varphi \wedge *\psi = \langle\varphi,\psi\rangle \cdot \omega, \qquad \text{for all } \varphi,\psi \in \Lambda^k V. \tag{1.19}$$

The isomorphism $*$ is called the ***Hodge star operator****.*

Proof.
Uniqueness:
Let $e_1, \ldots, e_n$ be a positively oriented orthonormal basis of V. Then the family

$$\{e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}\}_{I=(1\le i_1<\ldots<i_k\le n)}$$

is an orthonormal basis of $\Lambda^k V$. Let $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$ be a basis element. Writing $*e_I$ in the corresponding basis of $\Lambda^{n-k} V$, we have:

$$*e_I = \sum_{|J|=n-k} \alpha_J \cdot e_J. \tag{1.20}$$

Now for any ordered multi-index K with $|I| = |K| = k$, we have:

$$e_K \wedge *e_I = \sum_{|J|=n-k} \alpha_J \cdot e_K \wedge e_J = \alpha_{K^c} \cdot e_K \wedge e_{K^c} = \operatorname{sign}(K, K^c)\, \alpha_{K^c} \cdot \omega.$$

Here K^c denotes the multi-index complementary to K and $\operatorname{sign}(K, K^c)$ is defined by the equation $e_K \wedge e_{K^c} = \operatorname{sign}(K, K^c) \cdot \omega$. By equation (1.19), we have:

$$e_K \wedge *e_I = \langle e_K, e_I \rangle \cdot \omega = \delta_{KI} \cdot \omega.$$

Thus $\alpha_{K^c} = \delta_{KI} \operatorname{sign}(K, K^c)$ and equivalently, $\alpha_J = \delta_{J^c I} \operatorname{sign}(J^c, J)$. Inserting this into (1.20), we obtain:

$$*e_I = \operatorname{sign}(I, I^c) \cdot e_{I^c}. \tag{1.21}$$

Existence:
We define the Hodge star operator $*$ by formula (1.21) on the basis $\{e_I\}_I$ and extend by linearity to $\Lambda^k V$. Then equation (1.19) holds for basis vectors and hence, by linearity, for all $\varphi, \psi \in \Lambda^k V$. □

Lemma 1.3.16. *Let V be an n-dimensional oriented Euclidean vector space. Then the following holds:*

a) $*1 = \omega$ and $*\omega = 1$.

b) $\langle *\varphi, *\psi \rangle = \langle \varphi, \psi \rangle$ for all $\varphi, \psi \in \Lambda^k V$.

c) On $\Lambda^k V$ we have: $*^2 = (-1)^{k(n-k)} \cdot \mathrm{id}_{\Lambda^k V}$.

d) $\langle \varphi, *\psi \rangle = (-1)^{k(n-k)} \langle *\varphi, \psi \rangle$ for all $\varphi \in \Lambda^k V$, $\psi \in \Lambda^{n-k} V$.

Proof.

a) This is clear from the formula (1.21) for the Hodge star operator in terms of an orthonormal basis of V.

b) We write $\varphi = \sum_{|I|=k} \varphi_I\, e_I$ and $\psi = \sum_{|J|=k} \psi_J\, e_J$. Then $\langle \varphi, \psi \rangle = \sum_{|I|=k} \varphi_I \psi_I$. Applying the operator $*$ we have:

$$*\varphi = \sum_{|I|=k} \varphi_I \operatorname{sign}(I, I^c) \cdot e_{I^c} \qquad \text{and}$$
$$*\psi = \sum_{|J|=k} \psi_J \operatorname{sign}(J, J^c) \cdot e_{J^c}.$$

Now we compute:

$$\langle *\varphi, *\psi\rangle = \sum_{|I|=k} \varphi_I\, \psi_I\, \mathrm{sign}(I, I^c)^2 = \sum_{|I|=k} \varphi_I\, \psi_I = \langle \varphi, \psi\rangle.$$

c) Let $|I| = k$. Then by (1.21) we have:

$$\begin{aligned} *^2 e_I &= * \left(\mathrm{sign}(I, I^c) \cdot e_{I^c}\right) \\ &= \mathrm{sign}(I, I^c) \cdot \mathrm{sign}(I^c, I) \cdot e_I \\ &= (-1)^{k(n-k)} \mathrm{sign}(I, I^c)^2 \cdot e_I \\ &= (-1)^{k(n-k)} \cdot e_I. \end{aligned}$$

d) For any $\varphi \in \Lambda^k V$, $\psi \in \Lambda^{n-k} V$, we have:

$$\langle *\varphi, \psi\rangle \overset{b)}{=} \langle * * \varphi, *\psi\rangle \overset{c)}{=} \left\langle (-1)^{k(n-k)} \varphi, *\psi\right\rangle = (-1)^{k(n-k)} \langle \varphi, *\psi\rangle. \qquad \square$$

Let M be an oriented Riemannian manifold. Then we may apply the construction of the Hodge star operator to the Euclidean vector spaces $(T_x^*M, \langle\cdot,\cdot\rangle_x)$, where $x \in M$ and $\langle\cdot,\cdot\rangle_x$ denotes the scalar product on T_x^*M induced by the Riemannian metric. The resulting isomorphisms $* : \Lambda^k(T_x^*M) \to \Lambda^{n-k}(T_x^*M)$ depend smoothly on the point x. Thus we have the Hodge star operator $*\colon\ \Omega^k(M) \to \Omega^{n-k}(M)$ on differential forms. Note that the operator $*$ depends on the Riemannian metric.
Combining the Hodge star operator with the exterior derivative we obtain:

Lemma 1.3.17. *Let M be an oriented n-dimensional Riemannian manifold. Then the formal adjoint of $d\colon \Omega^{k-1}(M) \to \Omega^k(M)$ is given by*

$$d^* = (-1)^{n(k+1)+1} * d* : \quad \Omega^k(M) \to \Omega^{k-1}(M). \tag{1.22}$$

The operator d^* is called the ***codifferential****.*

Proof. Exercise. $\square$

Corollary 1.3.18. *For n even we have: $d^* = - * d *$.*

Example 1.3.19. Let M be an n-dimensional oriented Riemannian manifold and let $n = 2m$ be even. For any $k \in \{0, \dots, n\}$ and any $x \in M$ we define

$$\tau_x := i^{k(k-1)+m} * : \Lambda^k T_x^*M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{n-k} T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$$

and we obtain $\tau_x : \Lambda^\bullet T_x^* M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^\bullet T_x^* M \otimes_{\mathbb{R}} \mathbb{C}$ as an operator on forms of arbitrary degree. By Lemma 1.3.16, we have $\tau_x^2 = \mathrm{id}_{\Lambda^\bullet T_x^* M \otimes_{\mathbb{R}} \mathbb{C}}$. Thus, the only eigenvalues of τ_x are 1 and -1. We put

$$
\begin{aligned}
E^+ &:= \bigsqcup_{x \in M} \ker\left(\tau_x - \mathrm{id}_{\Lambda^\bullet T_x^* M \otimes_{\mathbb{R}} \mathbb{C}}\right), \\
E^- &:= \bigsqcup_{x \in M} \ker\left(\tau_x + \mathrm{id}_{\Lambda^\bullet T_x^* M \otimes_{\mathbb{R}} \mathbb{C}}\right)
\end{aligned}
$$

Thus $E^\pm$ denotes the bundle of ± 1-eigenvectors of τ. Then we have the decomposition

$$
\Lambda^\bullet T^* M \otimes_{\mathbb{R}} \mathbb{C} = E^+ \oplus E^-.
$$

A simple computation yields $(d + d^*)\,\tau = -\tau\,(d + d^*)$. Thus, we have:

$$
d + d^* : C^\infty\left(M, E^+\right) \to C^\infty\left(M, E^-\right).
$$

As for the Euler operator above, one can show that $d + d^* \in \mathscr{D}\!\mathit{iff}_1(E^+, E^-)$ is a Dirac-type operator. It is called the **signature operator**.

Example 1.3.20. Let M be a complex manifold of complex dimension m with a Hermitian metric on the tangent bundle (considered as a complex vector bundle), i.e., M is a Hermitian manifold. The complexified cotangent bundle $\Lambda^1 T^* M \otimes_{\mathbb{R}} \mathbb{C}$ has the following decomposition:

$$
\begin{aligned}
T^* M \otimes_{\mathbb{R}} \mathbb{C} &= \Lambda^{1,0}\, T^* M \oplus \Lambda^{0,1}\, T^* M \\
&= \{\mathbb{C}\text{-linear forms on } TM\} \oplus \{\mathbb{C}\text{-anti-linear forms on } TM\}.
\end{aligned}
$$

Given local coordinates $z^1, \ldots, z^m, \bar{z}^1, \ldots, \bar{z}^m$, we can write any complex-valued 1-form ω as

$$
\omega = \sum_{j=1}^m \alpha_j\, dz^j + \sum_{j=1}^m \beta_j\, d\bar{z}^j \quad \in \Lambda^{1,0}\, T^* M \oplus \Lambda^{0,1}\, T^* M.
$$

Now for any $f \in C^\infty(M, \mathbb{C})$, the differential $df \in \Lambda^1 T^* M \otimes_{\mathbb{R}} \mathbb{C}$ splits as

$$
df = \underbrace{\sum_{j=1}^m \frac{\partial f}{\partial z^j}\, dz^j}_{\in \Lambda^{1,0}\, T^* M} + \underbrace{\sum_{j=1}^m \frac{\partial f}{\partial \bar{z}^j}\, d\bar{z}^j}_{\in \Lambda^{0,1}\, T^* M} =: \partial f + \bar{\partial} f. \tag{1.23}
$$

From complex analysis we have:

$$
\bar{\partial} f = 0 \quad \Longleftrightarrow \quad f \text{ is holomorphic.}
$$

Similarly, any complex-valued k-form ω can be decomposed as

$$\omega = \sum_{p+q=k} \sum_{\substack{|I|=p \\ |J|=q}} \omega_{IJ} \underbrace{dz^{i_1} \wedge \ldots \wedge dz^{i_p}}_{=: dz^I} \wedge \underbrace{d\bar{z}^{j_1} \wedge \ldots \wedge d\bar{z}^{j_q}}_{=: d\bar{z}^J} .$$

This defines the decomposition

$$\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\substack{p+q=k \\ p,q \in \{0,\ldots,m\}}} \Lambda^{p,q} T^*M .$$

Now if ω is a (p,q)-form on M,

$$\omega = \sum_{\substack{|I|=p \\ |J|=q}} \omega_{IJ}\, dz^I \wedge d\bar{z}^J ,$$

its exterior derivative splits as

$$\begin{aligned} d\omega &= \sum_{\substack{|I|=p \\ |J|=q}} d\omega_{IJ} \wedge dz^I \wedge d\bar{z}^J \\ &\overset{(1.23)}{=} \underbrace{\sum_{\substack{|I|=p \\ |J|=q}} \partial\omega_{IJ} \wedge dz^I \wedge d\bar{z}^J}_{=: \partial\omega \in \Lambda^{p+1,q} T^*M} + \underbrace{\sum_{\substack{|I|=p \\ |J|=q}} \bar{\partial}\omega_{IJ} \wedge dz^I \wedge d\bar{z}^J}_{=: \bar{\partial}\omega \in \Lambda^{p,q+1} T^*M} . \end{aligned}$$

We have split d into $d = \partial + \bar{\partial}$ where

$$\begin{aligned} \partial &: C^\infty(M, \Lambda^{p,q} T^*M) \to C^\infty(M, \Lambda^{p+1,q} T^*M), \\ \bar{\partial} &: C^\infty(M, \Lambda^{p,q} T^*M) \to C^\infty(M, \Lambda^{p,q+1} T^*M). \end{aligned}$$

The operator $\bar{\partial}$ is called the **Dolbeault operator**. Using this decomposition, we have for any $\omega \in \Lambda^{p,q} T^*M$:

$$0 = d^2\omega = \left(\partial + \bar{\partial}\right)\left(\partial + \bar{\partial}\right)\omega = \underbrace{\partial^2\omega}_{\in \Lambda^{p+2,q}} + \underbrace{\partial\bar{\partial}\omega + \bar{\partial}\partial\omega}_{\in \Lambda^{p+1,q+1}} + \underbrace{\bar{\partial}^2\omega}_{\in \Lambda^{p,q+2}} .$$

We thus have:

$$\partial^2 = 0 \tag{1.24}$$
$$\partial\bar{\partial} + \bar{\partial}\partial = 0 \tag{1.25}$$
$$\bar{\partial}^2 = 0. \tag{1.26}$$

Hence the operators ∂ and $\bar{\partial}$ define complexes. The q-th cohomology of the complex $(\Omega^{p,\bullet}, \bar{\partial})$ is called the **Dolbeault cohomology** of M in the bidegree (p,q).

For the Hodge-Laplacian Δ_d on $\Omega^{p,q}(M)$, we obtain:

$$\begin{aligned}
\Delta_d &= (d+d^*)^2 = dd^* + d^*d \\
&= \left(\partial+\bar\partial\right)\left(\partial^*+\bar\partial^*\right) + \left(\partial^*+\bar\partial^*\right)\left(\partial+\bar\partial\right) \\
&= \underbrace{\partial\partial^* + \partial^*\partial}_{\substack{=:\Delta_\partial \\ \text{values in } \Omega^{p,q}(M)}} + \underbrace{\partial\bar\partial^* + \bar\partial^*\partial}_{\substack{\text{values in} \\ \Omega^{p+1,q-1}(M)}} + \underbrace{\bar\partial\partial^* + \partial^*\bar\partial}_{\substack{\text{values in} \\ \Omega^{p-1,q+1}(M)}} + \underbrace{\bar\partial\bar\partial^* + \bar\partial^*\bar\partial}_{\substack{=:\Delta_{\bar\partial} \\ \text{values in } \Omega^{p,q}(M)}} .
\end{aligned}$$

Thus the principal symbol of Δ_d splits into

$$\begin{aligned}
-|\xi|^2\cdot \mathrm{id}_{\Lambda^{p,q}T^*M} &= \sigma_2(\Delta_d,\xi) \\
&= \sigma_2\left(\Delta_\partial,\xi\right) + \sigma_2\left(\partial\bar\partial^* + \bar\partial^*\partial\right) + \sigma_2\left(\bar\partial\partial^* + \partial^*\bar\partial\right) + \sigma_2\left(\Delta_{\bar\partial},\xi\right). \quad (1.27)
\end{aligned}$$

Since the left hand side is an endomorphism of $\Lambda^{p,q}T^*M$, the two middle terms of the right hand side necessarily vanish. Hence the operators $\partial\bar\partial^* + \bar\partial^*\partial$ and $\bar\partial\partial^* + \partial^*\bar\partial$ are actually of first order, i.e.,

$$\begin{aligned}
\partial\bar\partial^* + \bar\partial^*\partial &\in \mathscr{Diff}_1\left(\Lambda^{p,q}T^*M, \Lambda^{p+1,q-1}T^*M\right) \qquad \text{and} \\
\bar\partial\partial^* + \partial^*\bar\partial &\in \mathscr{Diff}_1\left(\Lambda^{p,q}T^*M, \Lambda^{p-1,q+1}T^*M\right).
\end{aligned}$$

For $j = 1,\dots m$ we write $z^j = x^j + iy^j$ and we decompose $\xi \in T^*M$ as

$$\xi = \sum_{j=1}^m (\xi_{x_j}dx^j + \xi_{y_j}dy^j) = \frac{1}{2}\sum_{j=1}^m (\xi_j dz^j + \bar\xi_j d\bar z^j)$$

where $\xi_{x_j}, \xi_{y_j} \in \mathbb{R}$ and $\xi_j = \xi_{x_j} - i\xi_{y_j}$. We then compute for $\omega \in \Lambda^{p,q}T^*M$:

$$\begin{aligned}
\sigma_1(\partial,\xi)\omega &= \frac{1}{2}\sum_{j=1}^m \xi_j dz^j \wedge \omega, \quad & \sigma_1(\bar\partial,\xi)\omega &= \frac{1}{2}\sum_{j=1}^m \bar\xi_j d\bar z^j \wedge \omega, \\
\sigma_1(\partial^*,\xi)\omega &= -\frac{1}{2}\sum_{j=1}^m \bar\xi_j dz^j \lrcorner\, \omega, \quad & \sigma_1(\bar\partial^*,\xi)\omega &= -\frac{1}{2}\sum_{j=1}^m \xi_j d\bar z^j \lrcorner\, \omega.
\end{aligned}$$

It follows that

$$\sigma_2(\Delta_\partial,\xi) = \sigma_2(\Delta_{\bar\partial},\xi) = -\frac{1}{2}\,|\xi|^2\cdot \mathrm{id}_{\Lambda^{p,q}T^*M}\,.$$

Thus, $2\Delta_\partial$ and $2\Delta_{\bar\partial}$ are Laplace-type operators.

Now we look for a first order operator whose square is $2\Delta_{\bar\partial}$: Fix $p \in \{0,\dots,m\}$ and define

$$E := \Lambda^{p,\mathrm{even}}T^*M, \quad F := \Lambda^{p,\mathrm{odd}}T^*M.$$

In analogy to the Euler operator, we put

$$D_{\bar\partial} := \sqrt{2}\left(\bar\partial + \bar\partial^*\right) : C^\infty\left(M,E\right) \to C^\infty\left(M,F\right).$$

Then we get

$$\begin{aligned} D_{\bar\partial}^* D_{\bar\partial} &= 2\left(\bar\partial\bar\partial + \bar\partial\bar\partial^* + \bar\partial^*\bar\partial + \bar\partial^*\bar\partial^*\right) \\ &= 2\left(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial\right) \\ &= 2\Delta_{\bar\partial} \end{aligned}$$

and similarly $D_{\bar\partial} D_{\bar\partial}^* = 2\Delta_{\bar\partial}$. Hence $D_{\bar\partial}$ is a Dirac-type operator.

The operator

$$\Delta_{\bar\partial} : C^\infty\left(M, \Lambda^{p,q} T^* M\right) \to C^\infty\left(M, \Lambda^{p,q} T^* M\right)$$

is called the **Dolbeault Laplacian**. The Dirac-type operator $D_{\bar\partial}$ is called the **Dolbeault Dirac operator**.

Twisting of first order operators with coefficient bundles

Let E and F be Riemannian or Hermitian vector bundles over a manifold M and let $D \in \mathscr{Diff}_1(E, F)$. Then for each $x \in M$ the principal symbol yields a bilinear map

$$T_x^* M \times E_x \to F_x, \quad (\xi, e) \mapsto \sigma_1(D, \xi) e.$$

This corresponds uniquely to a linear map

$$T_x^* M \otimes E_x \to F_x, \quad \xi \otimes e \mapsto \sigma_1(D, \xi) e.$$

Hence the principal symbol $\sigma(D, \cdot)$ of D can be considered as an element of $C^\infty(M, \mathrm{Hom}(T^* M \otimes E, F))$.

Conversely, given a section $A \in C^\infty(M, \mathrm{Hom}(T^* M \otimes E, F))$ and a connection ∇ on E, we define $D_{A,\nabla} \in \mathscr{Diff}_1(E, F)$ by

$$D_{A,\nabla}\, e := \sum_{j=1}^n A\left(b_j^* \otimes \nabla_{b_j} e\right), \tag{1.28}$$

where $b_1, \dots, b_n$ is a local frame of TM and $b_1^*, \dots, b_n^*$ is the dual frame.

This definition is independent of the choice of the basis $b_1, \dots, b_n$:
Let $\widetilde{b}_1, \dots, \widetilde{b}_n$ be another local frame of TM. We express $\widetilde{b}_j$ and $\widetilde{b}_j^*$ by $b_1, \dots, b_n$ and $\widetilde{b}_1, \dots, \widetilde{b}_n$, respectively:

$$\widetilde{b}_j = \sum_{i=1}^n \alpha_{ij}\, b_i, \quad \text{and} \quad \widetilde{b}_j^* = \sum_{k=1}^n \beta_{kj}\, b_k^*.$$

Then we have:

$$\delta_{lj} = \widetilde{b}_l^*\left(\widetilde{b}_j\right) = \sum_{k=1}^n \beta_{kl}\, b_k^*\Big(\sum_{i=1}^n \alpha_{ij}\, b_i\Big) = \sum_{k,i=1}^n \beta_{kl}\, \alpha_{ij}\, \underbrace{b_k^*(b_i)}_{\delta_{ki}} = \sum_{i=1}^n \beta_{il}\, \alpha_{ij}. \tag{1.29}$$

We may write this in matrix form as $\mathbb{1} = \beta^\top \cdot \alpha$, or equivalently, $\mathbb{1} = \alpha \cdot \beta^\top$. Thus equation (1.29) is equivalent to

$$\delta_{ki} = \sum_{j=1}^{n} \alpha_{kj}\, \beta_{ij}. \tag{1.30}$$

Now we compute:

$$\begin{aligned}
\sum_{j=1}^{n} A\left(\widetilde{b}_j^* \otimes \nabla_{\widetilde{b}_j} e\right) &= \sum_{j=1}^{n} A\left(\sum_{k=1}^{n} \beta_{kj}\, b_k^* \otimes \nabla_{\sum_{i=1}^{n} \alpha_{ij}\, b_i} e\right) \\
&= \sum_{j,k,i=1}^{n} \beta_{kj}\, \alpha_{ij}\, A\left(b_k^* \otimes \nabla_{b_i} e\right) \\
&\overset{(1.30)}{=} \sum_{k,i=1}^{n} \delta_{ki}\, A\left(b_k^* \otimes \nabla_{b_i} e\right) \\
&= \sum_{i=1}^{n} A\left(b_i^* \otimes \nabla_{b_j} e\right).
\end{aligned}$$

Hence the definition of $D_{A,\nabla}$ is independent of the choice of the basis $b_1, \ldots, b_n$.

We compute the principal symbol of $D_{A,\nabla}$:
Let $\xi \in T_x^*M$ and $e \in E_x$. Choose a function $f \in C^\infty(M)$ such that $f(x) = 0$ and $df(x) = \xi$. Choose a section $\widetilde{e} \in C^\infty(M, E)$ with $\widetilde{e}(x) = e$. Then we have:

$$\begin{aligned}
\sigma(D_{A,\nabla}, \xi)e &= D_{A,\nabla}\big(f\widetilde{e}\big)(x) \\
&= \sum_{j=1}^{n} A\Big(b_j^* \otimes \nabla_{b_j}\big(f \cdot \widetilde{e}\,\big)\Big)(x) \\
&= \sum_{j=1}^{n} A\Big(b_j^* \otimes \big(b_j(f)\,\widetilde{e} + f \cdot \nabla_{b_j}\widetilde{e}\big)\Big)(x).
\end{aligned}$$

Using $b_j(f)(x) = df|_x(b_j) = \xi(b_j)$, and the properties $\widetilde{e}(x) = e$ and $f(x) = 0$, we get:

$$\begin{aligned}
\sigma(D_{A,\nabla}, \xi)e &= \sum_{j=1}^{n} A\left(b_j^* \otimes \xi(b_j)\, e\right) \\
&= A\Big(\sum_{j} \xi(b_j) b_j^* \otimes e\Big) \\
&= A(\xi \otimes e).
\end{aligned} \tag{1.31}$$

Thus, for a fixed connection ∇ and any operator $D \in \mathscr{D}\!\mathit{iff}_1(E, F)$, the operator $D_{\sigma(D,\cdot),\nabla} = \sum_{j=1}^{n} \sigma\big(D, b_j^*\big)\nabla_{b_j}$ has the same principal symbol as D. Hence it differs from

D by a zero-order operator $B \in C^\infty(M, \mathrm{Hom}(E,F))$, i.e.,

$$D = D_{\sigma(D,\cdot),\nabla} + B = \sum_{j=1}^{n} \sigma\big(D, b_j^*\big)\nabla_{b_j} + B.$$

Note that the operator B depends on the choice of the connection ∇.

Now let M be a manifold, let $E, F, C \to M$ be $\mathbb{K}$-vector bundles over M, and let ∇^C be a connection on C. For $D \in \mathscr{Diff}_1(E,F)$, choose a connection ∇^E on E and write D as

$$D = \sum_j \sigma\big(D, b_j^*\big)\nabla^E_{b_j} + B, \tag{1.32}$$

where the homomorphism field $B \in C^\infty(M, \mathrm{Hom}(E,F))$ depends on the choice of ∇^E. Now define $D^{\nabla^C} \in \mathscr{Diff}_1(E \otimes C, F \otimes C)$ by

$$D^{\nabla^C} := \sum_j \Big(\sigma\big(D, b_j^*\big) \otimes \mathrm{id}_C\Big)\nabla^{E\otimes C}_{b_j} + B \otimes \mathrm{id}_C\,. \tag{1.33}$$

Here $\nabla^{E\otimes C}$ is the tensor product connection on $E \otimes C$, defined by

$$\nabla^{E\otimes C}(e \otimes c) := \nabla^E e \otimes c + e \otimes \nabla^C c.$$

We check that the definition of D^{∇^C} does not depend on the choice of the connection ∇^E: For any $e \otimes c \in C^\infty(M, E \otimes C)$, we compute:

$$\begin{aligned}
D^{\nabla^C}(e\otimes c) &= \sum_j \Big(\sigma\big(D, b_j^*\big) \otimes \mathrm{id}_C\Big)\nabla^{E\otimes C}_{b_j}(e\otimes c) + (B \otimes \mathrm{id}_C)(e \otimes c) \\
&= \sum_j \Big(\sigma\big(D, b_j^*\big) \otimes \mathrm{id}_C\Big)\Big(\nabla^E_{b_j} e \otimes c + e \otimes \nabla^C_{b_j} c\Big) + (B\,e) \otimes c \\
&= \sum_j \sigma\big(D, b_j^*\big)\nabla^E_{b_j} e \otimes c + \sum_j \sigma\big(D, b_j^*\big) e \otimes \nabla^C_{b_j} c + (B\,e) \otimes c \\
&= D\,e \otimes c + \sum_j \sigma\big(D, b_j^*\big) e \otimes \nabla^C_{b_j} c.
\end{aligned}$$

In the last equality we used equation (1.32). We have obtained an expression for D^{∇^C} that is independent of the connection ∇^E and B.

Definition 1.3.21. Let M be a differentiable manifold, let $E, F, C \to M$ be $\mathbb{K}$-vector bundles over M and let ∇^C be a connection on C.
For a first order operator $D \in \mathscr{Diff}_1(E,F)$ we say that the operator

$$D^{\nabla^C} \in \mathscr{Diff}_1(E \otimes C, F \otimes C)$$

defined by (1.33) is obtained from D by **twisting with (C, ∇^C)**.

We compute the principal symbol of the twisted operator D^{∇^C}:
For $\xi \in T_x^*M$ choose a function $f \in C^\infty(M)$ with $f(x) = 0$ and $df(x) = \xi$. For $e \in E_x$ and $c \in C_x$ choose sections $\widetilde{e} \in C^\infty(M,E)$ and $\widetilde{c} \in C^\infty(M,C)$ with $\widetilde{e}(x) = e$ and $\widetilde{c}(x) = c$. Then we have:

$$\begin{aligned}
\sigma\big(D^{\nabla^C},\xi\big)(e\otimes c) &= D^{\nabla^C}\big(f\cdot(\widetilde{e}\otimes\widetilde{c})\big)(x)\\
&= D^{\nabla^C}\big((f\cdot\widetilde{e})\otimes\widetilde{c}\big)(x)\\
&= \Big(D(f\cdot\widetilde{e})\otimes\widetilde{c} + \sum_{j=1}^n \sigma(D,b_j^*)(f\cdot\widetilde{e})\otimes\nabla^C_{b_j}\widetilde{c}\Big)(x)\\
&= \big(D(f\cdot\widetilde{e})\big)(x)\otimes c + \sum_{j=1}^n \sigma(D,b_j^*)\underbrace{(f\cdot\widetilde{e})(x)}_{=0}\otimes\big(\nabla^C_{b_j}\widetilde{c}\big)(x)\\
&= \sigma(D,\xi)\,e\otimes c.
\end{aligned}$$

Hence

$$\sigma(D^{\nabla^C},\xi) = \sigma(D,\xi)\otimes \mathrm{id}_C\,. \tag{1.34}$$

Corollary 1.3.22. *Let (M,g) be a Riemannian manifold, let $E,F,C\to M$ be Riemannian or Hermitian vector bundles over M, and let ∇^C be a connection on C. If $D\in\mathscr{Diff}_1(E,F)$ is of Dirac-type then D^{∇^C} is also of Dirac-type.*

Proof. For any $\xi\in T^*M$, we have:

$$\begin{aligned}
\sigma_2\big((D^{\nabla^C})^*\circ D^{\nabla^C},\xi\big) &\overset{(1.5)}{=} -\sigma_1\big(D^{\nabla^C},\xi\big)^*\circ\sigma_1\big(D^{\nabla^C},\xi\big)\\
&\overset{(1.34)}{=} -\big(\sigma_1(D,\xi)\otimes\mathrm{id}_C\big)^*\circ\big(\sigma_1(D,\xi)\otimes\mathrm{id}_C\big)\\
&= -\sigma_1(D,\xi)^*\sigma_1(D,\xi)\otimes\mathrm{id}_C\\
&\overset{(1.5)}{=} \sigma_1(D^*,\xi)\sigma_1(D,\xi)\otimes\mathrm{id}_C\\
&= \sigma_2(D^*D,\xi)\otimes\mathrm{id}_C\\
&= \big(-|\xi|^2\cdot\mathrm{id}_E\big)\otimes\mathrm{id}_C\\
&= -|\xi|^2\cdot\mathrm{id}_{E\otimes C}\,.
\end{aligned}$$

Similarly, we find $\sigma_2(D^{\nabla^C}\circ(D^{\nabla^C})^*,\xi) = -|\xi|^2\cdot\mathrm{id}_{F\otimes C}$. ∎

Lemma 1.3.23. *Let (M,g) be a Riemannian manifold, let $E,F,C\to M$ be Riemannian or Hermitian vector bundles over M, and let ∇^C be a metric connection on C.*

Let $D \in \mathscr{Diff}_1(E,F)$*. Then*

$$\left(D^*\right)^{\nabla^C} = \left(D^{\nabla^C}\right)^*.$$

Proof. Exercise. ☐

Remark 1.3.24. If the connection ∇^C in Lemma 1.3.23 is not metric then we have:

$$\left(D^*\right)^{\nabla^C} = \left(D^{\nabla^C}\right)^* + B \otimes \mathrm{id}_C$$

for some $B \in C^\infty(M, \mathrm{Hom}(F,E))$.

Corollary 1.3.25. *Let* (M,g) *be a Riemannian manifold, let* $E, F, C \to M$ *be Riemannian or Hermitian vector bundles over* M*, and let* ∇^C *be a metric connection on* C*. Let* $D \in \mathscr{Diff}_1(E,F)$*.*
If D *is formally self-adjoint then so is* D^{∇^C}*.*

1.4. The analysis of Dirac-type operators

Throughout this section let M be a compact Riemannian manifold. Let $E, F \to M$ be Riemannian or Hermitian vector bundles over M and let $D \in \mathscr{Diff}_1(E,F)$. For any $s \in \mathbb{R}$ the differential operator D extends uniquely to a bounded linear map

$$D : H^{s+1}(M,E) \to H^s(M,F),$$

i.e., for every $u \in H^{s+1}(M,E)$ we have $\|Du\|_{H^s} \le C\|u\|_{H^{s+1}}$ with C independent of u. If D is of Dirac-type, we will get a kind of inverse to this inequality.

Proposition 1.4.1 (Gårding inequality). *Let* M *be a compact Riemannian manifold and let* $E, F \to M$ *be Riemannian or Hermitian vector bundles over* M*. Let* $D \in \mathscr{Diff}_1(E,F)$ *be a Dirac-type operator. Then there exists a constant* $C > 0$ *such that for all* $u \in H^1(M,E)$*, we have:*

$$\|u\|_{H^1} \le C\left(\|Du\|_{H^0} + \|u\|_{H^0}\right). \tag{1.35}$$

Proof. Since D is of Dirac-type, the formally self-adjoint operator D^*D is of Laplace-type. Thus, by Lemma 1.3.5, we may write

$$D^*D = \nabla^*\nabla + K \tag{1.36}$$

for some metric connection ∇ on E and some $K \in C^\infty(M, \mathrm{symEnd}(E))$. Now for any smooth section $u \in C^\infty(M,E)$ we have:

$$\begin{aligned}\|u\|^2_{H^1} &= \|u\|^2_{H^0} + \|\nabla u\|^2_{H^0}\\ &= (\nabla u, \nabla u)_{L^2} + \|u\|^2_{H^0}\\ &= (\nabla^*\nabla u, u)_{L^2} + \|u\|^2_{H^0}\end{aligned}$$

Inserting (1.36) yields

$$\begin{aligned}\|u\|^2_{H^1} &= ((D^*D - K)u, u)_{L^2} + \|u\|^2_{H^0}\\ &= \|Du\|^2_{H^0} - \int_M \langle Ku(x), u(x)\rangle\, dx + \|u\|^2_{H^0}\end{aligned}$$

Since M is compact, there exists a constant C_1 such that $|K|_{E_x,E_x} \le C_1$ uniformly in x. This yields

$$\begin{aligned}\|u\|^2_{H^1} &\le \|Du\|^2_{H^0} + (1+C_1)\|u\|^2_{H^0}\\ &\le \|Du\|^2_{H^0} + 2\sqrt{1+C_1}\,\|Du\|_{H^0}\cdot\|u\|_{H^0} + (1+C_1)\|u\|^2_{H^0}\\ &= \Big(\|Du\|_{H^0} + \sqrt{1+C_1}\,\|u\|_{H^0}\Big)^2\\ &\le (1+C_1)\big(\|Du\|_{H^0} + \|u\|_{H^0}\big)^2.\end{aligned}$$

Thus we have for any smooth section $u \in C^\infty(M,E)$ the inequality

$$\|u\|_{H^1} \le \sqrt{1+C_1}\big(\|Du\|_{H^0} + \|u\|_{H^0}\big).$$

Since $C^\infty(M,E)$ is dense in $H^1(M,E)$ and the Sobolev norms $\|\cdot\|_{H^0}$ and $\|\cdot\|_{H^1}$ are continuous on $H^1(M,E)$, this estimate holds for all $u \in H^1(M,E)$. □

Proposition 1.4.2 (Elliptic estimates). *Let M be a compact Riemannian manifold and let $E, F \to M$ be Riemannian or Hermitian vector bundles over M. Let $D \in \mathscr{D}\!\mathit{iff}_1(E,F)$ be a Dirac-type operator.*
Then for every $k \in \mathbb{N}_0$ there exists a constant $C > 0$ such that the inequality

$$\|u\|_{H^{k+1}} \le C\,(\|Du\|_{H^k} + \|u\|_{H^k}) \tag{1.37}$$

holds for all $u \in H^{k+1}(M,E)$.

Proof. We will prove the estimates by induction on k:
For $k = 0$, the elliptic estimate (1.37) is the Gårding inequality (1.35).

Let $\mathscr{D} \in \mathscr{D}\!\mathit{iff}_1(E \otimes T^*M, F \otimes T^*M)$ be the operator obtained from D by twisting with the Levi-Civita connection on T^*M. We choose connections ∇^E and ∇^F on E and F, respectively. Now we consider the operator

$$P := \mathscr{D} \circ \nabla^E - \nabla^F \circ D \in \mathscr{D}\!\mathit{iff}_2\big(E, F \otimes T^*M\big).$$

For any $\xi \in T^*M$, we have:

$$\begin{aligned}
\sigma_2(P,\xi)e &= \Big(\sigma_1(\mathscr{D},\xi) \circ \sigma_1(\nabla^E,\xi) - \sigma_1(\nabla^F,\xi) \circ \sigma_1(D,\xi)\Big)e \\
&\overset{(1.34)}{=} \big(\sigma_1(D,\xi) \otimes \mathrm{id}_{T^*M}\big)(e \otimes \xi) - \sigma_1(D,\xi)e \otimes \xi \\
&= 0.
\end{aligned}$$

Thus P is actually a first-order operator.

Now fix $k \in \mathbb{N}$ and assume that the elliptic estimate (1.37) holds for $k-1$. For any $u \in H^{k+1}(M,E)$ we have:

$$\begin{aligned}
\|u\|^2_{H^{k+1}} &= \|\nabla^{k+1}u\|^2_{H^0} + \|\nabla^k u\|^2_{H^0} + \ldots + \|u\|^2_{H^0} \\
&= \|\nabla^{k+1}u\|^2_{H^0} + \|u\|^2_{H^k},
\end{aligned} \tag{1.38}$$

and moreover

$$\|\nabla^{k+1}u\|^2_{H^0} = \|\nabla^k \nabla u\|^2_{H^0} \le \|\nabla u\|^2_{H^k}.$$

We now apply the induction hypothesis for the operator $\mathscr{D} \in \mathscr{D}\!\mathit{iff}_1(E \otimes T^*M, F \otimes T^*M)$ and the section $\nabla^E u \in C^\infty(M, E \otimes T^*M)$ and obtain:

$$\begin{aligned}
\|\nabla^E u\|^2_{H^k} &\le C_1 \left(\|\mathscr{D}\nabla^E u\|_{H^{k-1}} + \|\nabla^E u\|_{H^{k-1}}\right)^2 \\
&= C_1 \left(\|Pu + \nabla^F Du\|_{H^{k-1}} + \|\nabla^E u\|_{H^{k-1}}\right)^2 \\
&\le C_1 \left(\|Pu\|_{H^{k-1}} + \|\nabla^F Du\|_{H^{k-1}} + \|\nabla^E u\|_{H^{k-1}}\right)^2.
\end{aligned}$$

Since P is a first order operator there is a constant C_2 such that $\|Pu\|_{H^{k-1}} \le C_2\|u\|_{H^k}$. Moreover, $\|\nabla^F Du\|_{H^{k-1}} \le \|Du\|_{H^k}$. Hence

$$\|\nabla^E u\|^2_{H^k} \le C_3 \left(\|Du\|_{H^k} + \|u\|_{H^k}\right)^2$$

and thus

$$\|\nabla^{k+1} u\|^2_{H^0} \le C_3 \left(\|Du\|_{H^k} + \|u\|_{H^k}\right)^2.$$

Together with (1.38) we obtain the assertion. ☐

Lemma 1.4.3. *Let M be a compact manifold, and let $D \in \mathscr{Diff}_1(E,F)$ be a Dirac-type operator. Let*

$$\bar{D} : H^1(M,E) \to L^2(M,F)$$

be the unique bounded extension of D. Then the graph of $\bar{D}$,

$$\Gamma_{\bar{D}} := \left\{ \left(x, \bar{D}x\right) \mid x \in H^1(M,E) \right\} \subset L^2(M,E) \oplus L^2(M,F)$$

is a closed subspace. In other words: $\bar{D}$ is a closed operator.

Proof. Let $(x_j)_{j\in\mathbb{N}}$ be a sequence in $H^1(M,E)$ such that $(x_j, \bar{D}x_j) \to (x,y)$ in $L^2(M,E) \oplus L^2(M,F)$. We need to check that $(x,y) \in \Gamma_{\bar{D}}$. Since the sequences $(x_j)_{j\in\mathbb{N}}$ and $(\bar{D}x_j)_{j\in\mathbb{N}}$ converge in L^2 they are also bounded in L^2,

$$\|x_j\|_{L^2} \leq C_1 \quad \text{and} \quad \left\|\bar{D}x_j\right\|_{L^2} \leq C_2$$

for constants C_1, C_2, independent of j. From the Gårding inequality (1.35) for the Dirac-type operator D we obtain:

$$\|x_j\|_{H^1} \leq C_3 \left(\|\bar{D}x_j\|_{L^2} + \|x_j\|_{L^2}\right) \leq C_4.$$

By Lemma 1.2.16, we may pass to a weakly convergent subsequence $x_j \rightharpoonup z \in H^1(M,E)$. In particular, we have $x_j \rightharpoonup z$ in $L^2(M,E)$. On the other hand, we also have $x_j \rightharpoonup x \in L^2(M,E)$. Since by Lemma 1.2.16 weak limits are unique, we find $x = z \in H^1(M,E)$. Since $\bar{D}$ is bounded and $x_j \rightharpoonup x$, we also have $\bar{D}x_j \rightharpoonup \bar{D}x$ in $L^2(M,F)$. One the other hand, we also have $\bar{D}x_j \rightharpoonup y$. Therefore $y = \bar{D}x$. We conclude $(x,y) = (x,\bar{D}x) \in \Gamma_{\bar{D}}$. □

Let M be a compact Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian $\mathbb{K}$-vector bundles over M. Let $D \in \mathscr{Diff}_1(E,F)$.
Let us assume that $u \in H^1(M,E)$ and $\bar{D}u = f \in L^2(M,F)$. Choose a sequence $(u_j)_{j\in\mathbb{N}}$ of smooth sections in E such that $u_j \xrightarrow{H^1} u$. Then for all $\varphi \in C^\infty(M,F)$ we have:[2]

$$\begin{aligned}
(f,\varphi)_{L^2} &= \left(\bar{D}u, \varphi\right)_{L^2} \\
&= \left(\bar{D}\big(H^1\text{–}\lim_{j\to\infty} u_j\big), \varphi\right)_{L^2} \\
&= \left(L^2\text{–}\lim_{j\to\infty} (Du_j), \varphi\right)_{L^2} \\
&= \lim_{j\to\infty} (Du_j, \varphi)_{L^2} \\
&= \lim_{j\to\infty} (u_j, D^*\varphi)_{L^2}
\end{aligned}$$

[2] Here and in the following $H^1 - \lim\limits_{i\to\infty}$ and $L^2 - \lim\limits_{i\to\infty}$ denote the limits in $H^1(M,E)$ and $L^2(M,E)$, respectively.

$$= \Big(L^2\text{–}\lim_{j\to\infty} u_j, D^*\varphi \Big)_{L^2}$$
$$= \big(u, D^*\varphi\big)_{L^2}.$$

We have shown: If $\bar{D}u = f$ with $u \in H^1(M,E)$ and $f \in L^2(M,F)$ then

$$\big(u, D^*\varphi\big)_{L^2} = (f,\varphi)_{L^2}, \qquad \text{for all } \varphi \in C^\infty(M,F).$$

The last equation also makes sense for $u \in L^2(M,E)$. This motivates the following:

Definition 1.4.4. Let M be a compact Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian $\mathbb{K}$-vector bundles over M. Let $D \in \mathscr{Diff}_1(E,F)$. Let $u \in L^2(M,E)$ and $f \in L^2(M,F)$.
We say that the equation $Du = f$ holds **in the weak sense** if for all $\varphi \in C^\infty(M,F)$ we have:

$$\big(u, D^*\varphi\big)_{L^2} = (f,\varphi)_{L^2}.$$

Proposition 1.4.5. *Let M be a compact Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian vector bundles over M. Let $D \in \mathscr{Diff}_1(E,F)$ be a Dirac-type operator, and let $\bar{D} : H^1(M,E) \to L^2(M,F)$ be the unique bounded extension of D.*
If for $u \in L^2(M,E)$ and $f \in L^2(M,F)$ the equation $Du = f$ holds in the weak sense then actually $u \in H^1(M,E)$ and $\bar{D}u = f$ holds in the usual sense.

Remark 1.4.6. Let $f \in L^2(M,F)$. By Proposition 1.4.5, we have that for any Dirac-type operator $D \in \mathscr{Diff}_1(E,F)$, the equation

$$\bar{D}u = f, \quad \text{holds with } u \in H^1(M,E),$$

if and only if the equation $Du = f$ holds in the weak sense.

In order to prove Proposition 1.4.5, we first introduce smoothing kernels and Friedrichs mollifiers.

Definition 1.4.7. Let $V, W \to M$ be $\mathbb{K}$-vector bundles. Let $\mathrm{pr}_1, \mathrm{pr}_2 : M \times M \to M$ be the projections on the first or second factor, respectively. We define the **exterior tensor product** $V \boxtimes W$ by

$$V \boxtimes W := \mathrm{pr}_1^* V \otimes \mathrm{pr}_2^* W \to M \times M.$$

For $(x,y) \in M \times M$ the fiber of $V \boxtimes W$ over (x,y) is given by[3]

$$(V \boxtimes W)_{(x,y)} = (\mathrm{pr}_1^* V)_{(x,y)} \otimes (\mathrm{pr}_2^* W)_{(x,y)} = V_x \otimes W_y.$$

Let M be a Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian vector bundles over M. Then the vector bundle $F \boxtimes E^* \to M \times M$ has the fibers

$$(F \boxtimes E^*)_{(y,x)} = F_y \otimes E_x^* = \mathrm{Hom}(E_x, F_y).$$

Definition 1.4.8. An operator $A : C^\infty(M,E) \to C^\infty(M,F)$ of the form

$$(Au)(y) = \int\limits_M K(y,x) \cdot u(x)\, dvol(x) \tag{1.39}$$

with

$$K \in C^\infty(M \times M, F \boxtimes E^*)$$

is called a **smoothing operator**. The section K is called the **smoothing kernel** of A.

Remark 1.4.9

i) A smoothing operator operator A extends uniquely to a bounded operator $L^2(M,E) \to L^2(M,F)$. In fact, for any $u \in C^\infty(M,E)$, we have:

$$\begin{aligned}
\|Au\|_{L^2}^2 &= \int\limits_M |Au(y)|^2\, dvol(y) \\
&= \int\limits_M \Big| \int\limits_M K(y,x) u(x)\, dvol(x) \Big|^2\, dvol(y) \\
&\overset{\text{C.S.}}{\leq} \int\limits_M \Big[\int\limits_M |K(y,x)|^2\, dvol(x) \cdot \int\limits_M |u(x)|^2\, dvol(x) \Big]\, dvol(y) \\
&= \int\limits_M \int\limits_M |K(y,x)|^2\, dvol(x)\, dvol(y) \cdot \int\limits_M |u(x)|^2\, dvol(x) \\
&= \|K\|_{L^2}^2 \cdot \|u\|_{L^2}^2.
\end{aligned}$$

Thus $\|A\|_{L^2,L^2} \leq \|K\|_{L^2}$.

ii) A smoothing operator A maps L^2-sections to smooth sections, hence the name. In fact, Au as defined in (1.39) is smooth in y, since K is smooth.

[3] Let $f : X \to Y$ and let E be a vector bundle over Y. Let $x \in X$. Then the fiber $(f^*E)_x$ of the pull-back bundle f^*E is given by $(f^*E)_x = E_{f(x)}$.

iii) For a smoothing operator A, the induced operator $A : L^2(M,E) \to H^k(M,F)$ is bounded for any $k \in \mathbb{N}$:

For any $P \in \mathscr{D}\!\mathit{iff}_k(F,G)$ the composition $P \circ A : L^2(M,E) \to C^\infty(M,G)$ is again a smoothing operator: Since M is compact and $K(y,x)u(x)$ is smooth in y, the differentiations in y commute with integration in x, and we obtain:

$$(PAu)(y) = \int\limits_M P_y K(y,x)\, u(x)\, dvol(x),$$

where $P_y K \in C^\infty(M \times M, G \boxtimes E^*)$. In particular, for $P = \nabla^k$ we have:

$$\left\|(\nabla^k \circ A)u\right\|_{L^2} \leq C_k \|u\|_{L^2}.$$

Hence $A : L^2(M,E) \to H^k(M,F)$ is bounded because we have

$$\|Au\|_{H^k} \leq C\left(\left\|\nabla^k Au\right\|_{L^2} + \ldots + \|Au\|_{L^2}\right) \leq C\|u\|_{L^2}.$$

Next we want to approximate any section $u \in L^2(M,E)$ by smooth sections which are obtained from u by the application of a particular kind of smoothing operators. For this purpose, we introduce the notion of a Friedrichs mollifier.

Definition 1.4.10. A family of operators $J_\varepsilon : L^2(M,E) \to L^2(M,E)$, $0 < \varepsilon \leq \varepsilon_0$, $\varepsilon_0 > 0$, is called a **Friedrichs mollifier** if the following properties hold:

i) Each J_ε is a self-adjoint smoothing operator.

ii) There exists a constant $C > 0$ such that $\|J_\varepsilon\|_{L^2,L^2} \leq C$ holds for all $\varepsilon \in (0,\varepsilon_0]$.

iii) For any $k \in \mathbb{N}$ and any $P \in \mathscr{D}\!\mathit{iff}_k(E,E)$ the commutators $[P, J_\varepsilon]$ extend to bounded operators $H^{k-1}(M,E) \to L^2(M,E)$ and there exists a constant $C > 0$ such that
$$\left\|[P, J_\varepsilon]\right\|_{H^{k-1},L^2} \leq C \qquad \text{for all } \varepsilon \in (0,\varepsilon_0].$$

iv) For every $u \in L^2(M,E)$ we have $J_\varepsilon u \rightharpoonup u$ in $L^2(M,E)$ as $\varepsilon \to 0$.

Example 1.4.11. Choose a smooth function $j : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{supp}(j) \subset B_1(0)$, $j \geq 0$, $j(-x) = j(x)$ for all x and $\int_{\mathbb{R}^n} j(x)\, dx = 1$. For $\varepsilon > 0$ put

$$j_\varepsilon(x) = \varepsilon^{-n}\, j\left(\frac{x}{\varepsilon}\right).$$

Then $\mathrm{supp}(j_\varepsilon) \subset B_\varepsilon(0)$, $j_\varepsilon \geq 0$, $j_\varepsilon(-x) = j_\varepsilon(x)$ for all x and

$$\int\limits_{\mathbb{R}^n} j_\varepsilon(x)\, dx = \int\limits_{\mathbb{R}^n} j\left(\frac{x}{\varepsilon}\right) \frac{dx}{\varepsilon^n} = \int\limits_{\mathbb{R}^n} j(y)\, dy = 1.$$

Now put

$$(J_\varepsilon u)(x) := \int_{\mathbb{R}^n} j_\varepsilon(x-y)\,u(y)\,dy.$$

Smooth functions on the n-torus $T^n = \mathbb{R}^n/2\pi\,\mathbb{Z}^n$ are in $1-1$ correspondence to periodic smooth functions on $\mathbb{R}^n$, i.e., to smooth functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$u(x+k) = u(x), \qquad \text{for all } x \in \mathbb{R}^n, k \in 2\pi\mathbb{Z}^n.$$

Clearly, the operators J_ε preserve periodicity: If u is periodic then we have

$$\begin{aligned}(J_\varepsilon u)(x+k) &= \int_{\mathbb{R}^n} j_\varepsilon(x+k-y)\,u(y)\,dx = \int_{\mathbb{R}^n} j_\varepsilon(x-\xi)\,u(\xi+k)\,d\xi \\ &= \int_{\mathbb{R}^n} j_\varepsilon(x-\xi)\,u(\xi)\,d\xi = (J_\varepsilon u)(x).\end{aligned}$$

The family of operators $J_\varepsilon : C^\infty(T^n) \to C^\infty(T^n)$ is a Friedrichs mollifier on T^n. A proof of this fact can be found in Appendix A.

Remark 1.4.12. Using the example above, one can construct Friedrichs mollifiers on arbitrary compact manifolds and vector bundles with the help of a partition of unity and local trivializations of the bundle.

Proof of Prop. 1.4.5. Let $u \in L^2(M,E)$ and $f \in L^2(M,F)$ be such that the equation $Du = f$ holds in the weak sense. Let $(J_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ be a Friedrichs mollifier on E. Put $u_\varepsilon := J_\varepsilon u$.

a) We check that $u \in H^1(M,E)$: For any $\varepsilon \in (0,\varepsilon_0]$ and any $\varphi \in C^\infty(M,F)$ we have

$$\begin{aligned}(Du_\varepsilon,\varphi)_{L^2} &= \big(u_\varepsilon, D^*\varphi\big)_{L^2} \\ &= \big(J_\varepsilon u, D^*\varphi\big)_{L^2} \\ &= \big(u, J_\varepsilon D^*\varphi\big)_{L^2} \\ &= \big(u, D^*J_\varepsilon\varphi\big)_{L^2} + \big(u, \big[J_\varepsilon, D^*\big]\varphi\big)_{L^2} \\ &= \big(f, J_\varepsilon\varphi\big)_{L^2} + \big(u, \big[J_\varepsilon, D^*\big]\varphi\big)_{L^2}\end{aligned}$$

By properties ii) and iii) of Definition 1.4.10 we have:

$$\begin{aligned}|(Du_\varepsilon,\varphi)_{L^2}| &\le |(f, J_\varepsilon\varphi)_{L^2}| + \big|\big(u, \big[J_\varepsilon, D^*\big]\varphi\big)_{L^2}\big| \\ &\le \|f\|_{L^2}\cdot\|J_\varepsilon\varphi\|_{L^2} + \|u\|_{L^2}\cdot\big\|\big[J_\varepsilon, D^*\big]\varphi\big\|_{L^2} \\ &\le C_1\cdot\|f\|_{L^2}\cdot\|\varphi\|_{L^2} + C_2\cdot\|u\|_{L^2}\cdot\|\varphi\|_{L^2} \\ &= C_3(f,u)\cdot\|\varphi\|_{L^2}.\end{aligned}$$

Setting $\varphi = Du_\varepsilon$ we obtain:

$$\|Du_\varepsilon\|_{L^2} \leq C_3(f,u).$$

Moreover, by property ii) of Definition 1.4.10, we have:

$$\|u_\varepsilon\|_{L^2} = \|J_\varepsilon u\|_{L^2} \leq C_1 \|u\|_{L^2}.$$

Thus, by the Gårding inequality (1.35)

$$\begin{aligned} \|u_\varepsilon\|_{H^1} &\leq C_4 \left(\|Du_\varepsilon\|_{L^2} + \|u_\varepsilon\|_{L^2}\right) \\ &\leq C_4\big(C_3(f,u) + C_1 \|u\|_{L^2}\big) \\ &\leq C_5(f,u). \end{aligned}$$

Thus $\|u_\varepsilon\|_{H^1}$ is bounded uniformly in ε for $\varepsilon \in (0, \varepsilon_0]$. By Lemma 1.2.16, we may choose a sequence $\varepsilon_i \searrow 0$ and $v \in H^1(M,E)$ such that $u_{\varepsilon_i} \rightharpoonup v$ in $H^1(M,E)$ and thus also in $L^2(M,E)$.

On the other hand, by property iv) of Definition 1.4.10, we also have $u_{\varepsilon_i} \rightharpoonup u$ in $L^2(M,E)$. Again by Lemma 1.2.16, weak limits are unique, hence $u = v \in H^1(M,E)$.

b) We check that the equation $\bar{D}u = f$ holds in the usual sense: So let $\varphi \in C^\infty(M,F)$. Since $\bar{D} : H^1(M,E) \to L^2(M,E)$ is continuous, by Lemma 1.2.16, we have:[4]

$$\begin{aligned} \big(\bar{D}u, \varphi\big)_{L^2} &= \big(\bar{D}\big(w{-}H^1{-}\lim_{i\to\infty} u_{\varepsilon_i}\big), \varphi\big)_{L^2} \\ &= \big(w{-}L^2{-}\lim_{i\to\infty} \big(\bar{D}u_{\varepsilon_i}\big), \varphi\big)_{L^2} \\ &= \lim_{i\to\infty} \big(\bar{D}u_{\varepsilon_i}, \varphi\big)_{L^2}. \end{aligned}$$

Since u_{ε_i} is smooth, $\bar{D}u_{\varepsilon_i} = Du_{\varepsilon_i}$ and thus

$$\begin{aligned} \big(\bar{D}u, \varphi\big)_{L^2} &= \lim_{i\to\infty} \big(Du_{\varepsilon_i}, \varphi\big)_{L^2} \\ &= \lim_{i\to\infty} \big(u_{\varepsilon_i}, D^*\varphi\big)_{L^2} \\ &= \big(u, D^*\varphi\big)_{L^2} \\ &= (f, \varphi)_{L^2}. \end{aligned}$$

Since this is valid for all $\varphi \in C^\infty(M,F)$ we conclude that $\bar{D}u = f$. □

[4]Here and in the following $w-H^1-\lim\limits_{i\to\infty}$ and $w-L^2-\lim\limits_{i\to\infty}$ denote the weak limits in $H^1(M,E)$ and $L^2(M,E)$, respectively.

Reminder. Let $\mathscr{H}$ be a Hilbert space. An **unbounded operator** A in $\mathscr{H}$ is a linear map

$$A : \mathrm{dom}(A) \to \mathscr{H},$$

where $\mathrm{dom}(A) \subset \mathscr{H}$ is a dense linear subspace, called the **domain** of A. An operator A with $\mathrm{dom}(A) \subset \mathscr{H}$ dense is also called **densely defined**.

Let $\mathscr{H}$ be a Hilbert space and let $A : \mathrm{dom}(A) \subset \mathscr{H} \to \mathscr{H}$ be an unbounded operator. The adjoint operator A^* of A is the operator such that the relation

$$\big(A^*u, v\big) = \big(u, Av\big)$$

holds for all $u \in \mathrm{dom}(A^*)$ and all $v \in \mathrm{dom}(A)$. The domain of the adjoint operator is by definition the largest possible:

Definition 1.4.14. Let $\mathscr{H}$ be a Hilbert space and let $A : \mathscr{H} \supset \mathrm{dom}(A) \to \mathscr{H}$ be an unbounded operator. We set

$$\mathbf{dom}\big(\boldsymbol{A^*}\big) := \{u \in \mathscr{H} \mid \exists\, f \in \mathscr{H} \text{ with } (f, v) = (u, Av) \text{ for all } v \in \mathrm{dom}(A)\}.$$

On this domain, define $A^* : \mathscr{H} \supset \mathrm{dom}(A^*) \to \mathscr{H}$ by $A^*u := f$.

Remark 1.4.15. The adjoint operator is well defined: For a given $u \in \mathrm{dom}(A^*)$, the vector $f \in \mathscr{H}$ satisfying $(f, v) = (u, Av)$ for all $v \in \mathrm{dom}(A)$ is uniquely determined by u, since A is densely defined.
A densely defined operator A is called **symmetric** iff

$$(Au, v) = (u, Av) \qquad \text{for all } u, v \in \mathrm{dom}(A).$$

In this case, $\mathrm{dom}(A) \subset \mathrm{dom}(A^*)$ and $A^*|_{\mathrm{dom}(A)} = A$, i.e., A^* is an extension of A. We also write $A \subset A^*$.
If A is symmetric and $\mathrm{dom}(A) = \mathrm{dom}(A^*)$ then A is called **self-adjoint**.

Proposition 1.4.16. *Let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle over M, and let $D \in \mathscr{D}\!\mathit{iff}_1(E, E)$ be a formally self-adjoint Dirac-type operator.*
Then the operator $\bar{D} : H^1(M; E) \to L^2(M, E)$ is self-adjoint.

Proof. The fact that D is formally self-adjoint implies that $\bar{D}$ is symmetric. It remains to show that $\mathrm{dom}(\bar{D}^*) \subset \mathrm{dom}(\bar{D}) = H^1(M, E)$. Now for any $u \in \mathrm{dom}(\bar{D}^*)$, by definition, there is an element $f \in L^2(M, E)$ such that

$$(u, \bar{D}v)_{L^2} = (f, v)_{L^2} \qquad \forall v \in H^1(M, E)\,.$$

In particular, we have:

$$(u, \bar{D}v)_{L^2} = (f, v)_{L^2} \qquad \forall v \in C^\infty(M, E)\,.$$

Hence the equation $D^* u = f$ holds in the weak sense. Proposition 1.4.5 implies $u \in H^1(M, E) = \operatorname{dom}(\bar{D})$. □

Reminder. Let $A : \operatorname{dom}(A) \subset \mathscr{H} \to \mathscr{H}$ be an unbounded operator on a Hilbert space $\mathscr{H}$. The **resolvent set** $\operatorname{res}(A)$ of A is defined as:

$$\operatorname{res}(A) := \left\{\lambda \in \mathbb{C} \,|\, (A - \lambda) : \operatorname{dom}(A) \to \mathscr{H} \text{ is bijective and } (A-\lambda)^{-1} \text{ is bounded}\right\}. \tag{1.40}$$

Here $A - \lambda$ is a short-hand notation for $A - \lambda \cdot \mathrm{id}_{\mathscr{H}}$.
The **spectrum** $\operatorname{spec}(A)$ of A is defined as the complement of the resolvent set:

$$\operatorname{spec}(A) := \mathbb{C} \backslash \operatorname{res}(A) \tag{1.41}$$

For a self-adjoint operator A we have $\operatorname{spec}(A) \subset \mathbb{R}$ and $\operatorname{spec}(A^2) \subset [0, \infty)$. A proof of this fact can be found in books on functional analysis, e. g. in [6], Satz 30.5.

Now let $D \in \mathscr{D}\!\mathit{iff}_1(E, E)$ be a formally self-adjoint Dirac-type operator. Then the operator $\bar{D} : H^1(M; E) \to L^2(M, E)$ is self-adjoint. Since $-1 \notin \operatorname{spec}(\bar{D}^2)$, the operator

$$\bar{D}^2 + 1 : \operatorname{dom}(\bar{D}^2) \to L^2(M; E)$$

has an inverse

$$(\bar{D}^2 + 1)^{-1} : L^2(M, E) \to \operatorname{dom}(\bar{D}^2 + 1) \subset L^2(M, E),$$

bounded in L^2. Let C_0 be an L^2-L^2-bound for $(\bar{D}^2 + 1)^{-1}$. Then we also find an L^2-H^1-bound:

$$\begin{aligned}
&\left\| (\bar{D}^2 + 1)^{-1} u \right\|_{H^1}^2 \\
&\overset{(1.37)}{\leq} \quad C_1 \cdot \left\{ \left\| \bar{D}(\bar{D}^2 + 1)^{-1} u \right\|_{L^2}^2 + \left\| (\bar{D}^2 + 1)^{-1} u \right\|_{L^2}^2 \right\} \\
&\leq \quad C_1 \cdot \left\{ \left(\bar{D}^2 \left(\bar{D}^2 + 1\right)^{-1} u, (\bar{D}^2 + 1)^{-1} u \right)_{L^2} + C_0 \cdot \|u\|_{L^2}^2 \right\} \\
&\leq \quad C_1 \cdot \left\{ \left((\bar{D}^2 + 1)\left(\bar{D}^2 + 1\right)^{-1} u - \left(\bar{D}^2 + 1\right)^{-1} u, \left(\bar{D}^2 + 1\right)^{-1} u \right)_{L^2} + C_0 \cdot \|u\|_{L^2}^2 \right\} \\
&\leq \quad C_1 \cdot \left\{ \left(u, \left(\bar{D}^2 + 1\right)^{-1} u \right)_{L^2} + \left\| \left(\bar{D}^2 + 1\right)^{-1} u \right\|_{L^2}^2 + C_0 \|u\|_{L^2}^2 \right\} \\
&\leq \quad C_1 \cdot \left\{ \|u\|_{L^2} \cdot \underbrace{\left\| \left(\bar{D}^2 + 1\right)^{-1} u \right\|_{L^2}}_{\leq \sqrt{C_0} \cdot \|u\|_{L^2}} + 2\, C_0 \cdot \|u\|_{L^2}^2 \right\} \\
&\leq \quad C_2 \cdot \|u\|_{L^2}^2\,.
\end{aligned}$$

Hence $\left(\bar{D}^2+1\right)^{-1}$ factors through $H^1(M,E)$. Thus $K := \left(\bar{D}^2+1\right)^{-1} : L^2(M,E) \to L^2(M,E)$ is compact by the Rellich embedding theorem, see Remark 1.2.18 and Theorem 1.2.14.
From the spectral theorem for compact self-adjoint operators we know that the Hilbert space $\mathscr{H} = L^2(M,E)$ has an orthogonal decomposition

$$L^2(M,E) = \overline{\bigoplus_{n\in\mathbb{N}} E(\mu_n, K)}$$

into eigenspaces $E(\mu_n,K) = \{v \in L^2(M,E) \,|\, Kv = \mu_n v\}$ of K. Moreover, all eigenspaces are finite dimensional and the eigenvalues μ_n are real and $\mu_n \searrow 0$. Since $K = (\bar{D}^2+1)^{-1}$ is injective, we have $\mu_n \neq 0$ for all n and thus:

$$\begin{aligned} u \in E(\mu_n,K) &\Longleftrightarrow \left(\bar{D}^2+1\right)^{-1} u = \mu_n\, u \\ &\Longleftrightarrow \left(\bar{D}^2+1\right) u = \frac{1}{\mu_n}\, u \\ &\Longleftrightarrow \bar{D}^2 u = \left(\frac{1}{\mu_n} - 1\right)\cdot u\,. \end{aligned}$$

Moreover, the eigenspaces $E(\mu_n,K)$ are $\bar{D}$-invariant, since for $u \in E(\mu_n,K)$, we have:

$$\bar{D}^2\left(\bar{D}u\right) = \bar{D}\left(\bar{D}^2 u\right) = \bar{D}\left(\frac{1}{\mu_n}-1\right)\cdot u = \left(\frac{1}{\mu_n}-1\right)\cdot \bar{D}u\,.$$

Therefore, $\bar{D}|_{E(\mu_n,K)}$ is an endomorphism of $E(\mu_n,K)$ and self-adjoint with respect to $(\cdot,\cdot)_{L^2}$. Hence the K-eigenspaces $E(\mu_n,K)$ split L^2-orthogonally into eigenspaces for $\bar{D}$:

$$E(\mu_n,K) = E\big(\lambda_n,\bar{D}\big) \oplus E\left(-\lambda_n,\bar{D}\right)\,,$$

where $\lambda_n := \sqrt{\frac{1}{\mu_n}-1}$.

Summarizing the above discussion we obtain:

Theorem 1.4.18. *Let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle over M, and let $D \in \mathscr{Diff}_1(E,E)$ be a formally self-adjoint Dirac-type operator.*
Then the spectrum of $\bar{D} : H^1(M,E) \to L^2(M,E)$ consists of eigenvalues only. Its eigenvalues λ_n, $n \in \mathbb{Z}$, form a discrete subset of $\mathbb{R}$ and satisfy $\lambda_n \xrightarrow{n\to\infty} \infty$ and $\lambda_n \xrightarrow{n\to-\infty} -\infty$. The eigenspaces $E\big(\lambda_n,\bar{D}\big)$ are all finite dimensional, and we have the L^2-orthogonal decomposition:

$$L^2(M,E) = \overline{\bigoplus_{n\in\mathbb{Z}} E(\lambda_n,\bar{D})}\,. \tag{1.42}$$

Example 1.4.19. Let $M = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and let $E \to M$ be the trivial complex line bundle. Let $D = i\frac{d}{dx}$ and denote by $u_n(x) := e^{-inx}$, $n \in \mathbb{Z}$ the usual orthonormal basis of $\mathscr{H} = L^2(S^1)$. Then we have $Du_n = i \cdot (-in) \cdot e^{-inx} = n \cdot u_n$. Hence $\operatorname{spec}(D) = \mathbb{Z}$ and all eigenvalues have multiplicity 1. The orthogonal decomposition in (1.42) of a function in $L^2(S^1)$ is nothing but the Fourier decomposition of the corresponding 2π-periodic function on $\mathbb{R}$.

Functional calculus

Let M be a compact Riemannian manifold, let $E \to M$ be Riemannian or Hermitian vector bundle over M, and let $D \in \mathscr{D}\!\mathit{iff}_1(E,E)$ be a formally self-adjoint Dirac-type operator. Let $f : \operatorname{spec}(D) \to \mathbb{R}$ be a function on the spectrum of D.
We define an operator $f(\bar{D}) : \operatorname{dom}(f(\bar{D})) \subset L^2(M,E) \to L^2(M,E)$ as follows: By (1.42) we may decompose any $u \in L^2(M,E)$ into $u = \sum_{n\in\mathbb{Z}} u_n$ with $u_n \in E(\lambda_n, \bar{D})$. We then put

$$f(\bar{D})\, u := \sum_{n\in\mathbb{Z}} f(\lambda_n)\, u_n\,. \tag{1.43}$$

The largest possible domain of $f(\bar{D})$ is the set of those u for which the right hand side of (1.43) converges in $L^2(M,E)$. We thus set:

$$\begin{aligned} \operatorname{dom}\left(f(\bar{D})\right) &:= \Big\{ u = \sum_{n\in\mathbb{Z}} u_n \;\Big|\; \sum_{n\in\mathbb{Z}} f(\lambda_n)\, u_n \text{ converges in } L^2(M,E) \Big\} \\ &= \Big\{ u = \sum_{n\in\mathbb{Z}} u_n \;\Big|\; \sum_{n\in\mathbb{Z}} |f(\lambda_n)|^2 \cdot \|u_n\|^2_{L^2} < \infty \Big\}. \end{aligned} \tag{1.44}$$

The equality holds, since the eigenvectors u_n, $n \in \mathbb{Z}$, are mutually perpendicular and hence $\|\sum_{n\in\mathbb{Z}} f(\lambda_n) u_n\|^2_{L^2} = \sum_{n\in\mathbb{Z}} |f(\lambda_n)|^2 \|u_n\|^2_{L^2}$.

Examples 1.4.20
1) For $f \equiv 1$, we have $f(\bar{D}) = \mathrm{id}_{L^2(M,E)}$.

2) Let $f(\lambda) = a_k\lambda^k + \ldots + a_1\lambda + a_0$ be a polynomial. Then

$$f\left(\bar{D}\right) = a_k \bar{D}^k + \ldots + a_1 \bar{D} + a_0 \cdot \mathrm{id}_{L^2(M,E)}\,.$$

Here $\left(\bar{D}^k\right) := \underbrace{\bar{D} \circ \ldots \circ \bar{D}}_{k \text{ times}}$.

Proposition 1.4.21. *Let M be a compact Riemannian manifold, and let $E \to M$ be a Riemannian or Hermitian vector bundle over M. Let $D \in \mathscr{D}\!\mathit{iff}_1(E,E)$ be a formally self-adjoint Dirac-type operator, and let f be a bounded function on* $\operatorname{spec}(D)$*.*
Then we have:

$$\left\| f\left(\bar{D}\right) \right\|_{L^2,L^2} = \sup_{\lambda \in \operatorname{spec}(D)} |f(\lambda)|\,. \tag{1.45}$$

In particular, $\operatorname{dom}\big(f(\bar{D})\big) = L^2(M,E)$.

Proof. Let $u = \sum_{n\in\mathbb{Z}} u_n \in L^2(M,E)$. We compute:

$$\begin{aligned}\big\|f(\bar{D})u\big\|_{L^2}^2 &= \sum_{n\in\mathbb{Z}} |f(\lambda_n)|^2 \cdot \|u_n\|_{L^2}^2 \\ &\le \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)|^2 \cdot \sum_{n\in\mathbb{Z}} \|u_n\|_{L^2}^2 \\ &= \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)|^2 \cdot \|u\|_{L^2}^2 \,.\end{aligned}$$

This shows $\big\|f(\bar{D})\big\|_{L^2,L^2} \le \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)|$ and $\operatorname{dom}\big(f(\bar{D})\big) = L^2(M,E)$. Now assume $\big\|f(\bar{D})\big\|_{L^2,L^2} \le \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)| - \varepsilon$ for some $\varepsilon > 0$. Choose $\lambda_n \in \operatorname{spec}(D)$ such that

$$|f(\lambda_n)| > \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)| - \varepsilon.$$

Then for $u_n \in E(\lambda_n, \bar{D})\backslash\{0\}$ we find:

$$\begin{aligned}\big\|f(\bar{D})u_n\big\|_{L^2}^2 &= \|f(\lambda_n)\cdot u_n\|_{L^2}^2 \\ &= |f(\lambda_n)|^2 \cdot \|u_n\|_{L^2}^2 \\ &> (\sup_{\lambda\in\operatorname{spec}(D)} |f(\lambda)| - \varepsilon)^2 \cdot \|u_n\|_{L^2}^2 \\ &\ge \|f(\bar{D})\|_{L^2,L^2}^2 \cdot \|u_n\|_{L^2}^2 \,,\end{aligned}$$

a contradiction. Hence $\|f(\bar{D})\|_{L^2,L^2} = \sup_{\lambda\in\operatorname{spec}(D)} |f(\lambda)|$. □

Remark 1.4.22
For any two functions f_1, f_2 on $\operatorname{spec}(D)$, the operators $f_1(\bar{D})$ and $f_2(\bar{D})$ commute. This follows directly from the definition.

Example 1.4.23. For a fixed $t > 0$, put $f(\lambda) := e^{-t\lambda^2}$. Then we have

$$1 = \sup_{\lambda\,\in\,\mathbb{R}} |f(\lambda)| \ge \sup_{\lambda\,\in\,\operatorname{spec}(D)} |f(\lambda)|.$$

Hence the operator norm of $\exp(-t\bar{D}^2) : L^2(M,E) \to L^2(M,E)$ is bounded by 1.

Proposition 1.4.24. *Let M be a compact Riemannian manifold, and let $E \to M$ be a Riemannian or Hermitian vector bundle over M. Let $D \in \mathscr{Diff}_1(E,E)$ be a formally self-adjoint Dirac-type operator.*
For any $u \in L^2(M,E)$, we have:

$$\exp\big(-t\bar{D}^2\big)u \xrightarrow{L^2} u \qquad \text{as } t \searrow 0.$$

Proof. For any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\sum_{|n|>N} \|u_n\|^2_{L^2} < \varepsilon$. We compute:

$$\begin{aligned}
&\big\|\exp\big(-t\bar{D}^2\big)u - u\big\|^2_{L^2} \\
&= \sum_{|n|\le N} \big|\exp(-t\lambda_n^2) - 1\big|^2 \cdot \|u_n\|^2_{L^2} + \underbrace{\sum_{|n|>N} \underbrace{\big|\exp(-t\lambda_n^2) - 1\big|^2}_{\le 1} \cdot \|u_n\|^2_{L^2}}_{\le \varepsilon} \\
&\le \sum_{|n|\le N} \big|\exp(-t\lambda_n^2) - 1\big|^2 \cdot \|u_n\|^2_{L^2} + \varepsilon.
\end{aligned}$$

Since the first term is a finite sum, we obtain $\limsup_{t\to 0} \big\| \exp(-t\bar{D}^2)u - u\big\|^2_{L^2} \le 0 + \varepsilon$. By taking $\varepsilon \searrow 0$, we end up with:

$$\exp\big(-t\bar{D}^2\big)u \xrightarrow{L^2} u \qquad \text{as } t \searrow 0. \qquad \square$$

Now for any $u = \sum_{n\in\mathbb{Z}} u_n \in L^2(M,E)$, we have

$$\begin{aligned}
\Big\|\bar{D}^k \exp\big(-t\bar{D}^2\big)u\Big\|^2_{L^2} &= \sum_{n\in\mathbb{Z}} \underbrace{\lambda_n^{2k} \exp\big(-2t\lambda_n^2\big)}_{\le C(t,k)} \cdot \|u_n\|^2_{L^2} \\
&\le C(t,k) \cdot \sum_{n\in\mathbb{Z}} \|u_n\|^2_{L^2} \\
&= C(t,k) \cdot \|u\|^2_{L^2}\,. \qquad (1.46)
\end{aligned}$$

Remark 1.4.25. a) We have for all $k \in \mathbb{N}_0$:

$$H^k(M,E) = \operatorname{dom}\big(\bar{D}^k\big) = \Big\{ u = \sum_{n\in\mathbb{Z}} u_n \in L^2(M,E) \,\Big|\, \sum_{n\in\mathbb{Z}} \lambda_n^{2k} \|u_n\|^2_{L^2} < \infty \Big\}.$$

b) For every $t > 0$ the operator $\exp(-t\bar{D}^2)$ is a smoothing operator:

$$\exp\big(-t\bar{D}^2\big) : L^2(M,E) \to C^\infty(M,E).$$

Proof. a) The second equation and the inclusion "$\subset$" in the first equation are obvious. We prove the inclusion "$\supset$" in the first equation, i.e., we show by induction on k: if $u \in L^2(M,E)$ is such that $\bar{D}^k u \in L^2(M,E)$, then $u \in H^k(M,E)$.

For $k = 1$ this follows from Proposition 1.4.5. Therefore let $k \geq 2$ and assume that the assertion holds for $k-1$. We have $\bar{D}^k u = f \in H^k(M,E) \subset H^{k-1}(M,E)$, thus by the induction hypothesis we may assume that $u \in H^{k-1}(M,E)$. Let $(J_\varepsilon)_\varepsilon$ be a Friedrichs mollifier on M. We put $u_\varepsilon := J_\varepsilon u \in C^\infty(M,E)$. Then we have

$$\bar{D}^k u_\varepsilon = J_\varepsilon \bar{D}^k u + [\bar{D}^k, J_\varepsilon]u,$$

and since $\bar{D}^k u = f$ and by properties ii) and iii) of a Friedrichs mollifier we get

$$\|\bar{D}^k u_\varepsilon\|_{L^2} \leq \|J_\varepsilon \bar{D}^k u\|_{L^2} + \|[\bar{D}^k, J_\varepsilon]u\|_{L^2} \leq C_1 \|f\|_{L^2} + C_2 \|u\|_{H^{k-1}},$$

where $C_1, C_2 > 0$ are independent of ε. Moreover by property ii) of a Friedrichs mollifier we have

$$\|u_\varepsilon\|_{L^2} \leq C_1 \|u\|_{L^2}.$$

By applying the elliptic estimates (1.37) iteratively it follows that

$$\|u_\varepsilon\|_{H^k} \leq C_3(\|\bar{D}^k u_\varepsilon\|_{L^2} + \|u_\varepsilon\|_{L^2}) \leq C_4(\|f\|_{L^2} + \|u\|_{H^{k-1}}),$$

where $C_3, C_4 > 0$ are independent of ε as $\varepsilon \to 0$. Therefore, the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $H^k(M,E)$ as $\varepsilon \to 0$. Hence there exists a weakly convergent subsequence $u_{\varepsilon_j} \xrightharpoonup{H^k} w \in H^k(M,E)$. On the other hand, we have $u_{\varepsilon_j} \xrightarrow{L^2} u$ by property iv) of a Friedrichs mollifier. Since weak limits are unique, we conclude that $w = u$, and in particular that $u \in H^k(M,E)$.

b) The equation (1.46) now shows that for every $u \in L^2(M,E)$ we have $\exp(-t\bar{D}^2)u \in \mathrm{dom}(\bar{D}^k) = H^k(M,E)$ for all $k \in \mathbb{N}$ and thus $\exp(-t\bar{D}^2)u \in C^\infty(M,E)$ by the Sobolev embedding theorem. $\square$

Theorem 1.4.26 (Elliptic regularity). *Let M be a compact Riemannian manifold, and let $E, F \to M$ be Riemannian or Hermitian vector bundles over M. Let $D \in \mathscr{Diff}_1(E,F)$ be a Dirac-type operator. Let $u \in L^2(M,E)$ and $f \in L^2(M,F)$ be sections such that the equation $Du = f$ holds in the weak sense.*
If $f \in H^k(M,F)$ then $u \in H^{k+1}(M,E)$.

Proof. a) Let us first assume that $E = F$ and D is formally self-adjoint. Put $J_\varepsilon := \exp(-\varepsilon\bar{D}^2)$. By equation (1.45), we have $\|J_\varepsilon\|_{L^2,L^2} \leq 1$, since the function $\lambda \mapsto \exp(-\varepsilon\lambda^2)$ is bounded by 1. Further, $[J_\varepsilon, \bar{D}] = 0$ by Remark 1.4.22. For any $v \in L^2(M,E)$, we have $J_\varepsilon v \xrightarrow{L^2} v$ by Proposition 1.4.24. Finally, by Remark 1.4.25, we have $J_\varepsilon v \in C^\infty(M, E)$ for any $v \in L^2(M, E)$. Hence the family J_ε is a Friedrichs mollifier.

We show by induction on $k \in \mathbb{N}_0$ that the maps $J_\varepsilon : H^k(M, E) \to H^k(M, E)$ are bounded, uniformly in ε. We have just seen that $\|J_\varepsilon\|_{L^2, L^2} \leq 1$.

Assume that $\|J_\varepsilon\|_{H^k, H^k} \leq C_0$ for some constant C_0 independent of ε. For any $v \in H^{k+1}(M, E)$, we have by Proposition 1.4.2:

$$\begin{aligned}\|J_\varepsilon v\|_{H^{k+1}} &\leq C_1 \cdot \left(\left\|\bar{D} J_\varepsilon v\right\|_{H^k} + \|J_\varepsilon v\|_{H^k}\right) \\ &\leq C_1 \cdot \left(\left\|J_\varepsilon \bar{D} v\right\|_{H^k} + C_0 \|v\|_{H^k}\right) \\ &\leq C_2 \cdot \left(\left\|\bar{D} v\right\|_{H^k} + \|v\|_{H^k}\right) \\ &\leq C_3 \|v\|_{H^{k+1}} ,\end{aligned}$$

where C_3 is independent of ε.

b) Now let $u \in L^2(M, E)$ be a weak solution of $Du = f$ with $f \in H^k(M, E)$, where we still assume D to be formally self-adjoint. We prove by induction on $k \in \mathbb{N}_0$ that $f \in H^k(M, E)$ implies $u \in H^{k+1}(M, E)$.

For $k = 0$, the implication coincides with Proposition 1.4.5. So let $k \geq 1$ and let $f \in H^k(M, E)$. We have $f \in H^{k-1}(M, E)$ and thus by the induction hypothesis we have $u \in H^k(M, E)$. Then we have by Proposition 1.4.2

$$\begin{aligned}\|J_\varepsilon u\|_{H^{k+1}} &\leq C_4 \cdot \left(\|\bar{D} J_\varepsilon u\|_{H^k} + \|J_\varepsilon u\|_{H^k}\right) \\ &= C_4 \cdot \left(\|J_\varepsilon \bar{D} u\|_{H^k} + \|J_\varepsilon u\|_{H^k}\right) \\ &\overset{a)}{\leq} C_5 \cdot \left(\|\bar{D} u\|_{H^k} + \|u\|_{H^k}\right) \\ &= C_5 \cdot \left(\|f\|_{H^k} + \|u\|_{H^k}\right) \\ &\leq C_6(u, f) .\end{aligned}$$

Thus the family $J_\varepsilon u$ is bounded in H^{k+1}, uniformly in ε. Hence there exists a weakly convergent subsequence $J_{\varepsilon_j} u \overset{H^{k+1}}{\rightharpoonup} w \in H^{k+1}(M, E)$. On the other hand, we have $J_{\varepsilon_j} u \xrightarrow{L^2} u$. Since weak limits are unique, we conclude that $w = u$, and in particular that $u \in H^{k+1}(M, E)$.

c) Now we drop the assumption that $E = F$ and D be formally self-adjoint. Instead, we consider the operator

$$\mathcal{D} := \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \in \mathscr{D}\!\mathit{iff}_1(E \oplus F, E \oplus F) ,$$

which is obviously formally self-adjoint. We check that $\mathcal{D}$ is of Dirac-type:

$$\mathcal{D}^* \mathcal{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} = \begin{pmatrix} D^* D & 0 \\ 0 & D D^* \end{pmatrix} = \mathcal{D} \mathcal{D}^* .$$

Hence we have for the principal symbol:

$$\begin{aligned}\sigma_2(\mathcal{D}\mathcal{D}^*,\xi) = \sigma_2(\mathcal{D}^*\mathcal{D},\xi) &= \begin{pmatrix}\sigma_2(D^*D,\xi) & 0\\ 0 & \sigma_2(DD^*,\xi)\end{pmatrix}\\ &= \begin{pmatrix}-|\xi|^2\cdot \mathrm{id}_E & 0\\ 0 & -|\xi|^2\cdot \mathrm{id}_F\end{pmatrix}\\ &= -|\xi|^2 \cdot \mathrm{id}_{E\oplus F}\ .\end{aligned}$$

For the sections $\binom{u}{0} \in L^2(M, E\oplus F)$ and $\binom{0}{f} \in H^k(M, E\oplus F)$

$$\mathcal{D}\begin{pmatrix}u\\0\end{pmatrix} = \begin{pmatrix}0 & D^*\\ D & 0\end{pmatrix}\cdot\begin{pmatrix}u\\0\end{pmatrix} = \begin{pmatrix}0\\Du\end{pmatrix} = \begin{pmatrix}0\\f\end{pmatrix}$$

holds in the weak sense. We conclude from part b) that $\binom{u}{0} \in H^{k+1}(M, E\oplus F)$ and hence $u \in H^{k+1}(M,E)$. □

Corollary 1.4.27. *Let M be a compact Riemannian manifold, and let $E \to M$ be a Riemannian or Hermitian vector bundle over M. Let $D \in \mathscr{Diff}_1(E,E)$ be a formally self-adjoint Dirac-type operator.*
Then all eigensections of D are smooth.

Proof. If $Du = \lambda u$ holds in the weak sense with $u \in L^2(M,E)$ then Theorem 1.4.26 implies $u \in H^1(M,E)$. Similarly, if $Du = \lambda u$ with $u \in H^k(M,E)$, then Theorem 1.4.26 implies $u \in H^{k+1}(M,E)$.
Hence $u \in \bigcap_{k\in\mathbb{N}_0} H^k(M,E) \subset C^\infty(M,E)$ by the Sobolev embedding theorem. □

Corollary 1.4.28 (Fredholm alternative). *Let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle and let $D \in \mathscr{Diff}_1(E,E)$ be a formally self-adjoint Dirac-type operator. Then we have*

$$C^\infty(M,E) = \ker(D) \oplus D(C^\infty(M,E))$$

and the sum is orthogonal with respect to $(\cdot,\cdot)_{L^2}$.

Proof. One checks that $D(C^\infty(M,E)) = \{u \in C^\infty(M,E) \,|\, (u,v)_{L^2} = 0 \text{ for all } v \in \ker(D)\}$. The details are left as an exercise. □

1.5. Hodge theory

Definition 1.5.1. Let M be a smooth manifold, and let $E_j \to M$, $j = 0, \dots, N+1$, be vector bundles over M. Let $d_j \in \mathscr{Diff}_1(E_j, E_{j+1})$, $j = 0, \dots, N$, be first order operators satisfying $d_{j+1} \circ d_j \equiv 0$ for $j = 0, \dots, N-1$. Then the sequence

$$C^\infty(M, E_0) \xrightarrow{d_0} C^\infty(M, E_1) \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} C^\infty(M, E_N) \xrightarrow{d_N} C^\infty(M, E_{N+1}) \tag{1.47}$$

is called a **complex** of differential operators and is denoted by $(E_\bullet, d_\bullet)$.
The vector space

$$H^j(E_\bullet, d_\bullet) := \frac{\ker d_j : C^\infty(M, E_j) \to C^\infty(M, E_{j+1})}{\operatorname{im} d_{j-1} : C^\infty(M, E_{j-1}) \to C^\infty(M, E_j)} \tag{1.48}$$

is called the ***j*-th cohomology** of the complex $(E_\bullet, d_\bullet)$.
A complex $(E_\bullet, d_\bullet)$ is called a **Dirac complex**, iff the manifold M and the bundles $E_j \to M$, $j = 0, \dots, N+1$, carry metrics such that the operator

$$D := d + d^* := \bigoplus_{j=0}^{N} d_j \oplus \bigoplus_{j=0}^{N} d_j^* : C^\infty(M, E_0 \oplus \dots \oplus E_{N+1}) \to C^\infty(M, E_0 \oplus \dots \oplus E_{N+1})$$

is of Dirac-type.

Remark 1.5.2. The condition $d_{j+1} \circ d_j \equiv 0$ is equivalent to $\operatorname{im}(d_j) \subset \ker(d_{j+1})$; thus the definition of cohomology makes sense.

Example 1.5.3. The **de Rham complex** consists of the bundles $E_j := \Lambda^j T^* M$ with the exterior derivative $d_j : C^\infty(M, \Lambda^j T^* M) \to C^\infty(M, \Lambda^{j+1} T^* M)$ on j-forms as j-th differential operator. For any Riemannian metric on M and the induced metrics on the bundles $\Lambda^j T^* M$, the de Rham complex is a Dirac complex, since the Euler operator $D = d + d^*$ is of Dirac-type by Example 1.3.10.
Obviously, the j-th cohomology of the de Rham complex is nothing but the j-th de Rham cohomology of M.

Example 1.5.4. Let M be a complex manifold of complex dimension m. For a fixed $p \in \{0, \dots, m\}$ set $E_j := \Lambda^{p,j} T^* M$ and $d_j := \sqrt{2} \cdot \bar{\partial}$. This defines the **Dolbeault complex** of M. By Example 1.3.20, for any Hermitian metric on M and the induced metrics on the bundles $\Lambda^{p,j} T^* M$, the Dolbeault complex is a Dirac complex, since the Dolbeault operator $D = \sqrt{2} \cdot (\bar{\partial} + \bar{\partial}^*)$ is a Dirac-type operator.

In the following, let M be a compact manifold, and let $(E_\bullet, d_\bullet)$ be a Dirac complex on M. The aim of **Hodge theory** is to find particularly nice representatives of the cohomology classes of $(E_\bullet, d_\bullet)$. The idea is to do this by minimizing the L^2-norm: For any cohomology class $\omega \in H^j(E_\bullet, d_\bullet)$, we look for $\alpha \in C^\infty(M, E_j)$ that minimizes $\|\cdot\|_{L^2}$ on $\omega \subset C^\infty(M, E_j)$. Thus for any $\eta \in C^\infty(M, E_{j-1})$, we set

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \Big\|\alpha + t\, d_{j-1}\,\eta\Big\|^2_{L^2} \\ &= \frac{d}{dt}\Big|_{t=0} \int_M \big\langle \alpha + t\, d_{j-1}\,\eta, \alpha + t\, d_{j-1}\,\eta\big\rangle \, dvol \\ &= \int_M \big[\langle \alpha, d_{j-1}\eta\rangle + \langle d_{j-1}\eta, \alpha\rangle\big]\, dvol \\ &= 2\,\mathrm{Re} \int_M \langle \alpha, d_{j-1}\eta\rangle \, dvol \\ &= 2\,\mathrm{Re} \int_M \langle d^*_{j-1}\alpha, \eta\rangle \, dvol. \end{aligned}$$

Thus,

$$\int_M \langle d^*_{j-1}\alpha, \eta\rangle \, dvol = 0 \qquad \text{for all } \eta \in C^\infty(M, E_{j-1}),$$

hence $d^*_{j-1}\alpha = 0$. By assumption, α represents a cohomology class $\omega \in H^j(E_\bullet, d_\bullet)$, hence $d_j\alpha = 0$. We thus conclude

$$D\alpha = 0.$$

Definition 1.5.5. Let $(E_\bullet, d_\bullet)$ be a Dirac complex on M. The operator

$$\Delta_d := D^*D = D^2$$

is called the **Hodge Laplacian** of the Dirac complex.

For the de Rham complex, we had already defined the Hodge Laplacians Δ_d in degree k in (1.16).

Remark 1.5.6. Let M be a compact manifold, and let D be a self-adjoint Dirac-type operator, acting on sections of a vector bundle E over M. Then we have $D\alpha = 0 \iff D^2\alpha = 0$:
For, if $D^2\alpha = 0$, we have

$$0 = \big(D^2\alpha, \alpha\big)_{L^2} = (D\alpha, D\alpha)_{L^2} = \|D\alpha\|^2_{L^2},$$

and thus, $D\alpha = 0$. The reverse implication is obvious.

We have seen above that a d_j-closed section $\alpha \in C^\infty(M, E_j)$ that minimizes the L^2-norm in its cohomology class necessarily satisfies $\Delta_d \alpha = 0$. In the case of the de Rham complex, this means that α is a harmonic form. For the general case we define:

Definition 1.5.7. Let D be a formally self-adjoint Dirac-type operator, acting on sections of a vector bundle E over M.
A section $\alpha \in C^\infty(M, E)$ is called **harmonic** iff $D\alpha = 0$.

Remark 1.5.8. Let D be a self-adjoint Dirac-type operator, acting on sections of a vector bundle E over a compact manifold M. By Remark 1.5.6, a section $\alpha \in C^\infty(M, E)$ is harmonic iff $\Delta_d \alpha = 0$.

Theorem 1.5.9 (Hodge). *Let M be a compact Riemannian manifold, and let $(E_\bullet, d_\bullet)$ be a Dirac complex on M. Then any cohomology class in $H^*(E_\bullet, d_\bullet)$ has a unique harmonic representative.*
More precisely, the map

$$\begin{aligned} \ker\left(\Delta_d : C^\infty(M, E_j) \to C^\infty(M, E_j)\right) &\to H^j(E_\bullet, d_\bullet) \\ \alpha &\mapsto [\alpha] \end{aligned}$$

is a vector space isomorphism.

Remark 1.5.10. Let $\alpha \in C^\infty(M, E_j)$. By Remark 1.5.6, the condition $\Delta_d \alpha = 0$ is equivalent to α being harmonic, i.e. to

$$0 = D\alpha = \underbrace{d_j \alpha}_{\in E_{j+1}} + \underbrace{d^*_{j-1}\alpha}_{\in E_{j-1}} .$$

Thus $\Delta_d \alpha = 0$ yields $d_j \alpha = 0$ and $d^*_{j-1}\alpha = 0$. In particular, α represents a cohomology class.

Proof of Theorem 1.5.9.

a) By definition, $D = d + d^* : C^\infty(M; E) \to C^\infty(M, E)$ is a formally self-adjoint Dirac-type operator, where $E := \bigoplus_{j=0}^{N+1} E_j$.

Let f be the function

$$f(\lambda) := \begin{cases} \frac{1}{\lambda^2}, & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

Then $f|_{\mathrm{spec}(D)}$ is bounded. Thus $G := f(\bar{D}) : L^2(M,E) \to L^2(M,E)$ is a bounded operator. It is called the Green operator for D^2, i.e., G is an inverse to D^2 on the complement of its kernel.

Note that d and d^* commute with $D^2 = \Delta_d = dd^* + d^*d$. Thus the eigenspaces $E(D^2, \lambda^2)$ are invariant under d and d^*, and hence d and d^* both commute with G.

Now define for $j = 0, \ldots, N+1$:

$$\mathcal{H}^j := \ker\big(\Delta_d : C^\infty(M, E_j) \to C^\infty(M, E_j)\big)$$

and let

$$\pi : L^2(M,E) \to \mathcal{H}^0 \oplus \ldots \oplus \mathcal{H}^{N+1} =: \mathcal{H} = \ker(D)$$

be the orthogonal projection.

Then we have $D^2 G = \mathrm{id} - \pi$, since G is the Green operator for D^2. We now put $H := d^*G$. This yields

$$\mathrm{id} - \pi = D^2 G = \big(dd^* + d^*d\big)G = dH + Hd. \tag{1.49}$$

b) We show that the map $\mathcal{H}^j \to H^j(E_\bullet, d_\bullet)$, $\alpha \mapsto [\alpha]$, is injective:

For $\alpha \in \mathcal{H}^j$ with $[\alpha] = 0$, there exists a section $\beta \in C^\infty(M, E_{j-1})$ satisfying $\alpha = d\beta$. We then have

$$\alpha = d\beta = d(\mathrm{id}\,\beta) \overset{(1.49)}{=} d\big((dH + Hd + \pi)\beta\big) = dHd\beta = dH\alpha = dd^* \underbrace{G\alpha}_{=0} = 0.$$

c) We show that the map $\mathcal{H}^j \to H^j(E_\bullet, d_\bullet)$, $\alpha \mapsto [\alpha]$, is surjective:

Let $\omega \in H^j(E_\bullet, d_\bullet)$. Choosing any representative $\gamma \in \omega$ and putting $\alpha := \pi(\gamma) \in \mathcal{H}^j$, we obtain:

$$\gamma - \alpha = (\mathrm{id} - \pi)(\gamma) = (dH + Hd)(\gamma) = dH\gamma.$$

Thus, $\alpha = \gamma - dH\gamma$ and hence $[\alpha] = [\gamma] = \omega$. □

Corollary 1.5.11. *The cohomologies $H^j(E_\bullet, d_\bullet)$ of a Dirac-complex over a compact manifold are finite-dimensional.*

Proof. By Theorem 1.4.18, the eigenspace $\mathcal{H} = \ker(\Delta_d)$ is finite dimensional. By the Hodge Theorem 1.5.9, it is isomorphic to the direct sum of the cohomologies. □

Definition 1.5.12. Let M be a compact Riemannian manifold, and let $(E_\bullet, d_\bullet)$ be a Dirac complex on M. Let Δ_d be the Hodge Laplacian of the complex. The set of harmonic sections in degree j is denoted as

$$\mathscr{H}^j(E_\bullet, d_\bullet) := \ker\big(\Delta_d : C^\infty(M, E_j) \to C^\infty(M, E_j)\big) \tag{1.50}$$

Corollary 1.5.13 (Hodge decomposition). *Let M be a compact Riemannian manifold and let $(E_\bullet, d_\bullet)$ be a Dirac complex on M. Then for all j we have*

$$C^\infty(M, E_j) = \mathscr{H}^j(E_\bullet, d_\bullet) \oplus d_{j-1}(C^\infty(M, E_{j-1})) \oplus d_j^*(C^\infty(M, E_{j+1}))$$

and the sum is orthogonal with respect to $(\cdot,\cdot)_{L^2}$.

Proof. Put $E := \oplus_{j=0}^{N+1} E_j$. Then the Dirac-type operator $D \in \mathscr{D}\!\mathit{iff}_1(E, E)$ is formally self-adjoint. By the Fredholm alternative Corollary 1.4.28 we have

$$C^\infty(M, E) = \ker(D) \oplus D(C^\infty(M, E)) = \ker(\Delta_d) \oplus D(C^\infty(M, E))$$

and the sum is orthogonal with respect to $(\cdot,\cdot)_{L^2}$. The corollary now follows easily by considering the degree j. $\square$

Example 1.5.14. The dimension $\dim_{\mathbb{R}} H^j_{dR}(M) =: b_j(M)$ of the j-th cohomomology of the de Rham complex is called the j-th **Betti number** of M. The Betti numbers are topological invariants of the manifold M.

Example 1.5.15. Let M be a compact complex manifold of complex dimension m. For a fixed $p \in \{0, \dots, m\}$, the dimension $\dim H^{p,q}(M) =: h^{p,q}(M)$ of the q-th cohomology of the Dolbeault complex (as defined in Example 1.5.4) is called the (p, q)-th **Hodge number** of M.

Let M be a compact connected 2-dimensional Riemannian manifold, and let K be the Gauß curvature of M. Then by the Gauß-Bonnet Theorem we have

$$\int_M K\, dA = 2\pi\, \chi(M) = 4\pi\big(1 - g(M)\big)\,, \tag{1.51}$$

where $\chi(M)$ is the Euler characteristic and $g(M) = \frac{1}{2} b_1(M)$ is the genus of M. Both $\chi(M)$ and $g(M)$ are topological invariants of the manifold M, i.e. they only depend on the homotopy type of M. A proof of these statements is postponed to Section 3.4.

As a consequence of the Gauß-Bonnet theorem, we observe that if $g(M) \geq 2$ then M does not admit a metric of nonnegative Gauß curvature $K \geq 0$. If $g(M) \geq 1$ then M does not admit a metric of positive Gauß curvature $K > 0$.
The following theorem may be regarded as a generalization of this observation:

Theorem 1.5.16 (Bochner). *Let M be a compact connected n-dimensional Riemannian manifold. Then the following holds:*

a) If $\operatorname{Ric} > 0$ *then* $b_1(M) = 0$.

b) If $\operatorname{Ric} \geq 0$ *then* $b_1(M) \leq n$.

Proof. Let Δ_d be the Hodge Laplacian, restricted to $C^\infty\big(M, \Lambda^1 T^*M\big)$. By the Hodge Theorem 1.5.9, we have $b_1(M) = \dim\ker(\Delta_d)$.

a) Assume that $\operatorname{Ric} > 0$, and let α be a harmonic 1-form. By the compactness of M, there is a $\kappa > 0$ such that $\operatorname{Ric} \geq \kappa$. Using the Bochner formula (1.17) and integration by parts, we conclude:

$$\begin{aligned} 0 &= (\Delta_d \alpha, \alpha)_{L^2} \\ &= \big(\nabla^*\nabla\alpha + \operatorname{Ric}(\alpha), \alpha\big)_{L^2} \\ &= (\nabla\alpha, \nabla\alpha)_{L^2} + \big(\operatorname{Ric}(\alpha), \alpha\big)_{L^2} \\ &\geq \|\nabla\alpha\|_{L^2}^2 + \kappa\|\alpha\|_{L^2}^2 \\ &\geq \kappa\|\alpha\|_{L^2}^2. \end{aligned}$$

Thus, $\|\alpha\|_{L^2} = 0$ and hence $\alpha = 0$.

b) Now assume $\operatorname{Ric} \geq 0$, and let $\alpha_1, \ldots, \alpha_{n+1}$ be harmonic 1-forms. We show that they are linearly dependent. From the estimate in a), we conclude

$$0 = (\Delta_d \alpha_j, \alpha_j)_{L^2} \geq \|\nabla\alpha_j\|_{L^2}^2 \,.$$

Hence $\nabla\alpha_j = 0$.

Now fix $x_0 \in M$ and consider $\alpha_1(x_0), \ldots, \alpha_{n+1}(x_0) \in T^*_{x_0}M$. Since $\dim(T^*_{x_0}M) = n$, there exist $c_1, \ldots, c_{n+1} \in \mathbb{R}$ which are not all equal to 0 such that $\sum_{j=1}^{n+1} c_j\, \alpha_j = 0$.

Since M is connected, for any $x \in M$, we find a smooth curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$ and $\gamma(1) = x$. We then have

$$\frac{\nabla}{dt} \sum_{j=1}^{n+1} c_j\, \alpha_j(\gamma(t)) = \sum_{j=1}^{n+1} c_j \underbrace{\nabla_{\dot\gamma(t)}\alpha_j}_{=0} = 0.$$

Since $\sum_{j=1}^{n+1} c_j\, \alpha_j(\gamma(0)) = 0$, it follows that

$$\sum_{j=1}^{n+1} c_j\, \alpha_j(\gamma(t)) = 0, \quad \text{for all } t \in [0,1].$$

In particular, $\sum_{j=1}^{n+1} c_j\, \alpha_j(x) = 0$ holds for any $x \in M$. Thus, the 1-forms $\alpha_1, \dots, \alpha_{n+1}$ are linearly dependent. □

Remark 1.5.17. The inequality in b) is sharp: The n-torus $M = T^n$ carries a flat metric, in particular $\mathrm{Ric} = 0$, and it has $b_1(T^n) = n$. Moreover, by a small modification of the proof of Theorem 1.5.16 one can show a stronger statement: If $\mathrm{Ric} \geq 0$ on M and $\mathrm{Ric} > 0$ somewhere on M, then $b_1(M) = 0$.

Intersection form and signature

Definition 1.5.18. Let M be a compact oriented manifold of dimension n and let $k \in \{0, \dots, n\}$. The **intersection pairing** is the bilinear form

$$\begin{aligned} B : H^k_{dR}(M) \times H^{n-k}_{dR}(M) &\to \mathbb{R} \\ \big([\alpha], [\beta]\big) &\mapsto \int_M \alpha \wedge \beta. \end{aligned}$$

Remark 1.5.19. The intersection pairing is well-defined, since for closed forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^{n-k}(M)$, the integral $\int_M \alpha \wedge \beta$ only depends on the de Rham cohomology classes of α and β: Replacing α in the first entry by $\alpha + d\gamma$, where $\gamma \in \Omega^{k-1}(M)$, we find

$$\int_M (\alpha + d\gamma) \wedge \beta - \int_M \alpha \wedge \beta = \int_M d\gamma \wedge \beta = \underbrace{\int_M d(\gamma \wedge \beta)}_{=0 \text{ by Stokes}} \pm \int_M \gamma \wedge \underbrace{d\beta}_{=0} = 0$$

and similarly for the second entry.

Theorem 1.5.20 (Poincaré duality). *For a compact oriented manifold M of dimension n, the intersection pairing*

$$B : H^k_{dR}(M) \times H^{n-k}_{dR}(M) \to \mathbb{R}$$

is non-degenerate. In particular,

$$b_k(M) = b_{n-k}(M) \tag{1.52}$$

Proof. Let $[\alpha] \in H^k_{dR}(M)$ be a class such that for all $[\beta] \in H^{n-k}_{dR}(M)$, we have

$$B\big([\alpha],[\beta]\big) = 0.$$

We show that $[\alpha] = 0$.
Choose a Riemannian metric on M. By the Hodge Theorem 1.5.9 we choose a harmonic representative α of the class $[\alpha]$. Put $\beta := *\alpha \in \Omega^{n-k}(M)$. Then β is a harmonic representative[5] of the cohomology class $[\beta] \in H^{n-k}_{dR}(M)$. We thus have

$$0 = B\big([\alpha],[\beta]\big) = \int_M \alpha \wedge \beta = \int_M \alpha \wedge *\alpha = \int_M \langle \alpha, \alpha \rangle \, dvol = \|\alpha\|^2_{L^2}.$$

Hence $\alpha = 0$ and thus the map $H^k_{dR}(M) \to (H^{n-k}_{dR}(M))^*$, $[\alpha] \mapsto B([\alpha],\cdot)$ is injective. It follows that $b_k(M) \le b_{n-k}(M)$. Analogously one shows that the map $H^{n-k}_{dR}(M) \to (H^k_{dR}(M))^*$, $[\beta] \mapsto B(\cdot,[\beta])$ is injective and that $b_{n-k}(M) \le b_k(M)$. □

Corollary 1.5.21. *For a compact oriented manifold M of odd dimension n the* ***Euler characteristic***

$$\chi(M) := \sum_{k=0}^{n} (-1)^k \, b_k(M)$$

vanishes.

Proof. Since n is odd, we have that k is odd iff $n-k$ is even. By Poincaré duality, we have $b_k(M) = b_{n-k}(M)$, thus the k-th summand cancels with the $(n-k)$-th summand for $k = 0, \dots, \frac{n-1}{2}$. □

In the following let M be a compact oriented manifold of dimension $\dim M = n = 4k$. Then $\frac{n}{2} = 2k$ is even, so that the intersection pairing

$$\begin{aligned} B : H^{2k}_{dR}(M) \times H^{2k}_{dR}(M) &\to \mathbb{R}\,, \\ \big([\alpha],[\beta]\big) &\mapsto \int_M \alpha \wedge \beta\,, \end{aligned}$$

[5] The Hodge star operator $*$ maps harmonic forms to harmonic forms: If α is harmonic then we have

$$d(*\alpha) = \pm * \underbrace{*d*}_{=\pm d^*} \alpha = \pm * \underbrace{d^*\alpha}_{=0} = 0 \quad \text{and} \quad d^*(*\alpha) = \pm * d \underbrace{*(*}_{=\pm \mathrm{id}} \alpha) = \pm * \underbrace{d\alpha}_{=0} = 0.$$

is symmetric. The corresponding quadratic form on $H^{2k}_{dR}(M)$ is called the **intersection form**.

Choose a Riemannian metric on M and consider the Hodge star operator

$$* : \mathcal{H}^{2k}(M) \to \mathcal{H}^{2k}(M)$$

on the space $\mathcal{H}^{2k}(M)$ of harmonic $2k$-forms on M.
By Lemma 1.3.16 c), we have that $*^2 = 1$ on the space of $2k$-forms of a $4k$-dimensional manifold.
Thus we may put

$$\begin{aligned} \mathcal{H}^+ &:= +1\text{-eigenspace for } *, \\ \mathcal{H}^- &:= -1\text{-eigenspace for } *\,. \end{aligned}$$

Now for $\alpha \in \mathcal{H}^\pm$, we have

$$B\big([\alpha],[\alpha]\big) = \int\limits_M \alpha \wedge \alpha = \pm \int\limits_M \alpha \wedge *\alpha = \pm \|\alpha\|^2_{L^2}$$

For $\alpha \in \mathcal{H}^+$ and $\beta \in \mathcal{H}^-$, we have

$$\begin{aligned} B\big([\alpha],[\beta]\big) &= \int\limits_M \alpha \wedge \beta = \int\limits_M *\alpha \wedge \beta = \int\limits_M \beta \wedge *\alpha = \int\limits_M \langle \beta, \alpha \rangle \, dvol = \int\limits_M \alpha \wedge *\beta \\ &= -\int\limits_M \alpha \wedge \beta = -B\big([\alpha],[\beta]\big) \end{aligned}$$

and thus, $B\big([\alpha],[\beta]\big) = 0$. Hence in the splitting $H^{2k}_{dR}(M) = \mathcal{H}^+ \oplus \mathcal{H}^-$, the intersection form B is positive definite on $\mathcal{H}^+$, negative definite on $\mathcal{H}^-$, and the two subspaces are perpendicular with respect to B.

Definition 1.5.22. Let M be a compact oriented Riemannian $4k$-dimensional manifold. The harmonic $2k$-forms in $\mathcal{H}^+$ are called **self-dual**, those in $\mathcal{H}^-$ are called **anti-self-dual**. We set

$$\begin{aligned} b^+(M) &:= \dim \mathcal{H}^+, \\ b^-(M) &:= \dim \mathcal{H}^-. \end{aligned}$$

The **signature** of M is defined as

$$\operatorname{sign}(M) := b^+(M) - b^-(M)\,.$$

In the following let M be a compact complex manifold of complex dimension m and consider the map

$$\begin{aligned} B : H^{p,q}(M) \times H^{m-p,m-q}(M) &\to \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto \int_M \alpha \wedge \beta \end{aligned}$$

As for the intersection pairing in de Rham cohomology, the map B is well-defined: Replacing α in the first entry by $\alpha + \bar\partial\gamma$, with $\gamma \in \Omega^{p,q-1}(M)$, we find

$$\begin{aligned} \int_M (\alpha + \bar\partial\gamma) \wedge \beta - \int_M \alpha \wedge \beta &= \int_M (\bar\partial\gamma) \wedge \beta = \int_M (d\gamma) \wedge \beta - \int_M (\partial\gamma) \wedge \beta \\ &= \underbrace{\int_M d(\gamma \wedge \beta)}_{=0} \pm \int_M \gamma \wedge d\beta = \pm \int_M \gamma \wedge \underbrace{\bar\partial\beta}_{=0} \pm \underbrace{\int_M \gamma \wedge \partial\beta}_{\substack{=0 \text{ since} \\ \gamma\wedge\partial\beta \in \Omega^{m+1,m-1}}} . \end{aligned}$$

and similarly for the second entry. We have used that $\partial\gamma \wedge \beta, \gamma \wedge \partial\beta \in \Omega^{m+1,m-1} = \{0\}$. For any Hermitian metric on M, the Hodge star operator induces a map

$$* : \Lambda^{p,q}T^*M \to \Lambda^{m-p,m-q}T^*M.$$

For if $\alpha \in \Lambda^{p,q}T_x^*M$ and $\beta \in \Lambda^{k,l}T_x^*M$, with $(k,l) \neq (m-p, m-q)$, we have

$$\langle *\alpha, \beta\rangle\, vol = \pm\beta \wedge \alpha \in \Lambda^{k+p,q+l}T_x^*M = \{0\},$$

since $vol \in \Lambda^{k+p,q+l}T_x^*M$ implies $k+p+q+l = 2m$ and thus $k+p > m$ or $q+l > m$. Now let $\alpha \in \Omega^{p,q}(M)$. Then we have

$$d^*\alpha = \underbrace{\partial^*\alpha}_{\in \Omega^{p-1,q}(M)} + \underbrace{\bar\partial^*\alpha}_{\in \Omega^{p,q-1}(M)} .$$

On the other hand, we also have

$$\begin{aligned} d^*\alpha &= - * d * \alpha \\ &= - \underbrace{*\partial * \alpha}_{\in \Omega^{p-1,q}(M)} - \underbrace{*\bar\partial * \alpha}_{\in \Omega^{p,q-1}(M)} . \end{aligned}$$

Thus

$$\partial^* = - * \partial * \qquad \text{and} \quad \bar\partial^* = - * \bar\partial * .$$

Now the same argument as in the proof of the Poincaré duality Theroem 1.5.20 yields the following:

Theorem 1.5.23 (Kodaira-Serre duality). *Let M be a compact complex manifold of complex dimension m. Then the bilinear form*

$$B : H^{p,q}(M) \times H^{m-p,m-q}(M) \to \mathbb{C}$$

is non-degenerate. In particular,

$$h^{p,q}(M) = h^{m-p,m-q}(M).$$

2. Spinors and the classical Dirac operator

2.1. Clifford algebras

Definition 2.1.1. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let V be a finite-dimensional $\mathbb{K}$-vector space equipped with a symmetric bilinear form β.
A **Clifford algebra** for (V, β) is a unital $\mathbb{K}$-algebra A together with a linear map $\imath : V \to A$ such that the following properties hold:

i) $\imath(v)^2 = -\beta(v, v) \cdot 1, \quad$ for all $v \in V$,

ii) $(A, \imath)$ is universal with respect to i), i.e.:

Whenever A' is a unital $\mathbb{K}$-algebra with a linear map $\imath' : V \to A'$ satisfying i) then there exists a unique algebra homomorphism $\phi : A \to A'$ such that the diagram

$$\begin{tikzcd} & A \arrow[dd, "\phi"] \\ V \arrow[ur, "\imath"] \arrow[dr, "\imath'"'] & \\ & A' \end{tikzcd}$$

commutes. In other words: A is the smallest algebra that satisfies property i).

Remark 2.1.2. By a polarization argument, one immediately sees that property i) above is equivalent to the **Clifford relation**

$$\text{i}') \qquad \imath(v) \cdot \imath(w) + \imath(w) \cdot \imath(v) = -2\,\beta(v, w) \cdot 1, \qquad \text{for all } v, w \in V. \tag{2.1}$$

Remark 2.1.3. Let (M, g) be a Riemannian manifold and let $D \in \mathscr{Diff}_1(E, E)$ be a formally self-adjoint Dirac-type operator. By equation (1.18), property i) holds for $(V, \beta) = (T_x^*M, g|_x)$, $A = \mathrm{End}(E_x)$, and $\imath = \sigma_1(D, \cdot)$. However, ii), does not hold in general.

Example 2.1.4. Let $\beta = 0$. The Clifford relation (2.1) yields $\imath(v) \cdot \imath(w) = -\imath(w) \cdot \imath(v)$ for all $v, w \in V$. Let $n = \dim(V)$ and let $A := \Lambda^\bullet V = \bigoplus_{k=0}^n \Lambda^k V$ be the exterior algebra

of V. The map $\imath : V \xrightarrow{\cong} \Lambda^1 V \hookrightarrow \Lambda^\bullet V = A$ obviously satisfies property i).
Now let $\imath' : V \to A'$ be any map from V to a $\mathbb{K}$-algebra A' satisfying property i). A morphism $\phi : A = \Lambda^k V \to A'$ as in property ii) necessarily satisfies

$$\begin{aligned}
\phi(v_1 \wedge \ldots \wedge v_k) &= \phi(\imath(v_1) \wedge \ldots \wedge \imath(v_k)) \\
&= \phi(\imath(v_1)) \cdot \ldots \cdot \phi(\imath(v_k)) \\
&= \imath'(v_1) \cdot \ldots \cdot \imath'(v_k)\,.
\end{aligned}$$

Hence ϕ is uniquely determined. Clearly, $\phi : \Lambda^k V \to A'$ defined by this formula yields a homomorphism with $\phi \circ \imath = \imath'$.

Proposition 2.1.5. *Let V be a finite-dimensional $\mathbb{K}$-vector space and let β be a symmetric bilinear form on V.*
Then there exists a Clifford algebra $(A, \imath)$ for (V, β). The pair $(A, \imath)$ is unique up to isomorphism.

Proof.
Uniqueness: Let $(A, \imath)$ and $(A', \imath')$ be two Clifford algebras for (V, β).

By the universal property for A, there exists an algebra homomorphism

$$\phi : A \to A',$$

such that diagram 1 commutes.

$$\begin{tikzcd}
& A \arrow[dd, "\phi"] \\
V \arrow[ur, "\imath"] \arrow[dr, "\imath'"'] & \\
& A'
\end{tikzcd}$$

Diag. 1

Similarly, by the universal property for A', there exists an algebra homomorphism

$$\psi : A' \to A,$$

such that diagram 2 commutes.

$$\begin{tikzcd}
& A' \arrow[dd, "\psi"] \\
V \arrow[ur, "\imath'"] \arrow[dr, "\imath"'] & \\
& A
\end{tikzcd}$$

Diag. 2

We now combine both diagrams to the alongside commutative diagram. The uniqueness in the universal property of $(A, \imath)$ yields

$$\psi \circ \phi = \mathrm{id}_A \,.$$

$$\begin{tikzcd}
 & A \arrow[d, "\phi"] \\
V \arrow[ur, "\imath"] \arrow[r, "\imath'"] \arrow[dr, "\imath"'] & A' \arrow[d, "\psi"] \\
 & A
\end{tikzcd}$$

Analogously, we combine the first two diagrams to the alongside commutative diagram. Then the uniqueness in the universal property of $(A', \imath')$ yields

$$\phi \circ \psi = \mathrm{id}_{A'} \,.$$

$$\begin{tikzcd}
 & A' \arrow[d, "\psi"] \\
V \arrow[ur, "\imath'"] \arrow[r, "\imath"] \arrow[dr, "\imath'"'] & A \arrow[d, "\phi"] \\
 & A'
\end{tikzcd}$$

Thus ϕ is an isomorphism with inverse ψ.

Existence: We consider the tensor algebra $\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} \otimes^k V$ and define the inclusion $\imath_0 : V \xrightarrow{\cong} \otimes^1 V \hookrightarrow \mathcal{T}(V)$. Let $I \subset \mathcal{T}(V)$ be the two-sided ideal generated by all elements of the form

$$v \otimes w + w \otimes v + 2\beta(v, w) \cdot 1, \qquad v, w \in V.$$

Put $A := \mathcal{T}(V)/I$ and denote by $\pi : \mathcal{T}(V) \to A$ the quotient homomorphism. Then define $\imath$ by the alongside commutative diagram, i.e., $\imath = \pi \circ \imath_0$.

$$\begin{tikzcd}
V \arrow[r, "\imath_0"] \arrow[dr, "\imath"'] & \mathcal{T}(V) \arrow[d, "\pi"] \\
 & A
\end{tikzcd}$$

We check that the Clifford relation 2.1 (property i') in Definition 2.1.1) holds:

$$\begin{aligned}
\imath(v) \cdot \imath(w) + \imath(w) \cdot \imath(v) &= \pi(\imath_0(v)) \cdot \pi(\imath_0(w)) + \pi(\imath_0(w)) \cdot \pi(\imath_0(v)) \\
&= \pi\big(\imath_0(v) \otimes \imath_0(w) + \imath_0(w) \otimes \imath_0(v)\big) \\
&= \pi(v \otimes w + w \otimes v) \\
&= \pi(-2\beta(v, w) \cdot 1) \\
&= -2\beta(v, w) \cdot 1.
\end{aligned}$$

We check that property ii) of Definition 2.1.1 holds: Let A' be any unital $\mathbb{K}$-algebra, together with a linear map $\imath' : V \to A'$ satisfying property i) of Definition 2.1.1.

a) *Uniqueness of ϕ:* Let $\phi : A = \mathcal{T}(V)/I \to A'$ be a homomorphism satisfying $\phi \circ \imath = \imath'$. Then we have:

$$\begin{aligned}\phi(\pi(v_1 \otimes \ldots \otimes v_k)) &= \phi\big(\pi(\imath_0(v_1))\big) \cdot \ldots \cdot \phi\big(\pi(\imath_0(v_k))\big)\\ &= \phi\big(\imath(v_1)\big) \cdot \ldots \cdot \phi\big(\imath(v_k)\big)\\ &= \imath'(v_1) \cdot \ldots \cdot \imath'(v_k)\,.\end{aligned}$$

Thus ϕ is uniquely determined.

b) *Existence of ϕ:* Consider the unique homomorphism $\psi : \mathcal{T}(V) \to A'$, such that the following diagram commutes. Then we have:

$$\begin{tikzcd} V \arrow[r, "\imath_0"] \arrow[dr, "\imath'"'] & \mathcal{T}(V) \arrow[d, "\psi"] \\ & A' \end{tikzcd}$$

$$\psi\big(v_1 \otimes \ldots \otimes v_k\big) = \imath'(v_1) \cdot \ldots \cdot \imath'(v_k).$$

We need to check that the diagram factorizes through A, i.e., ψ vanishes on the ideal I and hence descends to the quotient $A = \mathcal{T}(V)/I$. For $v, w \in V$, we compute:

$$\begin{aligned}&\psi(v \otimes w + w \otimes v + 2\beta(v,w)1)\\ &\qquad = \psi(\imath_0(v) \otimes \imath_0(w) + \imath_0(w) \otimes \imath_0(v) + 2\beta(v,w)1)\\ &\qquad = \psi(\imath_0(v)) \cdot \psi(\imath_0(w)) + \psi(\imath_0(w)) \cdot \psi(\imath_0(v)) + 2\beta(v,w)1\\ &\qquad = \imath'(v) \cdot \imath'(w) + \imath'(w) \cdot \imath'(v) + 2\beta(v,w) \cdot 1\\ &\qquad = 0.\end{aligned}$$

In the last equality we have used property i) from Definition 2.1.1 for $\imath' : V \to A'$.

Thus $\psi : \mathcal{T}(V) \to A'$ descends to a homomorphism

$$\phi : A = \mathcal{T}(V)/I \to A',$$

such that the alongside diagram commutes. Clearly, the homomorphism ϕ is uniquely determined by the homomorphism ψ.

$$\begin{tikzcd} \mathcal{T}(V) \arrow[r, "\pi"] \arrow[dr, "\psi"'] & A \arrow[d, "\phi"] \\ & A' \end{tikzcd}$$

Moreover, the alongside diagram commutes. We have thus constructed a homomorphism $\phi : A \to A'$ satisfying $\phi \circ \imath = \imath'$.

$$\begin{tikzcd} V \arrow[r, "\imath_0"] \arrow[rr, bend left, "\imath"] \arrow[dr, "\imath'"'] & \mathcal{T}(V) \arrow[r, "\pi"] \arrow[d, "\psi"] & A \arrow[dl, "\phi"] \\ & A' & \end{tikzcd}$$

□

Definition 2.1.6. Let V be a finite-dimensional $\mathbb{K}$-vector space, equipped with a symmetric bilinear form β. We denote the Clifford algebra for (V, β) by $\mathbf{Cl}(\boldsymbol{V}, \boldsymbol{\beta})$.

Remark 2.1.7. Let V be a finite-dimensional $\mathbb{K}$-vector space, and let β be a symmetric bilinear form on V. For any Clifford algebra $(A, \imath)$ the map $\imath : V \to A$ is injective:

If the symmetric form β is definite then this is clear from i) of Definition 2.1.1:

$$\imath(v)^2 = -\beta(v, v) \cdot 1.$$

$$\begin{tikzcd} V \arrow[r, "\imath"] \arrow[dr, "\imath_1"'] & A \arrow[d, "\cong"] \\ & \mathcal{T}(V)/I \end{tikzcd}$$

In the general case, we have the alongside commutative diagram. Since $\imath_1(v) = v \notin I$, the map $\imath_1$ is injective. Hence so is $\imath$.

Remark 2.1.8. Let V be a $\mathbb{K}$-vector space, and let $b_1, \ldots, b_n \in V$ be a basis of V. Then

$$\{b_{i_1} \otimes \ldots \otimes b_{i_k}\}_{\substack{k \in \mathbb{N}_0 \\ 1 \le i_1, \ldots, i_k \le n}}$$

is a basis of the vector space $\mathcal{T}(V)$. Thus the elements

$$\{b_{i_1} \cdot \ldots \cdot b_{i_k}\}_{\substack{k \in \mathbb{N}_0 \\ 1 \le i_1, \ldots, i_k \le n}}$$

generate the Clifford algebra $\mathcal{T}(V)/I$ as a vector space. We use the Clifford relations $b_i \cdot b_j = -b_j \cdot b_i - 2\beta(b_i, b_j) \cdot 1$ to express all elements of $\mathcal{T}(V)/I$ as linear combinations of

$$\{b_{i_1} \cdot \ldots \cdot b_{i_k}\}_{\substack{k \in \mathbb{N}_0 \\ 1 \le i_1 \le i_2 \le \ldots \le i_k \le n}}.$$

Moreover, we use the relation $b_i \cdot b_i = -\beta(b_i, b_i) \cdot 1$ to express all elements of $\mathcal{T}(V)/I$ as linear combinations of

$$\{b_{i_1} \cdot \ldots \cdot b_{i_k}\}_{\substack{k = 0, 1, \ldots n \\ 1 \le i_1 < i_2 < \ldots < i_k \le n}}.$$

In particular, $\dim A \le 2^n < \infty$. We will see later that $\dim A = 2^n$, hence this generating system is a basis of the Clifford algebra $\mathrm{Cl}(V, \beta)$.

Example 2.1.9. Consider $V = \mathbb{R}$ with the symmetric bilinear form $\beta(x, y) := x \cdot y$. Let e_1 be the standard basis of $\mathbb{R}$.

- The elements 1 and e_1 generate $\mathrm{Cl}(\mathbb{R}, \beta)$ as a vector space. If $1, e_1 \in \mathrm{Cl}(\mathbb{R}, \beta)$ were linearly dependent, i.e., $e_1 = \alpha \cdot 1$ for some $\alpha \in \mathbb{R}$ then it would follow

$$\alpha^2 \cdot 1 = e_1^2 = -\beta(e_1, e_1) \cdot 1 = -1. \quad \lightning$$

- The vector space isomorphism $\phi : \mathrm{Cl}(\mathbb{R}, \beta) \to \mathbb{C}$, defined by

$$\phi(\alpha 1 + \beta e_1) = \alpha \cdot 1 + \beta \cdot i,$$

  is also an algebra homomorphism. Hence, $\mathrm{Cl}(\mathbb{R}, \beta) \cong \mathbb{C}$.

Example 2.1.10. Consider $V = \mathbb{R}$ with the symmetric bilinear form $\gamma(x, y) := -x \cdot y$. Let e_1 be the standard basis of $\mathbb{R}$. Then $e_1^2 = -\gamma(e_1, e_1)1 = 1$.

- Again, 1 and e_1 generate $\mathrm{Cl}(\mathbb{R}, \gamma)$ as a vector space. If $1, e_1 \in \mathrm{Cl}(\mathbb{R}, \gamma)$ were linearly dependent, i.e., $e_1 = \alpha \cdot 1$ for some $\alpha \in \mathbb{R}$ then it would follow that

  $$e_1 - \alpha \cdot 1 \text{ as an element of } \mathcal{T}(\mathbb{R}) \text{ was contained in } I,$$

  in other words,

  $$e_1 - \alpha \cdot 1 = x \otimes (e_1 \otimes e_1 - 1) \otimes y, \qquad \text{for some } x, y \in \mathcal{T}(\mathbb{R}).$$

  We write
  $$x = x_{\max} + x_{\text{lower}} \quad \text{and} \quad y = y_{\max} + y_{\text{lower}},$$
  where $x_{\max} \neq 0$ and $y_{\max} \neq 0$ are homogeneous of maximal degree. Then

  $$\underbrace{e_1 - \alpha \cdot 1}_{\text{degree } \leq 1} = \underbrace{x_{\max} \otimes e_1 \otimes e_1 \otimes y_{\max}}_{\text{degree } = \deg(x_{\max}) + 2 + \deg(y_{\max}) \geq 2} + \underbrace{l.o.t.}_{\text{lower degree}}$$

  Thus,
  $$x_{\max} \otimes e_1 \otimes e_1 \otimes y_{\max} = 0 \quad \lightning \text{ to } x_{\max} \neq 0, y_{\max} \neq 0.$$
  Hence, $1, e_1$ form a vector space basis of $\mathrm{Cl}(\mathbb{R}, \gamma)$. Thus, $\frac{1}{2}(1 + e_1), \frac{1}{2}(1 - e_1)$ is also a vector space basis. Moreover, we have:

  $$\begin{aligned} \frac{1}{2}(1 \pm e_1) \cdot \frac{1}{2}(1 \pm e_1) &= \frac{1}{4}\left(1 \pm e_1 \pm e_1 + e_1^2\right) = \frac{1}{2}(1 \pm e_1) \\ \frac{1}{2}(1 + e_1) \cdot \frac{1}{2}(1 - e_1) &= \frac{1}{4}\left(1 - e_1 + e_1 - e_1^2\right) = 0. \end{aligned} \tag{2.2}$$

- Consider the vector space isomorphism $\phi : \mathrm{Cl}(\mathbb{R}, \gamma) \to \mathbb{R} \oplus \mathbb{R}$,

  $$\alpha \cdot \frac{1}{2}(1 + e_1) + \beta \cdot \frac{1}{2}(1 - e_1) \longmapsto (\alpha, \beta).$$

  We check that ϕ is also an algebra homomorphism, where the multiplication in $\mathbb{R} \oplus \mathbb{R}$ is defined componentwise:

  $$\begin{aligned} &\phi\Big(\Big(\alpha \cdot \frac{1}{2}(1 + e_1) + \beta \cdot \frac{1}{2}(1 - e_1)\Big) \cdot \Big(\alpha' \cdot \frac{1}{2}(1 + e_1) + \beta' \cdot \frac{1}{2}(1 - e_1)\Big)\Big) \\ &\overset{(2.2)}{=} \phi\left(\frac{\alpha\alpha'}{2}(1 + e_1) + \frac{\beta\beta'}{2}(1 - e_1)\right) \\ &= \left(\alpha\alpha', \beta\beta'\right) \\ &= (\alpha, \beta) \cdot (\alpha', \beta') \\ &= \phi\left(\alpha \cdot \frac{1}{2}(1 + e_1) + \beta \cdot \frac{1}{2}(1 - e_1)\right) \cdot \phi\left(\alpha' \cdot \frac{1}{2}(1 + e_1) + \beta' \cdot \frac{1}{2}(1 - e_1)\right). \end{aligned}$$

The tensor algebra $\mathcal{T}(V)$ is $\mathbb{Z}$-graded, i.e., it admits a decomposition

$$\mathcal{T}(V) = \bigoplus_{i\in\mathbb{Z}} \mathcal{T}^i(V),$$

such that the multiplication of the homogeneous components corresponds to the addition of the degrees:

$$\mathcal{T}^i(V)\cdot\mathcal{T}^j(V) \subset \mathcal{T}^{i+j}(V),$$

where

$$\mathcal{T}^i(V) = \begin{cases} \otimes^i V, & i \geq 0 \\ 0 & i < 0. \end{cases}$$

This $\mathbb{Z}$-grading does not descend to a $\mathbb{Z}$-grading of $\mathrm{Cl}(V,\beta)$, unless $\beta = 0$, because the ideal I is not generated by homogeneous elements. In the generating elements $v\otimes w + w\otimes v + 2\beta(v,w)1$ of the ideal I, the part $v\otimes w + w\otimes v$ has degree 2, whereas $\beta(v,w)\cdot 1$ has degree 0.
Instead of the $\mathbb{Z}$-grading of the tensor algebra, consider the $\mathbb{Z}_2$-grading

$$\mathcal{T}(V) = \mathcal{T}^{\mathrm{even}}(V) \bigoplus \mathcal{T}^{\mathrm{odd}}(V),$$

where

$$\mathcal{T}^{\mathrm{even}}(V) = \bigoplus_{i \text{ even}} \mathcal{T}^i(V), \qquad \mathcal{T}^{\mathrm{odd}}(V) = \bigoplus_{i \text{ odd}} \mathcal{T}^i(V)\,.$$

This grading descends to the Clifford algebra $\mathrm{Cl}(V,\beta)$, since the ideal I is generated by elements in the even part.
A more intrinsic definition of the $\mathbb{Z}_2$-grading is given as follows: Let $\imath : V \to \mathrm{Cl}(V,\beta)$ be the standard embedding. Consider $\imath' : V \to \mathrm{Cl}(V,\beta)$, defined by

$$\imath'(v) = -\imath(v), \qquad \forall\, v \in V.$$

Then $\imath'$ satisfies

$$\imath'(v)^2 = \imath(v)^2 = -\beta(v,v)\cdot 1.$$

Thus, there exists an algebra homomorphism $\phi : \mathrm{Cl}(V,\beta) \to \mathrm{Cl}(V,\beta)$, such that the alongside diagram commutes.
Upon identification of $\imath(V) \subset \mathrm{Cl}(V,\beta)$ with V we have $\phi|_V = -\,\mathrm{id}$.

$$\begin{tikzcd}
 & \mathrm{Cl}(V,\beta) \arrow[dd, "\phi"] \\
V \arrow[ur, "\imath"] \arrow[dr, "\imath'"'] & \\
 & \mathrm{Cl}(V,\beta)
\end{tikzcd}$$

This yields a decomposition

$$\mathrm{Cl}(V,\beta) = \mathrm{Cl}^0(V,\beta) \oplus \mathrm{Cl}^1(V,\beta),$$

where

$$\begin{aligned}\mathrm{Cl}^0(V,\beta) &= +1 - \text{eigenspace of } \phi \\ \mathrm{Cl}^1(V,\beta) &= -1 - \text{eigenspace of } \phi.\end{aligned}$$

Definition 2.1.11. Let $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ be $\mathbb{Z}_2$-graded $\mathbb{K}$-algebras, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, i.e.

$$A^0A^0 + A^1A^1 \subset A^0, \qquad A^0A^1 + A^1A^0 \subset A^1.$$

Then the $\mathbb{Z}_2$-graded tensor product $A \,\hat{\otimes}\, B$ of $\mathbb{Z}_2$-graded algebras is given by

$$A \,\hat{\otimes}\, B \;=\; A \otimes B \text{ as a vector space,}$$

where

$$\begin{aligned}\big(A \,\hat{\otimes}\, B\big)^0 &= A^0 \otimes B^0 \oplus A^1 \otimes B^1 \\ \big(A \,\hat{\otimes}\, B\big)^1 &= A^0 \otimes B^1 \oplus A^1 \otimes B^0\end{aligned}$$

with the multiplication

$$(a \otimes b)(a' \otimes b') = (-1)^{ij} aa' \otimes bb', \quad a \in A, a' \in A^i, b \in B^j, b' \in B.$$

Let (V_i, β_i) , $i = 1, 2$, be finite dimensional $\mathbb{K}$-vector spaces, equipped with symmetric bilinear forms. By $\beta_1 \oplus \beta_2$, we denote the uniquely determined symmetric bilinear form on $V_1 \oplus V_2$ which restricts to β_i on V_i and for which the subspaces $V_i \subset V_1 \oplus V_2$, $i = 1, 2$, are mutually orthogonal.

Proposition 2.1.12. *Let V_i, $i = 1, 2$, be finite-dimensional $\mathbb{K}$-vector spaces, and let β_i be symmetric bilinear forms on V_i. Then we have:*

$$\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \cong \mathrm{Cl}(V_1, \beta_1) \,\hat{\otimes}\, \mathrm{Cl}(V_2, \beta_2).$$

Proof.

a) Consider the linear map

$$\begin{aligned} j : V_1 \oplus V_2 &\to \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2) \\ v_1 + v_2 &\mapsto \imath_1(v_1) \otimes 1 + 1 \otimes \imath_2(v_2). \end{aligned}$$

By Definition 2.1.11, we have:

$$\begin{aligned} j(v_1+v_2)^2 &= \big(\imath_1(v_1) \otimes 1 + 1 \otimes \imath_2(v_2)\big)^2 \\ &= \big(\imath_1(v_1)\otimes 1\big)\big(\imath_1(v_1)\otimes 1\big) + \big(\imath_1(v_1)\otimes 1\big)\big(1 \otimes \imath_2(v_2)\big) \\ &\quad + \big(1\otimes \imath_2(v_2)\big)\big(\imath_1(v_1)\otimes 1\big) + \big(1 \oplus \imath_2(v_2)\big)\big(1\otimes \imath_2(v_2)\big) \\ &= \imath_1(v_1)^2 \otimes 1 + \imath_1(v_1)\otimes \imath_2(v_2) - \imath_1(v_1)\otimes \imath_2(v_2) + 1 \otimes \imath_2(v_2)^2 \\ &= -\beta_1(v_1,v_1)1\otimes 1 + 1 \otimes \big(-\beta_2(v_2,v_2)1\big) \\ &= -\big(\beta_1(v_1,v_1)+\beta_2(v_2,v_2)\big)1\otimes 1 \\ &= -(\beta_1\oplus\beta_2)(v_1+v_2, v_1+v_2)1\otimes 1. \end{aligned}$$

In the last step we used the fact that the mixed terms in $\beta_1 \oplus \beta_2(v_1+v_2, v_1+v_2)$ cancel, since $V_1, V_2 \subset V_1 \oplus V_2$ are mutually perpendicular with respect to $\beta_1 \oplus \beta_2$.

Thus, by the universal property for $\mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ there exists an algebra homomorphism

$$\phi : \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \to \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2),$$

such that the diagram

$$\begin{tikzcd} & \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \arrow[dd, "\phi"] \\ V_1 \oplus V_2 \arrow[ur, "\imath"] \arrow[dr, "j"'] & \\ & \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2) \end{tikzcd}$$

commutes.

b) To show that ϕ is an isomorphism, we construct its inverse:

By the universal property for $\mathrm{Cl}(V_i, \beta_i)$, $i = 1, 2$, there exist unique algebra homomorphisms ψ_i such that the diagrams

$$\begin{tikzcd} & \mathrm{Cl}(V_i, \beta_i) \arrow[dd, "\psi_i"] \\ V_i \arrow[ur, "\imath_i"] \arrow[dr, "\imath|_{V_i}"'] & \\ & \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \end{tikzcd}$$

commute for $i = 1, 2$.

We define the map $\psi : \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2) \to \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ by

$$\psi(a \otimes b) := \psi_1(a) \cdot \psi_2(b), \qquad \text{for all } a \in \mathrm{Cl}(V_1, \beta_1), b \in \mathrm{Cl}(V_2, \beta_2).$$

The map ψ is linear and also multiplicative: By Definition 2.1.11, we have for all $a \in \mathrm{Cl}(V_1, \beta_1)$, $a' \in \mathrm{Cl}(V_1, \beta_1)^i$, $b \in \mathrm{Cl}(V_2, \beta_2)^j$, $b' \in \mathrm{Cl}(V_2, \beta_2)$:

$$\begin{aligned}
\psi\big((a \otimes b)(a' \otimes b')\big) &= \psi\big((-1)^{ij} aa' \otimes bb'\big) \\
&= (-1)^{ij} \psi_1(aa') \cdot \psi_2(bb') \\
&= (-1)^{ij} \psi_1(a) \cdot \psi_1(a') \cdot \psi_2(b) \cdot \psi_2(b') \\
&= \psi_1(a) \cdot \psi_2(b) \cdot \psi_1(a') \cdot \psi_2(b') \\
&= \psi(a \otimes b) \cdot \psi(a' \otimes b').
\end{aligned}$$

Hence, $\psi : \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2) \to \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2)$ is an algebra homomorphism.

c) We check that ψ is the inverse to ϕ. We have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2) \arrow[d, "\phi"] \\
V_1 \oplus V_2 \arrow[ur, "\imath"] \arrow[r, "j"] \arrow[dr, "\imath"'] & \mathrm{Cl}(V_1, \beta_1) \hat{\otimes} \mathrm{Cl}(V_2, \beta_2) \arrow[d, "\psi"] \\
 & \mathrm{Cl}(V_1 \oplus V_2, \beta_1 \oplus \beta_2)
\end{tikzcd}$$

The uniqueness in the universal property for $\mathrm{Cl}(V_1 \oplus V_2)$ yields $\psi \circ \phi = \mathrm{id}$.
To show that $\phi \circ \psi = \mathrm{id}$, we compute:

$$\begin{aligned}
\phi\big(\psi\big(\imath_1(v_1) \otimes \imath_2(v_2)\big)\big) &= \phi\big(\psi_1\big(\imath_1(v_1)\big) \cdot \psi_2\big(\imath_2(v_2)\big)\big) \\
&= \phi\big(\imath|_{V_1}(v_1) \cdot \imath|_{V_2}(v_2)\big) \\
&= \phi\big(\imath(v_1)\big) \cdot \phi\big(\imath(v_2)\big) \\
&= j(v_1) \cdot j(v_2) \\
&= \big(\imath_1(v_1) \otimes 1\big)\big(1 \otimes \imath_2(v_2)\big) \\
&= \imath_1(v_1) \otimes \imath_2(v_2).
\end{aligned}$$

□

Corollary 2.1.13. *Let (V, β) be a finite dimensional $\mathbb{K}$-vector space, equipped with a symmetric bilinear form β. Then we have $\dim_{\mathbb{K}} \mathrm{Cl}(V, \beta) = 2^{\dim_{\mathbb{K}} V}$.*

Proof. a) If (V_i, β_i), $i = 1, 2$, are $\mathbb{K}$-vector spaces with symmetric bilinear forms β_i and if $f\colon V_1 \to V_2$ is a $\mathbb{K}$-linear isomorphism such that for all $v, w \in V_1$ we have $\beta_2(f(v), f(w)) = \beta_1(v, w)$, then the Clifford algebras $\mathrm{Cl}(V_1, \beta_1)$ and $\mathrm{Cl}(V_2, \beta_2)$ are isomorphic. The proof of this statement is an exercise.

b) We prove the statement of the corollary by induction on $n := \dim_{\mathbb{K}} V$.

Let $n = 1$. By part a) we may assume that $V = \mathbb{K}$ and that there exists $a \in \mathbb{K}$ such that for all $v, w \in \mathbb{K}$ we have $\beta(v, w) = a \cdot v \cdot w$.
For $\mathbb{K} = \mathbb{R}$ and $a \neq 0$ the map $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \sqrt{|a|} \cdot x$ is an isomorphism such that $\beta(v, w) = \pm f(v) \cdot f(w)$ for all $v, w \in V$. By part a) it is thus sufficient to consider $\beta(x, y) = \pm x \cdot y$ and $\beta = 0$. We have already computed these Clifford algebras $\mathrm{Cl}(V, \beta)$ in Examples 2.1.4, 2.1.9 and 2.1.10. In either case, we obtained $\dim_{\mathbb{R}} \mathrm{Cl}(V, \beta) = 2 = 2^1$.
For $\mathbb{K} = \mathbb{C}$ it is by part a) sufficient to consider $\beta(x, y) = -x \cdot y$ and $\beta = 0$. For $\beta(x, y) = -x \cdot y$ an argument as in Example 2.1.10 shows that $\mathrm{Cl}(\mathbb{C}, \beta) \cong \mathbb{C} \oplus \mathbb{C}$, for $\beta = 0$ we have the result from Example 2.1.4. Again, we have $\dim_{\mathbb{C}} \mathrm{Cl}(V, \beta) = 2^1$.

Now let $n \in \mathbb{N}$ be arbitrary. Assume that $\dim_{\mathbb{K}} \mathrm{Cl}(W, \beta') = 2^{n-1}$ for any (W, β') with $\dim_{\mathbb{K}} W = n - 1$. Let $b_1, \ldots, b_n \in V$ be a basis such that the b_i are mutually perpendicular with respect to β. Consider the splitting

$$V = \underbrace{\mathbb{K} \cdot b_1}_{=:\, V_1} \oplus \underbrace{(\mathbb{K} \cdot b_2 \oplus \ldots \oplus \mathbb{K} \cdot b_n)}_{=:\, V_2}.$$

By Proposition 2.1.12, we have:

$$\mathrm{Cl}(V, \beta) \cong \mathrm{Cl}(V_1, \beta|_{V_1 \times V_1}) \,\hat{\otimes}\, \mathrm{Cl}(V_2, \beta|_{V_2 \times V_2}).$$

In particular, we have:

$$\begin{aligned} \dim \mathrm{Cl}(V, \beta) &= \dim \big(\mathrm{Cl}(V_1, \beta|_{V_1 \times V_1}) \,\hat{\otimes}\, \mathrm{Cl}(V_2, \beta|_{V_2 \times V_2})\big) \\ &= \dim \mathrm{Cl}(V_1, \beta|_{V_1 \times V_1}) \cdot \dim \mathrm{Cl}(V_2, \beta|_{V_2 \times V_2}) \\ &= 2 \cdot 2^{n-1} \\ &= 2^n. \end{aligned}$$

□

Example 2.1.14. Let β_{eucl} denote the standard Euclidean scalar product on $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Consider $\mathrm{Cl}(\mathbb{R}^2, \beta_{\mathrm{eucl}})$. By Proposition 2.1.12, we have:

$$\mathrm{Cl}(\mathbb{R}^2, \beta_{\mathrm{eucl}}) \cong \mathrm{Cl}(\mathbb{R}, \beta_{\mathrm{eucl}}) \,\hat{\otimes}\, \mathrm{Cl}(\mathbb{R}, \beta_{\mathrm{eucl}}) \cong \mathbb{C} \otimes \mathbb{C}.$$

Let e_1, e_2 be the standard othonormal basis of $\mathbb{R}^2$. Then a vector space basis of $\mathrm{Cl}(\mathbb{R}^2, \beta_{\mathrm{eucl}})$ is given by $1, e_1, e_2, e_1 \cdot e_2$. We then have the following identities for the basis elements:

$$\begin{aligned} e_1^2 &= -1 \\ e_2^2 &= -1 \\ (e_1 \cdot e_2)^2 &= e_1 \cdot e_2 \cdot e_1 \cdot e_2 = -e_1^2 \cdot e_2^2 = -1 \end{aligned}$$

Hence there is an algebra isomorphism $\mathrm{Cl}(\mathbb{R}^2, \beta_{\mathrm{eucl}}) \to \mathbb{H}$, given by

$$\begin{aligned} 1 &\mapsto 1 \\ e_1 &\mapsto i \\ e_2 &\mapsto j \\ e_1 \cdot e_2 &\mapsto k. \end{aligned}$$

Here $\mathbb{H}$ denotes the quaternions. Hence $\mathrm{Cl}(\mathbb{R}^2, \beta_{\mathrm{eucl}}) \cong \mathbb{H}$.

Remark 2.1.15. Let V be a $\mathbb{K}$-vector space with a symmetric bilinear form β and let $\Lambda^\bullet V := \oplus_{k=0}^n \Lambda^k V$ be the exterior algebra of V. If $v_1, \dots, v_n$ is a basis of V then the vectors

$$v_{i_1} \wedge \ldots \wedge v_{i_k} \in \Lambda^\bullet V, \quad v_{i_1} \cdot \ldots \cdot v_{i_k} \in \mathrm{Cl}(V, \beta),$$

$1 \leq i_1 < \ldots < i_k \leq n$, $0 \leq k \leq n$, form a basis of $\Lambda^\bullet V$ and $\mathrm{Cl}(V, \beta)$ respectively. One shows easily by induction on $n = \dim V$ that there exists a β-orthogonal basis $v_1, \dots, v_n$ of V, i.e., $\beta(v_i, v_j) = 0$ for $i \neq j$. For such a basis $v_1, \dots, v_n$ of V the map

$$\Phi : \quad \Lambda^\bullet V \to \mathrm{Cl}(V, \beta) \quad \text{given by} \quad v_{i_1} \wedge \ldots \wedge v_{i_k} \mapsto v_{i_1} \cdot \ldots \cdot v_{i_k}$$

and linear extension is independent of the choice of β-orthogonal basis. Φ is an isomorphism of vector spaces but not an isomorphism of algebras.

2.2. The Spin Group

Notation 2.2.1. In the following, we denote the Clifford algebra of $\mathbb{R}^n$ with the standard Euclidean scalar product by $\mathbf{Cl}_{\boldsymbol{n}} := \mathrm{Cl}(\mathbb{R}^n, \beta_{\mathrm{eucl}})$.

Remark 2.2.2. Upon identifying $\mathbb{R}^n$ with $\imath(\mathbb{R}^n) \subset \mathrm{Cl}_n$, for every $v \in \mathbb{R}^n \setminus \{0\}$, we have $v^2 = -|v|^2 \cdot 1$ and thus

$$-\frac{v}{|v|^2} \cdot v = v \cdot \left(-\frac{v}{|v|^2}\right) = 1.$$

Thus, $\mathbb{R}^n \setminus \{0\}$ is contained in the subgroup of (multiplicatively) invertible elements of Cl_n.

Definition 2.2.3. We define the **Pin group Pin(n)** by

$$\mathrm{Pin}(n) := \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_n \;\middle|\; v_j \in S^{n-1} \subset \mathbb{R}^n, m \in \mathbb{N}_0 \right\}.$$

Remark 2.2.4. The subset $\mathrm{Pin}(n) \subset \mathrm{Cl}_n$ is a group with respect to the multiplication in Cl_n. The inverse element to $v_1 \cdot \ldots \cdot v_m$ is given by

$$(v_1 \cdot \ldots \cdot v_m)^{-1} = (-v_m) \cdot \ldots \cdot (-v_1) \in \mathrm{Pin}(n).$$

Definition 2.2.5. We define the **Spin group Spin(n)** by

$$\begin{aligned} \mathrm{Spin}(n) &:= \mathrm{Pin}(n) \cap \mathrm{Cl}_n^0 \\ &= \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_n \;\middle|\; v_j \in S^{n-1}, m \in 2\mathbb{N}_0 \right\}. \end{aligned}$$

Remark 2.2.6. By the argument from Remark 2.2.4, $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Pin}(n)$.

For a fixed $v \in S^{n-1} \subset \mathbb{R}^n$ and any $x \in \mathbb{R}^n$, we have:

$$v \cdot x \cdot v^{-1} = -v \cdot x \cdot v = -\big(-x \cdot v - 2\langle x, v\rangle 1\big) \cdot v = -\big(x - 2\langle x, v\rangle\, v\big).$$

The map $x \mapsto \big(x - 2\langle x, v\rangle\, v\big)$ is the reflection about the hyperplane $v^\perp$ perpendicular to v. In particular, $(x \mapsto v \cdot x \cdot v^{-1}) \in \mathrm{O}(n)$. For any $a := v_1 \cdot \ldots \cdot v_m \in \mathrm{Spin}(n)$, the map

$$x \mapsto a \cdot x \cdot a^{-1} = v_1 \cdot \ldots \cdot v_m \cdot x \cdot v_m^{-1} \cdot \ldots \cdot v_1^{-1}$$

consists of an even number of hyperplane reflections and is thus contained in $\mathrm{SO}(n)$. We have thus defined a group homomorphism $\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ by

$$\varrho(a)\, x := a \cdot x \cdot a^{-1}. \tag{2.3}$$

Example 2.2.7. Let $n = 1$. Then we have $\mathrm{SO}(1) = \{1\}$ and

$$\begin{aligned} \mathrm{Spin}(1) &= \big\{ v_1 \cdot \ldots \cdot v_m \,|\, v_j \in S^0, m \in 2\mathbb{N}_0 \big\} \\ &= \{\varepsilon_1 e_1 \cdot \ldots \cdot \varepsilon_m e_m \,|\, \varepsilon_j = \pm 1, m \in 2\mathbb{N}_0\} \\ &= \{-1, 1\} \\ &\cong \mathbb{Z}_2. \end{aligned}$$

Example 2.2.8. Let $n = 2$. Then $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1 \subset \mathbb{C}$ and we have:

$$\mathrm{Spin}(2) = \{(\cos\theta_1 e_1 + \sin\theta_1 e_2) \cdot \ldots \cdot (\cos\theta_m e_1 + \sin\theta_m e_2) \;\big|\; \theta_j \in \mathbb{R}, j = 1, \ldots, m,\ m \in 2\mathbb{N}_0\}.$$

Now we compute:

$$\begin{aligned}(\cos\theta e_1 + \sin\theta e_2)&(\cos\varphi e_1 + \sin\varphi e_2)\\ &= -\cos\theta\cos\varphi - \sin\theta\sin\varphi + (\cos\theta\sin\varphi - \sin\theta\cos\varphi)e_1\cdot e_2\\ &= -\big(\cos(\theta-\varphi) + \sin(\theta-\varphi)e_1\cdot e_2\big)\end{aligned}$$

and

$$\begin{aligned}(\cos(\alpha) + \sin(\alpha)e_1\cdot e_2)&\cdot(\cos(\beta)+\sin(\beta)e_1\cdot e_2)\\ &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) + (\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta))e_1\cdot e_2\\ &= \cos(\alpha+\beta)+\sin(\alpha+\beta)e_1\cdot e_2.\end{aligned}$$

We thus obtain:

$$\begin{aligned}\mathrm{Spin}(2) &= \Big\{(-1)^{\frac{m}{2}}\big(\cos(\theta_1-\theta_2)+\sin(\theta_1-\theta_2)e_1\cdot e_2\big)\cdot\ldots\\ &\qquad\cdot\big(\cos(\theta_{m-1}-\theta_m)+\sin(\theta_{m-1}-\theta_m)e_1\cdot e_2\big)\;\Big|\;\theta_j\in\mathbb{R}, m\in 2\mathbb{N}_0\Big\}\\ &= \Big\{(-1)^{\frac{m}{2}}\big(\cos(\theta_1-\theta_2+\theta_3-\theta_4+\ldots+\theta_{m-1}-\theta_m)\\ &\qquad+\sin(\theta_1-\theta_2+\theta_3-\theta_4+\ldots+\theta_{m-1}-\theta_m)e_1\cdot e_2\big)\;\Big|\;\theta_j\in\mathbb{R}, m\in 2\mathbb{N}_0\Big\}\\ &= \big\{\pm(\cos\alpha+\sin\alpha e_1\cdot e_2)\;\big|\;\alpha\in\mathbb{R}\big\}\\ &= \big\{\cos\alpha+\sin\alpha e_1\cdot e_2\;\big|\;\alpha\in\mathbb{R}\big\}\\ &\cong \mathrm{U}(1)\\ &\cong \mathrm{SO}(2).\end{aligned}$$

We compute the group homomorphism $\varrho : \mathrm{Spin}(2)\to\mathrm{SO}(2)$: For $j=1,2$, we have:

$$\begin{aligned}\varrho(&\cos\alpha+\sin\alpha\, e_1\cdot e_2)(e_j)\\ &= (\cos\alpha+\sin\alpha\, e_1\cdot e_2)\cdot e_j\cdot(\cos\alpha+\sin\alpha\, e_1\cdot e_2)^{-1}\\ &= (\cos\alpha+\sin\alpha\, e_1\cdot e_2)\cdot e_j\cdot(\cos(-\alpha)+\sin(-\alpha)\, e_1\cdot e_2)\\ &= \cos^2\alpha\, e_j - \cos\alpha\sin\alpha\, e_j\cdot e_1\cdot e_2 + \cos\alpha\sin\alpha\, e_1\cdot e_2\cdot e_j - \sin^2\alpha\, e_1\cdot e_2\cdot e_j\cdot e_1\cdot e_2\\ &= \left(\cos^2\alpha-\sin^2\alpha\right)\, e_j + \begin{cases}2\cos\alpha\sin\alpha\, e_2 & j=1,\\ -2\cos\alpha\sin\alpha\, e_1 & j=2\end{cases}\\ &= \cos(2\alpha)\, e_j + \begin{cases}\sin(2\alpha)\, e_2 & j=1,\\ -\sin(2\alpha)\, e_1 & j=2\end{cases}.\end{aligned}$$

Thus,

$$\varrho(\cos\alpha+\sin\alpha\, e_1\cdot e_2) = \begin{pmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{pmatrix}.$$

In summary, we have the commutative diagram:

$$\begin{tikzcd}\mathrm{Spin}(2) \arrow[r, "\cong"] \arrow[d, "\varrho"'] & \mathrm{U}(1) \arrow[d, "z\mapsto z^2"]\\ \mathrm{SO}(2) \arrow[r, "\cong"'] & \mathrm{U}(1)\end{tikzcd}$$

Remark 2.2.9. Recall from Remark 2.1.15 that there is a canonical isomorphism of vector spaces $\Phi\colon \Lambda^\bullet\mathbb{R}^n \cong \mathrm{Cl}_n$. We equip the Clifford algebra Cl_n with the unique scalar product such that Φ is an isometry. If $v_1, \ldots, v_n$ is any orthonormal basis of $\mathbb{R}^n$, then the elements $v_{i_1} \cdot \ldots \cdot v_{i_k}$, $1 \leq i_1 < \ldots < i_k \leq n$, $0 \leq k \leq n$, form an orthonormal basis of Cl_n with respect to this scalar product. Moreover, for any unit vector $v \in S^{n-1}$ the map $\mu_v\colon \mathrm{Cl}_n \to \mathrm{Cl}_n$, $X \mapsto X \cdot v$ is an isometry. In order to see this extend v to an orthonormal basis of $\mathbb{R}^n$ and use that the map μ_v acts by permuting the corresponding basis vectors of Cl_n.

Proposition 2.2.10. *For any $n \in \mathbb{N}$, the sequence*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\varrho} \mathrm{SO}(n) \to 1$$

is exact.

Proof. a) The map $\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is surjective:

By a classical result of Elie Cartan, every $A \in \mathrm{O}(n)$ is the composition of at most n hyperplane reflections. Thus, any given $A \in \mathrm{SO}(n)$ is the product of an even number of hyperplane reflections. Let the i-th hyperplane be the orthogonal complement to $v_i \in S^{n-1}$. Then we have $v_1 \cdot \ldots \cdot v_{2k} \in \mathrm{Spin}(n)$ and $\varrho(v_1 \cdot \ldots \cdot v_{2k}) = A$.

b) It remains to show that $\ker(\varrho) = \mathbb{Z}_2 = \{1, -1\}$:

Writing $-1 = e_1 \cdot e_1 \in \mathrm{Spin}(n)$ and applying ϱ, we obtain:

$$\varrho(-1)(x) = (-1) \cdot x \cdot (-1)^{-1} = x.$$

Thus, $\{1, -1\} \subset \ker(\varrho)$.

Conversely, let $a \in \ker(\varrho)$. Then for all $x \in \mathbb{R}^n$, we have:

$$x = \varrho(a)(x) = a \cdot x \cdot a^{-1}.$$

Equivalently, we have $x \cdot a = a \cdot x$ for all $x \in \mathbb{R}^n$ and in particular, $x \cdot a = a \cdot x$ for all $x \in \mathrm{Cl}_n$. Hence, a is contained in the center $\mathcal{Z}(\mathrm{Cl}_n)$ of Cl_n. Moreover, we have $a \in \mathrm{Spin}(n) \subset \mathrm{Cl}_n^0$. Now for any $n \in \mathbb{N}_0$ we have

$$\mathcal{Z}(\mathrm{Cl}_n) \cap \mathrm{Cl}_n^0 = \mathbb{R} \cdot 1, \qquad \text{(exercise !)},$$

hence $a = \alpha\, 1$ for some $\alpha \in \mathbb{R}$. Since $a \in \mathrm{Spin}(n)$, we can write $a = v_1 \cdot \ldots \cdot v_m$ for some $v_j \in S^{n-1}$ and $m \in 2\mathbb{N}_0$. We denote by $|\cdot|$ the norm induced by the scalar product on Cl_n constructed in Remark 2.2.9. Using that Clifford multiplication by v_m is an isometry we get:

$$|\alpha| = |v_1 \cdot \ldots \cdot v_{m-1} \cdot v_m| = |v_1 \cdot \ldots \cdot v_{m-1}|.$$

Now we proceed inductively and obtain $|\alpha| = |v_1| = 1$. Hence, $\alpha = \pm 1$ and thus $a \in \{1, -1\}$. Therefore, $\ker(\varrho) \subset \{1, -1\}$. □

Remark 2.2.11.
Let $\mathrm{Cl}_n^\times := \left\{x \in \mathrm{Cl}_n \,\middle|\, \exists y \in \mathrm{Cl}_n \text{ s.t. } x \cdot y = 1\right\}$ be the group of invertible elements in the Clifford algebra Cl_n. Then we have:

i) The map

$$\begin{array}{rcl} \mathrm{Cl}_n \times \mathrm{Cl}_n & \to & \mathrm{Cl}_n \\ (a,b) & \mapsto & a \cdot b \end{array}$$

is a bilinear map on a finite-dimensional $\mathbb{R}$-vector space, hence it is smooth.

ii) The map

$$\begin{array}{rcl} \mathrm{Cl}_n^\times & \to & \mathrm{Cl}_n^\times \\ a & \mapsto & a^{-1} \end{array}$$

is also smooth.

Thus $\mathrm{Cl}_n^\times$ is a Lie group.

Remark 2.2.12. Part a) of the proof of Proposition 2.2.10 shows that every element $a \in \mathrm{Spin}(n)$ is of the form

$$a = \pm\, v_1 \cdot \ldots \cdot v_m \quad \text{with } m = 2k \leq n,\ v_j \in S^{n-1}.$$

We may drop the minus sign by replacing v_1 by $-v_1$ if necessary, to obtain

$$a = v_1 \cdot \ldots \cdot v_m \quad \text{with } m = 2k \leq n.$$

By multiplying with $1 = e_1 \cdot (-e_1)$, we can increase the number of factors by 2. Thus, we can assume w.l.o.g. that

$$a = v_1 \cdot \ldots \cdot v_m \quad \text{with} \quad m = 2k = \begin{cases} n & n \text{ even} \\ n+1 & n \text{ odd} \end{cases}.$$

Hence, we have a surjective continuous map

$$\begin{array}{rcl} \overbrace{S^{n-1} \times \ldots \times S^{n-1}}^{m \text{ times}} & \to & \mathrm{Spin}(n) \\ (v_1, \ldots, v_m) & \mapsto & v_1 \cdot \ldots \cdot v_m\,. \end{array}$$

It follows that $\mathrm{Spin}(n)$ is compact. In particular, $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^\times$ is a closed subgroup. Thus, $\mathrm{Spin}(n)$ is a Lie group and the homomorphism

$$\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

is a 2-fold covering.

Proposition 2.2.13. *The Spin group* Spin(n) *is connected, if* $n \geq 2$ *and simply-connected, if* $n \geq 3$.

Proof. Assume $n \geq 2$.

a) From the exact sequence in Proposition 2.2.10 we get the long exact homotopy sequence (base point = 1):

$$\to \underbrace{\pi_1(\mathbb{Z}_2)}_{=\{1\}} \to \pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n)) \to \underbrace{\pi_0(\mathbb{Z}_2)}_{=\mathbb{Z}_2} \to \pi_0(\mathrm{Spin}(n)) \to \underbrace{\pi_0(\mathrm{SO}(n))}_{=\{1\}}.$$

Claim: The map $\pi_0(\mathbb{Z}_2) \xrightarrow{\psi} \pi_0(\mathrm{Spin}(n))$ is trivial, that is, the image of ψ is $\{1\}$.

In fact, 1 and -1 can be connected by a continuous path in Spin(n): Since $n \geq 2$, we have at least two orthonormal vectors $e_1, e_2 \in S^{n-1}$ and we can define the smooth curve $c : \mathbb{R} \to \mathrm{Spin}(n)$,

$$t \mapsto (\cos(t)\, e_1 + \sin(t)\, e_2) \cdot e_1,$$

satisfying $c(0) = -1$ and $c(\pi) = 1$.

b) By exactness at $\pi_0(\mathrm{Spin}(n))$ and the claim, the map

$$\pi_0(\mathrm{Spin}(n)) \to \pi_0(\mathrm{SO}(n)) = \{1\}$$

is injective. Hence, $\pi_0(\mathrm{Spin}(n)) = \{1\}$, that is, Spin($n$) is connected.

c) We have $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$. Namely, the long exact homotopy sequence for the fiber bundle $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to S^n$ for $n \geq 3$ yields isomorphisms $\pi_1(\mathrm{SO}(n)) \xrightarrow{\cong} \pi_1(\mathrm{SO}(n+1))$. The long exact homotopy sequence of the fiber bundle $\mathbb{Z}_2 \to S^3 \to \mathbb{R}P^3$ yields the isomorphism $\pi_1(\mathbb{R}P^3) \xrightarrow{\cong} \pi_0(\mathbb{Z}_2) = \mathbb{Z}_2$. Finally, we have the identification $\mathbb{R}P^3 \cong \mathrm{SO}(3)$ as follows: We identify $\mathbb{R}P^3$ with the quotient of the upper hemisphere S^3_+ obtained by identifying antipodal points on the equator. Now, S^3_+ is homeomorphic to a closed unit ball $\bar{B}^3(0) \subset \mathbb{R}^3$ and thus $\mathbb{R}P^3$ is homeomorphic to $\bar{B}^3(0)$ with antipodal boundary points identified. The map sending the equivalence class of a point $x \in \bar{B}^3(0) \setminus \{0\}$ to the rotation with axis x and angle $\|x\|\pi$ is then a homeomorphism.

d) Now assume that $n \geq 3$. By exactness at $\pi_0(\mathbb{Z}_2)$ and the claim in a), the map $\pi_1(\mathrm{SO}(n)) \to \pi_0(\mathbb{Z}_2)$ is surjective. Exactness at $\pi_1(\mathrm{SO}(n))$, together with the fact that $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$ implies that the map $\pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n))$ is trivial. By exactness at $\pi_1(\mathrm{Spin}(n))$, the map $\pi_1(\mathrm{Spin}(n)) \to \pi_1(\mathrm{SO}(n))$ is also injective. Hence $\pi_1(\mathrm{Spin}(n)) = \{1\}$, that is, Spin($n$) is simply-connected. ☐

The Lie algebra of $\mathrm{SO}(n)$ is given by

$$\mathfrak{so}(n) = \left\{ A \in \mathrm{Mat}(n \times n; \mathbb{R}) \mid A^\top = -A \right\}$$

and $\dim \mathrm{SO}(n) = \dim \mathfrak{so}(n) = \frac{1}{2}n(n-1)$.

For the Lie algebra of the Spin group, we have $\dim \mathfrak{spin}(n) = \dim \mathrm{Spin}(n) = \dim \mathrm{SO}(n) = \frac{1}{2}n(n-1)$. We want to identify the Lie algebra $\mathfrak{spin}(n)$ of $\mathrm{Spin}(n)$ as a vector subspace of Cl_n:

For $i \neq j$ consider the smooth curve $c : \mathbb{R} \to \mathrm{Spin}(n)$, defined by

$$t \mapsto \big(\cos(t)\, e_i + \sin(t)\, e_j \big) \cdot (-e_i).$$

Then $c(0) = e_i \cdot (-e_i) = 1$ and $\dot{c}(0) = e_j \cdot (-e_i) = e_i \cdot e_j$. We thus have $e_i \cdot e_j \in T_1\mathrm{Spin}(n) \cong \mathfrak{spin}(n)$ for all $i \neq j$.
The products $\{e_i \cdot e_j\}$, $1 \le i < j \le n$ are linearly independent and there are $\frac{1}{2}n(n-1)$ of them. Since $\dim(\mathfrak{spin}(n)) = \frac{1}{2}n(n-1)$, we conclude that $\{e_i \cdot e_j\}_{i<j}$ is a basis of $\mathfrak{spin}(n)$.

We compute the Lie algebra homomorphism $\varrho_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$:

$$\begin{aligned}
\varrho_*(e_i \cdot e_j)(e_k) &= \frac{d}{dt}\Big|_{t=0} \varrho\Big((\cos(t)\, e_i + \sin(t)\, e_j) \cdot (-e_i) \Big)(e_k) \\
&= \frac{d}{dt}\Big|_{t=0} \Big((\cos(t)\, e_i + \sin(t)\, e_j) \cdot (-e_i) \cdot e_k \cdot e_i \cdot (-\cos(t)\, e_i - \sin(t)\, e_j) \Big) \\
&= \frac{d}{dt}\Big|_{t=0} \big(\cos^2(t)\, e_i \cdot e_i \cdot e_k \cdot e_i \cdot e_i + \cos(t)\sin(t)\, e_i \cdot e_i \cdot e_k \cdot e_i \cdot e_j \\
&\qquad\qquad + \sin(t)\cos(t)\, e_j \cdot e_i \cdot e_k \cdot e_i \cdot e_i + \sin^2(t)\, e_j \cdot e_i \cdot e_k \cdot e_i \cdot e_j \big) \\
&= e_i \cdot e_i \cdot e_k \cdot e_i \cdot e_j + e_j \cdot e_i \cdot e_k \cdot e_i \cdot e_i \\
&= -e_k \cdot e_i \cdot e_j - e_j \cdot e_i \cdot e_k. \\
&= \begin{cases} 0 & \text{for } k \notin \{i,j\} \\ 2e_j & \text{for } k = i \\ -2e_i & \text{for } k = j. \end{cases}
\end{aligned}$$

We thus have for $i < j$

$$\varrho_*(e_i \cdot e_j) = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & & \cdots & -2 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 2 & \cdots & \cdots & \cdots \\ & \vdots & & \vdots & \end{pmatrix}.$$

2.3. Spinors

Definition 2.3.1. A **representation** of a group G on a vector space V is a group homomorphism $\lambda : G \to \mathrm{GL}(V)$.
A representation $\lambda : G \to \mathrm{GL}(V)$ on a Euclidean or Hermitian vector space (V, β) is called **orthogonal** or **unitary**, if $\lambda(G) \subset \mathrm{O}(V, \beta)$ or $\lambda(G) \subset \mathrm{U}(V, \beta)$, respectively.

We will only consider real or complex finite-dimensional representations, i.e., representations, where V is a $\mathbb{K}$-vector space, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $\dim_{\mathbb{K}} V < \infty$.

Example 2.3.2. Let G be any group.

a) The **trivial representation** is the trivial group homomorphism:

$$\begin{aligned} \lambda : G &\to \mathrm{GL}(V) \\ g &\mapsto \mathrm{id}_V . \end{aligned}$$

b) Let $\lambda : G \to \mathrm{GL}(V)$ be a representation. The **dual representation** λ^* is defined by:

$$\begin{aligned} \lambda^* : G &\to \mathrm{GL}(V^*) \\ g &\mapsto \lambda\left(g^{-1}\right)^* . \end{aligned}$$

c) Let $\lambda : G \to \mathrm{GL}(V)$ be a representation. There is an induced representation $\Lambda^k\lambda$ of G on $\Lambda^k V$, defined by:

$$\begin{aligned} \Lambda^k\lambda : G &\to \mathrm{GL}\big(\Lambda^k V\big) \\ g &\mapsto \big(\Lambda^k\lambda\big)(g), \end{aligned}$$

where

$$\big(\Lambda^k\lambda\big)(g)(v_1 \wedge \ldots \wedge v_k) := \lambda(g)v_1 \wedge \ldots \wedge \lambda(g)v_k.$$

The representation $\Lambda^k\lambda$ is called the $\mathbf{k}^{\text{th}}$ **exterior power** of the representation λ.

d) Let $\lambda_1 : G \to \mathrm{GL}(V_1)$ and $\lambda_2 : G \to \mathrm{GL}(V_2)$ be representations of G on V_1 and V_2, respectively. There is an induced representation of G on $V_1 \oplus V_2$, defined by:

$$\begin{aligned} \lambda_1 \oplus \lambda_2 : G &\to \mathrm{GL}\left(V_1 \oplus V_2\right) \\ g &\mapsto \left(\lambda_1 \oplus \lambda_2\right)(g), \end{aligned}$$

where

$$\left(\lambda_1 \oplus \lambda_2\right)(g)(v_1 \oplus v_2) := \lambda_1(g)v_1 \oplus \lambda_2(g)v_2.$$

The representation $\lambda_1 \oplus \lambda_2$ is called the **direct sum** of the representations λ_1 and λ_2.

e) Let $\lambda_1 : G \to \mathrm{GL}(V_1)$ and $\lambda_2 : G \to \mathrm{GL}(V_2)$ be representations of G on V_1 and V_2, respectively. There is an induced representation of G on $V_1 \otimes V_2$, defined by:

$$\begin{aligned} \lambda_1 \otimes \lambda_2 : G &\to \mathrm{GL}\left(V_1 \otimes V_2\right) \\ g &\mapsto \left(\lambda_1 \otimes \lambda_2\right)(g), \end{aligned}$$

where

$$\left(\lambda_1 \otimes \lambda_2\right)(g)(v_1 \otimes v_2) := \lambda_1(g)v_1 \otimes \lambda_2(g)v_2.$$

The representation $\lambda_1 \otimes \lambda_2$ is called the **tensor product** of the representations λ_1 and λ_2.

Example 2.3.3. Let $G = \mathrm{SO}(n)$.

a) We have the standard representation

$$\lambda_{\mathrm{st}} : \mathrm{O}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(\mathbb{R}^n).$$

b) Then

$$\begin{aligned} \Lambda^n \lambda_{\mathrm{st}} : \mathrm{O}(n) &\to \mathrm{GL}\big(\Lambda^n \mathbb{R}^n\big) = \mathrm{GL}(1) = \mathbb{R} \setminus \{0\}, \\ g &\mapsto \big(\Lambda^n \lambda_{\mathrm{st}}\big)(g), \end{aligned}$$

is given by

$$\big(\Lambda^n \lambda_{\mathrm{st}}\big)(g) = \det g = \pm 1.$$

If we restrict $\Lambda^n \lambda_{\mathrm{st}}$ to $\mathrm{SO}(n)$ then $\Lambda^n \lambda_{\mathrm{st}} : \mathrm{SO}(n) \to \mathrm{GL}(\Lambda^n \mathbb{R}^n)$ is given by $g \mapsto 1$, i.e., $\Lambda^n \lambda_{\mathrm{st}} : \mathrm{SO}(n) \to \mathrm{GL}(\Lambda^n \mathbb{R}^n)$ is the trivial representation.

Remark 2.3.4. Given a representation $\lambda' : \mathrm{SO}(n) \to \mathrm{GL}(V)$ then

$$\lambda := \lambda' \circ \varrho : \mathrm{Spin}(n) \to \mathrm{GL}(V)$$

yields a representation of $\mathrm{Spin}(n)$ on V. Here, $\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ denotes the Lie group homomorphism defined by equation (2.3).

One may wonder whether every representation λ of $\mathrm{Spin}(n)$ on V is of the form $\lambda = \lambda' \circ \varrho$, where λ' is a representation of $\mathrm{SO}(n)$ on V.
Now, if $\lambda = \lambda' \circ \varrho : \mathrm{Spin}(n) \to \mathrm{GL}(V)$, we have:

$$\lambda(-1) = \lambda'(\varrho(-1)) = \lambda'(1) = \mathrm{id}_V.$$

Hence a representation $\lambda : \mathrm{Spin}(n) \to \mathrm{GL}(V)$ that is induced by a representation of $\mathrm{SO}(n)$ on V necessarily satisfies $\lambda(-1) = \mathrm{id}_V$.

Consider the representation

$$\begin{aligned} \lambda : \mathrm{Spin}(n) &\to \mathrm{GL}(\mathrm{Cl}_n) \\ a &\mapsto \lambda(a), \end{aligned}$$

given by the multiplication in Cl_n, i.e., for any $a \in \mathrm{Spin}(n)$ and $x \in \mathrm{Cl}_n$, we have

$$\lambda(a)(x) := a \cdot x.$$

Then we compute

$$\lambda(-1)(x) = -1 \cdot x = -x,$$

i.e., $\lambda(-1) = -\mathrm{id}_{\mathrm{Cl}_n}$. Thus, this representation cannot be induced by a representation of $\mathrm{SO}(n)$ on Cl_n.

Remark 2.3.5. If $\lambda : \mathrm{Spin}(n) \to \mathrm{GL}(V)$ is a representation of $\mathrm{Spin}(n)$ on V such that $\lambda(-1) = \mathrm{id}_V$ then $\ker(\varrho) = \{-1, 1\} \subset \ker(\lambda)$.

Since ϱ is surjective, there is a map $\lambda' : \mathrm{SO}(n) \to \mathrm{GL}(V)$ such that the alongside diagram commutes.

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[d, "\varrho"'] \arrow[dr, "\lambda"] & \\ \mathrm{SO}(n) \arrow[r, "\lambda'"'] & \mathrm{GL}(V) \end{tikzcd}$$

The even dimensional case

In the following, let $n = 2m$. Let Cl_n be the Clifford algebra of $\mathbb{R}^n$ with the standard Euclidean scalar product and let $\mathbb{C}\mathrm{l}_n := \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Let $e_1, \ldots, e_{2m}$ be the standard basis of $\mathbb{R}^n$. For $j = 1, \ldots, m$ define

$$z_j := \frac{1}{2}(e_{2j-1} - i\, e_{2j}) \in \mathbb{C}\mathrm{l}_n, \quad \bar{z}_j := \frac{1}{2}(e_{2j-1} + i\, e_{2j}) \in \mathbb{C}\mathrm{l}_n.$$

Then products of the form

$$\begin{aligned} z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_{i_1} \cdot \ldots \cdot \bar{z}_{i_l}, \qquad & k, l = 0, \ldots, m \\ & 1 \le j_1 < \ldots < j_k \le m,\ 1 \le i_1 < \ldots < i_l \le m, \end{aligned}$$

form a vector space basis of $\mathbb{C}\mathrm{l}_n$. Put

$$z(j_1, \ldots, j_k) := z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m.$$

Then

$$\Sigma_n := \mathrm{span}\{z(j_1, \ldots, j_k) \,|\, k = 0, \ldots, m,\ 1 \le j_1 < \ldots < j_k \le m\} \subseteq \mathbb{C}\mathrm{l}_n$$

is a complex vector subspace of $\mathbb{C}\mathrm{l}_n$ of dimension 2^m. We call $\boldsymbol{\Sigma_n}$ the **spinor space** in dimension n. Elements of Σ_n are called **spinors**.

For later purposes we want to compute $e_{2l} \cdot z(j_1, \ldots, j_k)$ and $e_{2l-1} \cdot z(j_1, \ldots, j_k)$. We have to distinguish two cases: e_{2l} and e_{2l-1} can be contained in $z_{j_1} \cdot \ldots \cdot z_{j_k}$ or not.

1) Let e_{2l} and e_{2l-1} not be contained in $z_{j_1} \cdot \ldots \cdot z_{j_k}$.

$$\begin{aligned} e_{2l} \cdot z(j_1, \ldots, j_k) &= e_{2l} \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\ &= (-1)^{k+(l-1)} z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_{l-1} \cdot e_{2l} \cdot \bar{z}_l \cdot \bar{z}_{l+1} \cdot \ldots \cdot \bar{z}_m. \end{aligned}$$

Because of

$$\begin{aligned} e_{2l} \cdot \bar{z}_l &= \frac{1}{2}\, e_{2l} \cdot (e_{2l-1} + i e_{2l}) \\ &= \frac{1}{2}\, (e_{2l} \cdot e_{2l-1} - i) \\ &= \frac{1}{2}\, (-e_{2l-1} \cdot e_{2l} + i e_{2l-1} \cdot e_{2l-1}) \\ &= i\, e_{2l-1} \cdot \frac{1}{2} (e_{2l-1} + i e_{2l}) \\ &= i\, e_{2l-1} \cdot \bar{z}_l, \end{aligned}$$

it follows that

$$\begin{aligned} e_{2l} \cdot z(j_1, \ldots, j_k) &= (-1)^{k+l-1}\, i\, z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_{l-1} \cdot e_{2l-1} \cdot \bar{z}_l \cdot \bar{z}_{l+1} \cdot \ldots \cdot \bar{z}_m \\ &= (-1)^{k+l-1}\, i\, (-1)^{k+l-1}\, e_{2l-1} \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\ &= i\, e_{2l-1} \cdot z(j_1, \ldots, j_k). \end{aligned} \tag{2.4}$$

Let ν such that $j_\nu < l < j_{\nu+1}$. Then we have:

$$\begin{aligned} e_{2l} \cdot z(j_1, \ldots, j_k) &= \frac{1}{2}\, e_{2l} \cdot z(j_1, \ldots, j_k) + \frac{1}{2}\, e_{2l} \cdot z(j_1, \ldots, j_k) \\ &\overset{(2.4)}{=} \frac{1}{2}\, e_{2l} \cdot z(j_1, \ldots, j_k) + \frac{i}{2}\, e_{2l-1} \cdot z(j_1, \ldots, j_k) \\ &= i\, \underbrace{\frac{1}{2}\, (e_{2l-1} - i e_{2l})}_{= z_l} \cdot z(j_1, \ldots, j_k) \\ &= i\, (-1)^\nu\, z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k). \end{aligned} \tag{2.5}$$

Moreover, it follows from equations (2.4) and (2.5) that

$$\begin{aligned} e_{2l-1} \cdot z(j_1, \ldots, j_k) &\overset{(2.4)}{=} -i\, e_{2l} \cdot z(j_1, \ldots, j_k) \\ &\overset{(2.5)}{=} (-1)^\nu\, z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k). \end{aligned} \tag{2.6}$$

2) Now let e_{2l} and e_{2l-1} be contained in $z_{j_1} \cdot \ldots \cdot z_{j_k}$.
   Multiplying equation (2.5) with e_{2l} we obtain:
$$e_{2l} \cdot z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k) = (-1)^\nu \, i \, z(j_1, \ldots, j_\nu, j_{\nu+1}, \ldots, j_k) \tag{2.7}$$
   Multiplying equation (2.6) with e_{2l-1} we obtain:
$$e_{2l-1} \cdot z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k) = (-1)^{\nu+1} \, z(j_1, \ldots, j_\nu, j_{\nu+1}, \ldots, j_k). \tag{2.8}$$

Hence the spinor space $\Sigma_n \subset \mathbb{C}\mathrm{l}_n$ is invariant under **Clifford multiplication** by vectors in $\mathbb{R}^n$. Since the Clifford algebra $\mathbb{C}\mathrm{l}_n$ is generated by $\mathbb{R}^n$, the same holds for Clifford multiplication by elements of $\mathbb{C}\mathrm{l}_n$, thus $\Sigma_n \subset \mathbb{C}\mathrm{l}_n$ is a left ideal. In particular, Σ_n is invariant under multiplication by elements of $\mathrm{Spin}(n)$.
We define:
$$\Sigma_n^+ := \mathrm{span}\{z(j_1, \ldots, j_k) \,|\, k = 0, \ldots, m \text{ even}\}$$
$$\Sigma_n^- := \mathrm{span}\{z(j_1, \ldots, j_k) \,|\, k = 0, \ldots, m \text{ odd}\}.$$

The spinor space Σ_n has the decomposition $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$. Elements in $\Sigma_n^\pm$ are called spinors of positive and negative **chirality** respectively.
The equations (2.5)–(2.8) show that the Clifford multiplication by elements of $\mathbb{R}^n$ satisfies:
$$\mathbb{R}^n \cdot \Sigma_n^+ \subset \Sigma_n^-, \quad \mathbb{R}^n \cdot \Sigma_n^- \subset \Sigma_n^+.$$
However, Clifford multiplication by elements of $\mathbb{C}\mathrm{l}_n^0$ satisfies:
$$\mathbb{C}\mathrm{l}_n^0 \cdot \Sigma_n^+ \subset \Sigma_n^+, \quad \mathbb{C}\mathrm{l}_n^0 \cdot \Sigma_n^- \subset \Sigma_n^-.$$
Thus, the restriction to $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^0 \subset \mathbb{C}\mathrm{l}_n^0$ yields representations of $\mathrm{Spin}(n)$ on Σ_n^+ and Σ_n^- and thus on Σ_n.

Definition 2.3.6. The representation $\sigma_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$ is called the **spinor representation**.
The representations $\sigma_n^\pm : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n^\pm)$ are called the **positive** and **negative** spinor representation, respectively.

Remark 2.3.7. The element
$$\omega := e_1 \cdot \ldots \cdot e_n \in \mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$$
is called the **volume element**. The equations (2.5)–(2.8) show that
$$e_{2l-1} \cdot e_{2l} \cdot z(j_1, \ldots, j_k) = \begin{cases} -i \, z(j_1, \ldots, j_k) & \text{if } e_{2l}, e_{2l-1} \text{ are not contained in } z_{j_1} \cdot \ldots \cdot z_{j_k} \\ i \, z(j_1, \ldots, j_k) & \text{if } e_{2l}, e_{2l-1} \text{ are contained in } z_{j_1} \cdot \ldots \cdot z_{j_k}. \end{cases}$$

Thus we get

$$\omega \cdot z(j_1, \ldots, j_k) = (-1)^{m-k}\, i^m\, z(j_1, \ldots, j_k)$$

and therefore

$$i^m\, \omega \cdot z(j_1, \ldots, j_k) = (-1)^k\, z(j_1, \ldots, j_k).$$

It follows that

$$\Sigma_n^{\pm} = \{z \in \Sigma_n \,|\, i^m \omega \cdot z = \pm z\}. \tag{2.9}$$

Example 2.3.8. Let $n = 2$, i.e., $m = 1$. Then we have:

$$\Sigma_2^+ = \mathbb{C} \cdot z() \qquad \text{and} \qquad \Sigma_2^- = \mathbb{C} \cdot z(1).$$

By the equations (2.6) and (2.8), we have

$$\left.\begin{aligned} e_1 \cdot z(\,) &= z(1) \\ e_1 \cdot z(1) &= -z(\,), \end{aligned}\right\} \quad \text{thus } e_1 \text{ acts on } \Sigma_2 \cong \mathbb{C}^2 \text{ as } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By the equations (2.5) and (2.7), we have

$$\left.\begin{aligned} e_2 \cdot z(\,) &= iz(1) \\ e_2 \cdot z(1) &= iz(\,), \end{aligned}\right\} \quad \text{thus } e_2 \text{ acts on } \Sigma_2 \cong \mathbb{C}^2 \text{ as } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Furthermore, from Example 2.2.8, we have the isomorphism

$$\begin{aligned} \mathrm{Spin}(2) &\to \mathrm{U}(1)\,, \\ \cos\psi + \sin\psi\, e_1 \cdot e_2 &\mapsto \cos\psi + i\,\sin\psi = e^{i\psi}\,. \end{aligned}$$

The element $\cos\psi + \sin\psi\, e_1 \cdot e_2 \in \mathrm{Spin}(2)$ acts on Σ_2 as

$$\begin{aligned} \cos\psi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &+ \sin\psi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi & 0 \\ 0 & \cos\psi \end{pmatrix} + \sin\psi \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi - i\sin\psi & 0 \\ 0 & \cos\psi + i\sin\psi \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\psi} & 0 \\ 0 & e^{i\psi} \end{pmatrix}. \end{aligned}$$

Thus, the action of the Spin group Spin(2) on Σ_2^- is given by the standard representation of U(1) on $\mathbb{C}$ whereas the action of Spin(2) on Σ_2^+ is given by the dual of the standard representation of U(1) on $\mathbb{C}$. In particular, the action of Spin(2) on $\Sigma_2^+ \otimes \Sigma_2^-$ is the trivial representation.

We equip the spinor space Σ_n with the Hermitian scalar product $\langle\cdot,\cdot\rangle$, for which the vectors $z(j_1,\ldots,j_k)$, $k=0,\ldots,m$, form an orthonormal basis. Then the decomposition $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ is orthogonal. By our convention $\langle\cdot,\cdot\rangle$ is complex linear in the first argument and complex antilinear in the second argument.

Lemma 2.3.9. *Let $n=2m$ be even. Then the Clifford multiplication of spinors by vectors is skew-symmetric, i.e., for any $X \in \mathbb{R}^n$ and any $\phi,\psi \in \Sigma_n$, we have:*

$$\langle X\cdot\phi,\psi\rangle = -\langle\phi, X\cdot\psi\rangle\,. \tag{2.10}$$

Proof. It suffices to prove this for any basis vector $X=e_j$ in $\mathbb{R}^n$ and any two basis elements $\phi = z(j_1,\ldots,j_k)$, $\psi = z(i_1,\ldots,i_l)$ in Σ_n. For $X=e_{2l}$ and $\phi = z(j_1,\ldots,j_k)$ with $l=j_{\nu+1}$, the scalar products are non-zero only if $\psi = z(j_1,\ldots,j_\nu,\widehat{l},j_{\nu+2},\ldots,j_k)$. If ψ is any other basis vector of Σ_n then both sides in (2.10) vanish.
We then have:

$$\begin{aligned}
\langle X\cdot\phi,\psi\rangle &= \big\langle e_{2l}\cdot z(j_1,\ldots,j_\nu,l,j_{\nu+2},\ldots,j_k), z(j_1,\ldots,j_\nu,\widehat{l},j_{\nu+2},\ldots,j_k)\big\rangle\\
&\overset{(2.7)}{=} \big\langle(-1)^\nu\, i\, z(j_1,\ldots,j_\nu,j_{\nu+2},\ldots,j_k), z(j_1,\ldots,j_\nu,j_{\nu+2},\ldots,j_k)\big\rangle\\
&= (-1)^\nu\, i
\end{aligned}$$

and

$$\begin{aligned}
\langle\phi, X\cdot\psi\rangle &= \big\langle z(j_1,\ldots,j_\nu,l,j_{\nu+2},\ldots,j_k), e_{2l}\cdot z(j_1,\ldots,j_\nu,\widehat{l},j_{\nu+2},\ldots,j_k)\big\rangle\\
&\overset{(2.5)}{=} \big\langle z(j_1,\ldots,j_\nu,l,j_{\nu+2},\ldots,j_k), (-1)^\nu\, i\, z(j_1,\ldots,j_\nu,l,j_{\nu+2},\ldots,j_k)\big\rangle\\
&= (-1)^{\nu+1}\, i.
\end{aligned}$$

Thus, we have

$$\langle X\cdot\phi,\psi\rangle = -\langle\phi, X\cdot\psi\rangle.$$

The computations for the remaining basis vectors $X\in\mathbb{R}^n$ and $\phi\in\Sigma_n$ are entirely analogous. □

Remark 2.3.10. For any unit vector $X \in S^{n-1}\subset\mathbb{R}^n$ and any two spinors $\phi,\psi\in\Sigma_n$ we have:

$$\langle X\cdot\phi, X\cdot\psi\rangle = -\langle X\cdot X\cdot\phi,\psi\rangle = -\langle -|X|^2\phi,\psi\rangle = \langle\phi,\psi\rangle\,.$$

Hence the Clifford multiplication by unit vectors $X\in S^{n-1}$ is an isometry on the spinor space. The action of $\mathrm{Spin}(n)$ on Σ_n is thus a unitary representation.

Proposition 2.3.11. *Let $n = 2m$ be even. Then the map*

$$\Phi : \quad \mathbb{C}l_n \to \mathrm{End}(\Sigma_n), \quad \Phi(X)(z) := X \cdot z$$

is an isomorphism of complex algebras.

Proof. Obviously Φ is a homomorphism of complex algebras. We prove that Φ is surjective. Note first that for all $\ell \in \{1, \dots, m\}$ we have

$$z_\ell \cdot \bar{z}_\ell + \bar{z}_\ell \cdot z_\ell = -1 \tag{2.11}$$
$$\bar{z}_\ell \cdot \bar{z}_\ell = 0 \tag{2.12}$$
$$z_\ell \cdot z_\ell = 0. \tag{2.13}$$

Let $i, \ell \in \{1, \dots, m\}$ and let $z(j_1, \dots, j_k) \in \Sigma_n$.
a) Assume $\ell \in \{j_1, \dots, j_k\}$. From the equations (2.11) and (2.12) we get

$$\begin{aligned}
\Phi(\bar{z}_\ell)(z(j_1, \dots, j_k)) &= \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_\ell \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_\ell \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot z_\ell \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= \pm(-1 - z_\ell \cdot \bar{z}_\ell) \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m + 0 \\
&= \pm z(j_1, \dots, \widehat{\ell}, \dots, j_k),
\end{aligned}$$

where the signs $\pm$ may change in every line.
b) Assume $\ell \notin \{j_1, \dots, j_k\}$. Then by the equation (2.12) we get

$$\begin{aligned}
\Phi(\bar{z}_\ell)(z(j_1, \dots, j_k)) &= \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_\ell \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= 0.
\end{aligned}$$

c) Assume $i \in \{j_1, \dots, j_k\}$. By the equation (2.13) we get

$$\begin{aligned}
\Phi(z_i)(z(j_1, \dots, j_k)) &= z_i \cdot z_{j_1} \cdot \ldots \cdot z_i \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\
&= \pm z_i \cdot z_i \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_i} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\
&= 0.
\end{aligned}$$

d) If $i \notin \{j_1, \dots, j_k\}$ then we get

$$\Phi(z_i)(z(j_1, \dots, j_k)) = z_i \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m = \pm z(j_1, \dots, i, \dots, j_k).$$

For any multi-index $I = \{i_1, \dots, i_s\}$ we write

$$z_I := z_{i_1} \cdot \ldots \cdot z_{i_s}, \quad \bar{z}_I := \bar{z}_{i_1} \cdot \ldots \cdot \bar{z}_{i_s}, \quad z(I) := z(i_1, \dots, i_s)$$

and we denote by I^c the complementary multi-index of I. Let now I and K be multi-indices. The calculations in a) - d) show that for all multi-indices J we have

$$z_1 \cdot \ldots \cdot z_m \cdot z(J) = \begin{cases} 0 & \text{if } J \neq \emptyset \\ \pm z_1 \cdot \ldots \cdot z_m & \text{if } J = \emptyset \end{cases}$$

and thus

$$z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I \cdot z(J) = \begin{cases} 0 & \text{if } J \neq I \\ \pm z_1 \cdot \ldots \cdot z_m & \text{if } J = I \end{cases}$$

and therefore

$$\bar{z}_{K^c} \cdot z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I \cdot z(J) = \begin{cases} 0 & \text{if } J \neq I \\ \pm z(K) & \text{if } J = I. \end{cases}$$

Thus every endomorphism of Σ_n can be obtained by composing endomorphisms of the form $\Phi(\bar{z}_{K^c} \cdot z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I)$. This shows that Φ is surjective. Since $\mathbb{C}\mathrm{l}_n$ and $\mathrm{End}(\Sigma_n)$ have the same dimension we conclude that Φ is an isomorphism. $\square$

The odd dimensional case

In the following, let $n = 2m-1$. To construct the spinor space Σ_n, we make the following observation:

Lemma 2.3.12. *Let $n \in \mathbb{N}$. The linear map $j : \mathbb{R}^n \to \mathrm{Cl}^0_{n+1}$,*

$$X \mapsto j(X) := X \cdot e_{n+1},$$

induces an algebra isomorphism $\mathrm{Cl}_n \to \mathrm{Cl}^0_{n+1}$.

Remark 2.3.13. Lemma 2.3.12 also holds for $\mathbb{C}\mathrm{l}_n$ instead of Cl_n.

Proof. We see immediately that the map j satisfies

$$j(X)^2 = X \cdot e_{n+1} \cdot X \cdot e_{n+1} = -X \cdot X \cdot e_{n+1} \cdot e_{n+1} = -|X|^2.$$

Thus, the universal property for Cl_n yields an algebra homomorphism $\alpha : \mathrm{Cl}_n \to \mathrm{Cl}^0_{n+1}$ such that the alongside diagram commutes.

$$\begin{tikzcd} & \mathrm{Cl}_n \arrow[dd, "\alpha"] \\ \mathbb{R}^n \arrow[ur, "\imath"] \arrow[dr, "j"'] & \\ & \mathrm{Cl}^0_{n+1} \end{tikzcd}$$

We first show that α is surjective. The elements

$$e_{i_1} \cdot \ldots \cdot e_{i_{2k}}, \quad 1 \le i_1 < \ldots < i_{2k} \le n+1$$

form a vector space basis of Cl^0_{n+1}.

a) First we assume that $i_{2k} \le n$. Then we have $e_{i_1} \cdot \ldots \cdot e_{i_{2k}} \in \mathrm{Cl}_n$ and

$$\begin{aligned}
\alpha(e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_{2k-1}} \cdot e_{i_{2k}}) &= \alpha(e_{i_1}) \cdot \alpha(e_{i_2}) \cdot \ldots \cdot \alpha(e_{i_{2k-1}}) \cdot \alpha(e_{i_{2k}}) \\
&= j(e_{i_1}) \cdot j(e_{i_2}) \cdot \ldots \cdot j(e_{i_{2k-1}}) \cdot j(e_{i_{2k}}) \\
&= \underbrace{e_{i_1} \cdot e_{n+1} \cdot e_{i_2} \cdot e_{n+1}}_{=e_{i_1} \cdot e_{i_2}} \cdot \ldots \cdot \underbrace{e_{i_{2k-1}} \cdot e_{n+1} \cdot e_{i_{2k}} \cdot e_{n+1}}_{=e_{i_{2k-1}} \cdot e_{i_{2k}}} \\
&= e_{i_1} \cdot \ldots \cdot e_{i_{2k}}
\end{aligned} \tag{2.14}$$

b) Now we assume that $i_{2k} = n+1$. Then we have $e_{i_1} \cdot \ldots \cdot e_{i_{2k-1}} \in \mathrm{Cl}_n$ and

$$\begin{aligned}
\alpha(e_{i_1} \cdot \ldots \cdot e_{i_{2k-1}}) &= \alpha(e_{i_1}) \cdot \ldots \cdot \alpha(e_{i_{2k-1}}) \\
&= j(e_{i_1}) \cdot \ldots \cdot j(e_{i_{2k-1}}) \\
&= (e_{i_1} \cdot e_{n+1} \cdot \ldots \cdot e_{i_{2k-2}} \cdot e_{n+1}) \cdot e_{i_{2k-1}} \cdot e_{n+1} \\
&\overset{(2.14)}{=} (e_{i_1} \cdot \ldots \cdot e_{i_{2k-2}}) \cdot e_{i_{2k-1}} \cdot e_{n+1} \\
&= e_{i_1} \cdot \ldots \cdot e_{i_{2k-1}} \cdot e_{n+1} \\
&= e_{i_1} \cdot \ldots \cdot e_{i_{2k}}.
\end{aligned}$$

Thus, the map α is surjective.

Since $\dim \mathrm{Cl}^0_{n+1} = \frac{2^{n+1}}{2} = 2^n = \dim \mathrm{Cl}_n$, the map α is an isomorphism. □

For n odd we define the spinor space Σ_n by:

$$\Sigma_n := \Sigma^+_{n+1}.$$

In particular, we have $\dim \Sigma_n = 2^{\lfloor \frac{n}{2} \rfloor}$ for both even and odd n. The Clifford algebra Cl_n acts on the spinor space Σ_n via the map α: For $X \in \mathrm{Cl}_n$ and $\phi \in \Sigma_n$ put

$$X \bullet \phi := \alpha(X) \cdot \phi \in \Sigma^+_{n+1} = \Sigma_n.$$

The restriction of this action to $\mathrm{Spin}(n) \subset \mathrm{Cl}_n \subset \mathbb{C}\mathrm{l}_n$ defines the **spinor representation** $\sigma_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$ in odd dimensions.

Lemma 2.3.14. *Let n be odd. Then the Clifford multiplication of spinors by vectors is skew-symmetric, i.e., for any vector $X \in \mathbb{R}^n$ and any two spinors $\phi, \psi \in \Sigma_n$, we have:*

$$\langle X \bullet \phi, \psi \rangle = -\langle \phi, X \bullet \psi \rangle . \tag{2.15}$$

Proof. We compute:

$$\begin{aligned}
\langle X \bullet \phi, \psi\rangle &= \langle \alpha(X)\cdot\phi,\psi\rangle \\
&= \langle X \cdot e_{n+1}\cdot\phi,\psi\rangle \\
&\overset{(2.10)}{=} -\langle e_{n+1}\cdot\phi, X\cdot\psi\rangle \\
&\overset{(2.10)}{=} -\langle \phi, X\cdot e_{n+1}\cdot\psi\rangle \\
&= -\langle \phi, \alpha(X)\cdot\psi\rangle \\
&= -\langle \phi, X\bullet\psi\rangle.
\end{aligned}$$

□

Remark 2.3.15. As in Remark 2.3.10 one concludes that in odd dimensions Clifford multiplication by unit vectors is an isometry and thus the spinor representation is unitary.

Example 2.3.16. Let $n = 1$. Then we have

$$\Sigma_1 = \Sigma_2^+ = \mathbb{C}\cdot z() \cong \mathbb{C}$$

and

$$e_1 \bullet z(\,) = e_1\cdot e_2\cdot z(\,) = e_1\cdot i\,z(1) = -i\,z(\,).$$

Example 2.3.17. Let $n = 3$. Then we have

$$\Sigma_3 = \Sigma_4^+ = \mathbb{C}\cdot z(\,)\oplus\mathbb{C}\cdot z(1,2) \cong \mathbb{C}^2.$$

By the equations (2.5) and (2.6), we have

$$\begin{aligned}
& e_1\bullet z(\,) = e_1\cdot e_4\cdot z(\,) \overset{(2.5)}{=} e_1\cdot i\,z(2) \overset{(2.6)}{=} i\,z(1,2) \\
\text{and}\qquad & e_1\bullet z(1,2) = e_1\bullet(-i)e_1\bullet z(\,) = i\,z(\,)\,.
\end{aligned}$$

From this (and similar computations for e_2, e_3), we thus have:

$$\begin{aligned}
e_1 \text{ acts on } \Sigma_3 \cong \mathbb{C}^2 \text{ as } &\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
e_2 \text{ acts on } \Sigma_3 \cong \mathbb{C}^2 \text{ as } &\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
e_3 \text{ acts on } \Sigma_3 \cong \mathbb{C}^2 \text{ as } &\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}
\end{aligned}$$

2.4. Spin Structures

Let M be an n-dimensional oriented Riemannian manifold. For $x \in M$ put

$$P_x^{\mathrm{SO}}(M) := \big\{h : \mathbb{R}^n \to T_xM \,|\, h \text{ orientation preserving isometry}\big\}.$$

Each element $h \in P^{\mathrm{SO}}_x(M)$ induces an oriented orthonormal basis $h(e_1), \dots, h(e_n)$ of T_xM. Conversely, for any oriented orthonormal basis $b_1, \dots, b_n$ of T_xM, there is a unique $h \in P^{\mathrm{SO}}_x(M)$ such that $b_1 = h(e_1), \dots, b_n = h(e_n)$.
The special orthogonal group $\mathrm{SO}(n)$ acts on $P^{\mathrm{SO}}_x(M)$ from the right:

$$\begin{aligned} P^{\mathrm{SO}}_x(M) \times \mathrm{SO}(n) &\to P^{\mathrm{SO}}_x(M) \\ (h, A) &\mapsto h \circ A. \end{aligned}$$

This action is simply transitive, i.e., for any two elements h_1, $h_2 \in P^{\mathrm{SO}}_x(M)$, there is a unique $A \in \mathrm{SO}(n)$ such that $h_2 = h_1 \circ A$ (namely, $A = h_1^{-1} \circ h_2$).
Thus, for a fixed $h_0 \in P^{\mathrm{SO}}_x(M)$, the map

$$\begin{aligned} \mathrm{SO}(n) &\to P^{\mathrm{SO}}_x(M) \\ A &\mapsto h_0 \circ A \end{aligned}$$

is bijective.

Now put

$$P^{\mathrm{SO}}(M) := \bigsqcup_{x \in M} P^{\mathrm{SO}}_x(M).$$

Let $\pi : P^{\mathrm{SO}}(M) \to M$ be such that $\pi^{-1}(x) = P^{\mathrm{SO}}_x(M)$, thus $\pi(h) = x$, where $h : \mathbb{R}^n \to T_xM$. There is a canonical smooth structure on $P^{\mathrm{SO}}(M)$ such that the projection map $\pi : P^{\mathrm{SO}}(M) \to M$ and the group action $P^{\mathrm{SO}}(M) \times \mathrm{SO}(n) \to P^{\mathrm{SO}}(M)$ are smooth maps. In other words: $(P^{\mathrm{SO}}(M), \pi, M; \mathrm{SO}(n))$ is an $\mathrm{SO}(n)$-principal bundle over M.

Definition 2.4.1. Let M be an n-dimensional oriented Riemannian manifold. The $\mathrm{SO}(n)$-principal bundle $P^{\mathrm{SO}}(M)$ is called the **(oriented orthonormal) frame bundle** of M.

Using a representation $\lambda : \mathrm{SO}(n) \to \mathrm{GL}(V)$, we can construct a vector bundle by glueing the vector space V onto the fibers of the frame bundle:

Definition 2.4.2. Let $\lambda : \mathrm{SO}(n) \to \mathrm{GL}(V)$ be a representation of $\mathrm{SO}(n)$ on a $\mathbb{K}$-vector space V, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The **associated vector bundle** to $P^{\mathrm{SO}}(M)$ is defined as:

$$P^{\mathrm{SO}}(M) \times_\lambda V := P^{\mathrm{SO}}(M) \times V/_\sim .$$

Here, the equivalence relation $\sim$ is defined as:

$$(h_1, v_1) \sim (h_2, v_2) :\Longleftrightarrow \exists\, A \in \mathrm{SO}(n) : h_2 = h_1 \circ A \text{ and } v_2 = \lambda\big(A^{-1}\big) v_1.$$

We denote the equivalence class of (h, v) in $P^{\mathrm{SO}}(M) \times_\lambda V$ by $[\![h, v]\!]$.

The induced projection $\widetilde{\pi} : P^{\mathrm{SO}}(M) \times_\lambda V \to M$, given as

$$\widetilde{\pi}\big([\![h, v]\!]\big) := \pi(h)\,,$$

is well-defined: If $(\bar{h}, \bar{v}) \sim (h, v)$, that is, $\bar{h} = h \circ A$ and $\bar{v} = \lambda(A^{-1})v$ for some $A \in \mathrm{SO}(n)$ then we have:

$$\widetilde{\pi}([\![\bar{h}, \bar{v}]\!]) = \widetilde{\pi}\big(\big[\!\!\big[h \circ A, \lambda\big(A^{-1}\big)v\big]\!\!\big]\big) = \pi(h \circ A) = \pi(h).$$

The vector space structure on the fibers of $P^{\mathrm{SO}}(M) \times_\lambda V$ is defined by:

$$\alpha \cdot [\![h, v_1]\!] + \beta \cdot [\![h, v_2]\!] := [\![h, \alpha\, v_1 + \beta\, v_2]\!],$$

where $[\![h, v_1]\!], [\![h, v_2]\!] \in \widetilde{\pi}^{-1}(x)$ and $\alpha, \beta \in \mathbb{K}$. This operation is well-defined:

$$\begin{aligned}
\alpha \cdot \big[\!\!\big[h \circ A, \lambda\big(A^{-1}\big)v_1\big]\!\!\big] + \beta \cdot \big[\!\!\big[h \circ A, \lambda\big(A^{-1}\big)v_2\big]\!\!\big]& \\
= \big[\!\!\big[h \circ A, \lambda\big(A^{-1}\big)v_1 + \beta\, \lambda\big(A^{-1}\big)v_2\big]\!\!\big]& \\
= \big[\!\!\big[h \circ A, \lambda\big(A^{-1}\big)(\alpha\, v_1 + \beta\, v_2)\big]\!\!\big]& \\
= \big[\!\!\big[h, \alpha\, v_1 + \beta\, v_2\big]\!\!\big]& \\
= \alpha \cdot [\![h, v_1]\!] + \beta \cdot [\![h, v_2]\!].&
\end{aligned}$$

For the standard representation $\lambda : \mathrm{SO}(n) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$, the map

$$\begin{array}{rcl}
P^{\mathrm{SO}}(M) \times_\lambda \mathbb{R}^n & \overset{\cong}{\to} & TM \\
[\![h, v]\!] & \mapsto & h(v)
\end{array}$$

is an isomorphism of vector bundles.

Similarly, we have the following canonical isomorphisms of vector bundles, associated to representations of $\mathrm{SO}(n)$:

vector space V	representation λ of $\mathrm{SO}(n)$ on V	$P^{\mathrm{SO}}(M) \times_\lambda V$
$\mathbb{R}$	id_V	$M \times \mathbb{R}$
$\mathbb{R}^n$	standard representation	TM
$\big(\mathbb{R}^n\big)^*$	dual of standard representation	T^*M
$\Lambda^k\big(\mathbb{R}^n\big)^*$	Λ^k(dual of standard representation)	$\Lambda^k T^*M$
$(\otimes^k \mathbb{R}^n) \otimes (\otimes^l (\mathbb{R}^n)^*)$	$(\otimes^k$(std rep.)$)\otimes$ $(\otimes^l$(dual of std rep.)$)$	$(\otimes^k TM) \otimes (\otimes^l T^*M)$

Now we want to construct a $\mathrm{Spin}(n)$-principal bundle $P^{\mathrm{Spin}}(M)$ such that for any representation $\lambda : \mathrm{Spin}(n) \to \mathrm{GL}(V)$ of the form $\lambda = \lambda' \circ \varrho$, with $\lambda' : \mathrm{SO}(n) \to \mathrm{GL}(V)$, the associated vector bundles $P^{\mathrm{Spin}}(M) \times_\lambda V$ and $P^{\mathrm{SO}}(M) \times_{\lambda'} V$ coincide.

Definition 2.4.3. Let M be an oriented Riemannian manifold. A **spin structure** on M is a pair $(P^{\mathrm{Spin}}(M), \bar{\varrho})$, consisting of

a) a $\mathrm{Spin}(n)$-principal bundle $P^{\mathrm{Spin}}(M)$ over M and

b) a twofold covering $\bar{\varrho} : P^{\mathrm{Spin}}(M) \to P^{\mathrm{SO}}(M)$ such that the diagram

$$\begin{tikzcd} P^{\mathrm{Spin}}(M) \times \mathrm{Spin}(n) \arrow[r] \arrow[d, "\bar{\varrho}\times\varrho"'] & P^{\mathrm{Spin}}(M) \arrow[d, "\bar{\varrho}"'] \arrow[dr] & \\ & & M \\ P^{\mathrm{SO}}(M) \times \mathrm{SO}(n) \arrow[r] & P^{\mathrm{SO}}(M) \arrow[ur] & \end{tikzcd}$$

commutes. Here $\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the twofold covering of $\mathrm{SO}(n)$ defined in equation (2.3). The horizontal maps are the group operations on the principal bundles.

If $\lambda = \lambda' \circ \varrho : \mathrm{Spin}(n) \to \mathrm{GL}(V)$ is a representation induced by $\lambda' : \mathrm{SO}(n) \to \mathrm{GL}(V)$ then

$$\begin{array}{ccc} P^{\mathrm{Spin}}(M) \times_\lambda V & \xrightarrow{\simeq} & P^{\mathrm{SO}}(M) \times_{\lambda'} V \\ [\![H, v]\!] & \longmapsto & [\![\bar{\varrho}(H), v]\!] \end{array}$$

is well-defined: For any $a \in \mathrm{Spin}(n)$, we have:

$$[\![H \cdot a, \lambda\big(a^{-1}\big)v]\!] \longmapsto [\![\bar{\varrho}(Ha), \lambda\big(a^{-1}\big)v]\!] = [\![\bar{\varrho}(H)\varrho(a), \lambda'\big(\varrho(a)^{-1}\big)v]\!] = [\![\bar{\varrho}(H), v]\!] .$$

Obviously, this map is an isomorphism of vector bundles.

Definition 2.4.4. An oriented Riemannian manifold equipped with a spin structure is called a Riemannian **spin manifold**.
An oriented Riemannian manifold is called **spinnable** if it admits a spin structure.

A detailed discussion of existence and uniqueness of spin structures on oriented Riemannian manifolds can be found in Chapter II of the book [7] by Lawson and Michelsohn.

Definition 2.4.5. Two spin structures $P_1^{\mathrm{Spin}}(M)$ and $P_2^{\mathrm{Spin}}(M)$ are called **equiva-**

lent, if there exists a diffeomorphism

$$\phi : P_1^{\mathrm{Spin}}(M) \longrightarrow P_2^{\mathrm{Spin}}(M),$$

such that the diagram

$$\begin{tikzcd}
P_1^{\mathrm{Spin}}(M) \times \mathrm{Spin}(n) \arrow[rr, "\phi\times\mathrm{Id}"] \arrow[d] & & P_2^{\mathrm{Spin}}(M) \times \mathrm{Spin}(n) \arrow[d] \\
P_1^{\mathrm{Spin}}(M) \arrow[rr, "\phi"] \arrow[dr, "\bar{\varrho}_1"'] & & P_2^{\mathrm{Spin}}(M) \arrow[dl, "\bar{\varrho}_2"] \\
& P^{\mathrm{SO}}(M) &
\end{tikzcd}$$

commutes.

Example 2.4.6. For $M = S^1$, we have $\mathrm{SO}(1) = \{1\}$, thus $\mathrm{Spin}(1) = \mathbb{Z}_2$ and $P^{\mathrm{SO}}(S^1) = S^1$. A spin structure on S^1 is thus a two-fold covering of S^1. There are two possibilites:

1) The **trivial** spin structure on S^1 is the trivial covering

$$P_{\mathrm{triv}}^{\mathrm{Spin}}(S^1) = S^1 \sqcup S^1 = S^1 \times \mathbb{Z}_2.$$

Let $\bar{\varrho} : P^{\mathrm{Spin}}(S^1) = S^1 \times \mathbb{Z}_2 \to P^{\mathrm{SO}}(S^1) = S^1$ be the projection on the first factor.

To see that this defines a spin structure on S^1, we need to check that the diagram

$$\begin{tikzcd}
S^1 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mathrm{pr}_1\times\varrho"'] & S^1 \times \mathbb{Z}_2 \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_1"] & \\
& & S^1 \\
S^1 \times \{1\} \arrow[r, "\mathrm{pr}_1"] & S^1 \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

as in Definition 2.4.3 commutes. But this is obvious.

2) The **non-trivial** spin structure on S^1 is the non-trivial covering $P_{\mathrm{non-triv}}^{\mathrm{Spin}}(S^1) = S^1$ with $\bar{\varrho} : S^1 \to S^1$, $z \mapsto z^2$. The action of $\mathrm{Spin}(1) = \mathbb{Z}_2$ on $P_{\mathrm{non-triv}}^{\mathrm{Spin}}(S^1)$ is given by $z \mapsto -z$.

This defines a spin structure on S^1, since the diagram

$$\begin{tikzcd}
S^1 \times \mathbb{Z}_2 \arrow[r, "(z,\varepsilon)\mapsto\varepsilon z"] \arrow[dd, "(z,\varepsilon)\mapsto(z^2,1)"'] & S^1 \arrow[dr, "z\mapsto z^2"] \arrow[dd, "z\mapsto z^2"'] & \\
 & & S^1 \\
S^1 \times \{1\} \arrow[r, "\mathrm{pr}_1"] & S^1 \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd}$$

commutes.

Obviously, the spin structures $P^{\mathrm{Spin}}_{\mathrm{triv}}(S^1)$ and $P^{\mathrm{Spin}}_{\mathrm{non-triv}}(S^1)$ are not equivalent, since the total spaces are not even diffeomorphic: $P^{\mathrm{Spin}}_{\mathrm{non-triv}}(S^1)$ is connected whereas $P^{\mathrm{Spin}}_{\mathrm{triv}}(S^1)$ is not.

Definition 2.4.7. Let $\sigma_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$ be the spinor representation. Let M be a Riemannian spin manifold of dimension n with a spin structure $P^{\mathrm{Spin}}(M)$. The **spinor bundle** of M for the spin structure $P^{\mathrm{Spin}}(M)$ is the associated vector bundle

$$\boldsymbol{\Sigma M} := P^{\mathrm{Spin}}(M) \times_{\sigma_n} \Sigma_n.$$

Sections of ΣM are called **spinor fields** on M.
If n is even and $\sigma_n^{\pm} : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n^{\pm})$ are the positive and negative spinor representation respectively, then the vector bundles

$$\boldsymbol{\Sigma^{\pm} M} := P^{\mathrm{Spin}}(M) \times_{\sigma_n^{\pm}} \Sigma_n^{\pm}$$

are called the **positive** and the **negative spinor bundle** of M for the spin structure $P^{\mathrm{Spin}}(M)$ respectively.

If n is even we have the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$.

Example 2.4.8
1) For the trivial spin structure $P^{\mathrm{Spin}}_{\mathrm{triv}}(S^1)$ of S^1 we get

$$\Sigma_{\mathrm{triv}} S^1 = P^{\mathrm{Spin}}_{\mathrm{triv}}\big(S^1\big) \times_{\sigma_1} \Sigma_1 = \big(S^1 \sqcup S^1\big) \times \Sigma_1/_\sim \cong S^1 \times \Sigma_1.$$

The action of $\mathrm{Spin}(1) = \mathbb{Z}_2$ identifies the two copies of $S^1 \times \Sigma_1$. Thus the identification of $\big(S^1 \sqcup S^1\big) \times \Sigma_1/_\sim \cong S^1 \times \Sigma_1$ is by projection onto the first factor, or by the embedding

$$S^1 \times \Sigma_1 \hookrightarrow \big(S^1 \times \Sigma_1\big) \sqcup \big(S^1 \times \Sigma_1\big) = \big(S^1 \times \mathbb{Z}_2\big) \times \Sigma_1,$$

into the first factor, respectively.

Since the spinor bundle is trivial, spinor fields correspond to $\Sigma_1 = \mathbb{C}$-valued functions on S^1 or periodic $\mathbb{C}$-valued functions on $\mathbb{R}$. As a convention, we choose the period 1, i.e. , we use $\exp : \mathbb{R} \to S^1$, $\exp(t) = e^{2\pi i\, t}$ for the identification.

2) For the non-trivial spin structure $P^{\mathrm{Spin}}_{\mathrm{non-triv}}(S^1)$ of S^1 we have the diagram

$$\begin{tikzcd} \mathbb{R} \times \Sigma_1 \arrow[r, "\mathrm{EXP}"] \arrow[d, "\mathrm{pr}"] & P^{\mathrm{Spin}}_{\mathrm{non-triv}}(S^1) \times_{\sigma_1} \Sigma_1 \arrow[d, "{[z,v] \mapsto z^2}"] \\ \mathbb{R} \arrow[r, "\exp"] & S^1 \end{tikzcd}$$

where $\exp(t) = e^{2\pi i\, t}$ and $\mathrm{EXP}(t, \varphi) = [\![e^{\pi i\, t}, \varphi]\!]$.

Spinor fields correspond to functions $f : \mathbb{R} \to \mathbb{C} = \Sigma_1$, satisfying

$$f(t+k) = (-1)^k f(t), \qquad \text{for all } t \in \mathbb{R}, k \in \mathbb{Z}. \tag{2.16}$$

Functions satisfying (2.16) will be called **$\mathbb{Z}$-anti-periodic**.

The spinor field corresponding to the function f is defined as

$$t \mapsto \mathrm{EXP}\big(t, f(t)\big) = [\![e^{\pi i\, t}, f(t)]\!].$$

The periodicity condition (2.16) guarantees that this mapping descends to S^1: for any $e^{\pi i\, k} \in \mathrm{Spin}(1)$, we have

$$\begin{aligned} \big[\!\big[e^{\pi i\,(t+k)}, f(t+k) \big]\!\big] &= \big[\!\big[e^{\pi i\, t} \cdot e^{\pi i\, k}, (-1)^k f(t) \big]\!\big] \\ &= \big[\!\big[e^{\pi i\, t} \cdot e^{\pi i\, k}, \sigma_1(e^{-\pi i\, k}) f(t) \big]\!\big] \\ &= \big[\!\big[e^{\pi i\, t}, f(t) \big]\!\big]. \end{aligned}$$

Now let M be a Riemannian spin manifold with spin structure $P^{\mathrm{Spin}}(M)$. The spinor bundle ΣM carries a canonical Hermitian bundle metric defined as

$$\big\langle [\![H, \varphi]\!], [\![H, \psi]\!] \big\rangle := \langle \varphi, \psi \rangle, \qquad \text{for } H \in P^{\mathrm{Spin}}(M), \varphi, \psi \in \Sigma_n.$$

This assignment is well-defined, since for any $a \in \mathrm{Spin}(n)$, we have:

$$\big\langle \big[\!\big[H \cdot a, \sigma_n(a^{-1})\varphi \big]\!\big], \big[\!\big[H \cdot a, \sigma_n(a^{-1})\psi \big]\!\big] \big\rangle = \big\langle \sigma_n(a^{-1})\varphi, \sigma_n(a^{-1})\psi \big\rangle = \langle \varphi, \psi \rangle.$$

Clifford multiplication

In the following definition of the Clifford multiplication, we use the fact that for an oriented Riemannian manifold, the tangent bundle TM is isomorphic to the vector bundle associated to $P^{\mathrm{SO}}(M)$ via the standard representation λ_{st} of $\mathrm{SO}(n)$ on $\mathbb{R}^n$.

Definition 2.4.9. Let M be a Riemannian spin manifold with spinor bundle ΣM, and let $x \in M$. The map

$$\begin{aligned} \cdot : T_xM \times \Sigma_x M &\longrightarrow \Sigma_x M \\ (\llbracket \bar{\varrho}(H), X \rrbracket, \llbracket H, \varphi \rrbracket) &\longmapsto \llbracket H, X \cdot \varphi \rrbracket, \end{aligned} \tag{2.17}$$

is called **Clifford multiplication**.

Remark 2.4.10. (i) The Clifford multiplication is well defined: for any $a \in \mathrm{Spin}(n)$, we have by equation (2.3):

$$\begin{aligned} \llbracket \bar{\varrho}(H \cdot a), \lambda_{\mathrm{st}}(\varrho(a^{-1}))X \rrbracket \cdot \llbracket H \cdot a, \sigma_n(a^{-1})\varphi \rrbracket &= \llbracket H \cdot a, \lambda_{\mathrm{st}}(\varrho(a^{-1}))X \cdot \sigma_n(a^{-1})\varphi \rrbracket \\ &= \llbracket H \cdot a, a^{-1} \cdot X \cdot a \cdot a^{-1} \cdot \varphi \rrbracket \\ &= \llbracket H \cdot a, \sigma_n(a^{-1})(X \cdot \varphi) \rrbracket \\ &= \llbracket H, X \cdot \varphi \rrbracket. \end{aligned}$$

(ii) The Clifford multiplication satisfies the Clifford relation:

$$\begin{aligned} &\llbracket \bar{\varrho}(H), X \rrbracket \cdot \Big(\llbracket \bar{\varrho}(H), Y \rrbracket \cdot \llbracket H, \varphi \rrbracket \Big) + \llbracket \bar{\varrho}(H), Y \rrbracket \cdot \Big(\llbracket \bar{\varrho}(H), X \rrbracket \cdot \llbracket H, \varphi \rrbracket \Big) \\ &\qquad = \llbracket H, X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi \rrbracket \\ &\qquad = \llbracket H, -2\langle X, Y \rangle \varphi \rrbracket \\ &\qquad = -2 \, \langle X, Y \rangle \llbracket H, \varphi \rrbracket \\ &\qquad = -2 \, \Big\langle \llbracket \bar{\varrho}(H), X \rrbracket, \llbracket \bar{\varrho}(H), Y \rrbracket \Big\rangle \llbracket H, \varphi \rrbracket \end{aligned}$$

Upon writing $X' := \llbracket \bar{\varrho}(H), X \rrbracket$, $Y' := \llbracket \bar{\varrho}(H), Y \rrbracket$ and $\phi := \llbracket H, \varphi \rrbracket$, the Clifford relation reads

$$X' \cdot Y' \cdot \phi + Y' \cdot X' \cdot \phi = -2 \big\langle X', Y' \big\rangle \phi.$$

(iii) The Clifford multiplication is bilinear.

(iv) The Clifford multiplication is skew-symmetric: for any tangent vectors $X' \in T_xM$ and spinors $\phi, \psi \in \Sigma_x M$, we have

$$\big\langle X' \cdot \phi, \psi \big\rangle = -\big\langle \phi, X' \cdot \psi \big\rangle.$$

The spinor connection

As above, let M be a Riemannian spin manifold. The Levi-Civita connection ∇ on TM induces a connection 1-form $\omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{SO}}(M), \mathfrak{so}(n))$. By pull-back with $\bar{\varrho}$, we obtain an $\mathfrak{so}(n)$-valued 1-form $\bar{\varrho}^* \omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{Spin}}(M), \mathfrak{so}(n))$. Applying the isomorphism $\varrho_*^{-1} : \mathfrak{so}(n) \to \mathfrak{spin}(n)$ yields the connection 1-form

$$\widetilde{\omega}^{\mathrm{LC}} := \varrho_*^{-1} \bar{\varrho}^* \omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{Spin}}(M), \mathfrak{spin}(n))$$

and a corresponding **spinor connection** ∇^Σ on ΣM. The covariant derivative with respect to ∇^Σ of a local section $[\![H,\varphi]\!] \in C^\infty(U,\Sigma M)$ is given by:

$$\nabla^\Sigma_X [\![H,\varphi]\!] = [\![H, \partial_X\varphi + (\sigma_n)_*(\widetilde{\omega}^{\mathrm{LC}}(dH(X))) \cdot \varphi]\!]. \tag{2.18}$$

Here $U \subset M$ is an open subset, $x \in U$, $X \in T_xM$, and $H : U \to P^{\mathrm{Spin}}(M)$ is a local smooth section, and $\varphi : U \to \Sigma_n$ a smooth function.
In order to write the spinor connection in terms of Christoffel symbols, we fix a local smooth section $H : U \to P^{\mathrm{Spin}}(M)$. Then $h := \bar{\varrho} \circ H : U \to P^{\mathrm{SO}}(M)$ is a smooth local oriented orthonormal tangent frame and the vector fields

$$b_1 := h(e_1), \ldots, b_n := h(e_n)$$

form an oriented orthonormal basis of T_xM at each $x \in U$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. The Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ of this orthonormal frame are defined by the equation

$$\nabla^{\mathrm{LC}}_{b_i} b_j = \sum_{k=1}^n \Gamma^k_{ij}\, b_k \quad \text{for all } i,j \in \{1,\ldots,n\}.$$

Note that unlike the Christoffel symbols of a local coordinate system the Γ^k_{ij} are in general not symmetric in i,j. Instead we have $\Gamma^k_{ij} = -\Gamma^j_{ik}$ for all i,j,k. We compute the covariant derivative of $b_j = [\![h,e_j]\!]$ in terms of the connection 1-form ω^{LC}:

$$\begin{aligned}
\left[\!\!\left[h, \sum_{k=1}^n \Gamma^k_{ij}\, e_k\right]\!\!\right] &= \nabla^{\mathrm{LC}}_{b_i} b_j \\
&= \nabla^{\mathrm{LC}}_{b_i} [\![h, e_j]\!] \\
&= \big[\!\big[h, \underbrace{\partial_{b_i} e_j}_{=0} + \lambda_*\big(\omega^{\mathrm{LC}}(dh(b_i))\big) e_j\big]\!\big] \\
&= \left[\!\!\left[h, \sum_{k=1}^n \omega^{\mathrm{LC}}_{kj}(dh(b_i)) e_k\right]\!\!\right].
\end{aligned}$$

Hence

$$\Gamma^k_{ij} = \omega^{\mathrm{LC}}_{kj}(dh(b_i)). \tag{2.19}$$

For the local section $H : U \to P^{\mathrm{Spin}}(M)$ with $\bar{\varrho} \circ H = h$, we then have:

$$\bar{\varrho}^*\omega^{\mathrm{LC}}\big(dH(b_i)\big) = \omega^{\mathrm{LC}}\big(d\bar{\varrho} \circ dH(b_i)\big) = \omega^{\mathrm{LC}}\big(d(\bar{\varrho} \circ H)(b_i)\big) = \omega^{\mathrm{LC}}\big(dh(b_i)\big).$$

Upon writing

$$\varrho_*^{-1}\big(\bar{\varrho}^*\omega^{\mathrm{LC}}(dH(b_i))\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, e_\mu \cdot e_\nu \ \in \mathfrak{spin}(n), \tag{2.20}$$

we obtain

$$\omega^{\mathrm{LC}}\big(dh(b_i)\big) = \bar{\varrho}^*\omega^{\mathrm{LC}}\big(dH(b_i)\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, \varrho_*(e_\mu \cdot e_\nu).$$

We apply this to $e_j \in \mathbb{R}^n$ and obtain

$$\begin{aligned}
\omega^{\mathrm{LC}}\big(dh(b_i)\big)(e_j) &= \sum_{\mu<\nu} \gamma_{\mu\nu i}\, \varrho_*(e_\mu \cdot e_\nu)(e_j) \\
&= \sum_{\mu<\nu} \gamma_{\mu\nu i} \begin{cases} 2\, e_\nu, & j = \mu \\ -2\, e_\mu, & j = \nu \\ 0 & \text{otherwise} \end{cases} \\
&= 2 \sum_{\nu>j} \gamma_{j\nu i}\, e_\nu - 2 \sum_{\mu<j} \gamma_{\mu j i}\, e_\mu.
\end{aligned}$$

Comparing the coefficients with equation (2.19) yields

$$\Gamma_{ij}^k = \begin{cases} 2\, \gamma_{jki} & k > j \\ -2\, \gamma_{kji} & k < j \\ 0 & k = j \end{cases}.$$

Thus, we can replace the coefficients in (2.20) by Christoffel symbols and obtain:

$$\widetilde{\omega}^{\mathrm{LC}}\big(dH(b_i)\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, e_\mu \cdot e_\nu = \frac{1}{2} \sum_{\mu<\nu} \Gamma_{i\mu}^\nu\, e_\mu \cdot e_\nu = \frac{1}{4} \sum_{\mu,\nu=1}^{n} \Gamma_{i\mu}^\nu\, e_\mu \cdot e_\nu$$

The covariant derivative of a local section $[\![H, \varphi]\!] \in C^\infty(U, \Sigma M)$ can be written in terms of Christoffel symbols:

$$\begin{aligned}
\nabla_{b_i}^\Sigma [\![H, \varphi]\!] &= \big[\!\big[H, \partial_{b_i}\varphi + (\sigma_n)_*\big(\widetilde{\omega}^{\mathrm{LC}}(dH(b_i))\big) \cdot \varphi\big]\!\big] \\
&= \Big[\!\Big[H, \partial_{b_i}\varphi + \frac{1}{4} \sum_{j,k=1}^{n} \Gamma_{ij}^k\, e_j \cdot e_k \cdot \varphi\Big]\!\Big]. \qquad (2.21)
\end{aligned}$$

Remark 2.4.11

a) The spinor connection ∇^Σ is a metric connection on the spinor bundle ΣM. Hence for all smooth spinor fields ϕ, ψ and every tangent vector X, we have

$$\partial_X \langle \phi, \psi \rangle = \big\langle \nabla_X^\Sigma \phi, \psi \big\rangle + \big\langle \phi, \nabla_X^\Sigma \psi \big\rangle. \qquad (2.22)$$

This is a general fact: for a G-principal bundle $P \to M$ with connection 1-form $\widetilde{\omega}$ and a unitary representation $\lambda : G \to \mathrm{U}(V)$, the induced connection on $P \times_\lambda V$ preserves the induced metric: the covariant derivative of a local section $[\![H, \phi]\!]$ reads

$$\nabla_X [\![H, \phi]\!] = \big[\!\big[H, \partial_X \phi + \underbrace{\lambda_*\big(\widetilde{\omega}(dH(X))\big)}_{\in \mathfrak{u}(V)} \cdot \phi\big]\!\big]$$

and thus

$$
\begin{aligned}
&\big\langle \nabla_X [\![H,\phi]\!], [\![H,\psi]\!] \big\rangle + \big\langle [\![H,\phi]\!], \nabla_X [\![H,\psi]\!] \big\rangle \\
&= \big\langle \partial_X \phi + \lambda_*\big(\widetilde{\omega}(dH(X))\big) \cdot \phi, \psi \big\rangle + \big\langle \phi, \partial_X \psi + \lambda_*\big(\widetilde{\omega}(dH(X))\big) \cdot \psi \big\rangle \\
&= \langle \partial_X \phi, \psi \rangle + \langle \phi, \partial_X \psi \rangle \\
&= \partial_X \langle \phi, \psi \rangle \\
&= \partial_X \big\langle [\![H,\phi]\!], [\![H,\psi]\!] \big\rangle.
\end{aligned}
$$

b) On an even dimensional Riemannian spin manifold M, the spinor connection ∇^Σ preserves the chirality: for every vector field $X \in C^\infty(M, TM)$ and every spinor field $\phi \in C^\infty(M, \Sigma^\pm M)$, we have $\nabla^\Sigma_X \phi \in C^\infty(M, \Sigma^\pm M)$. This follows immediately from equation (2.21).

Now we prove a Leibniz rule for the Clifford multiplication:

Lemma 2.4.12. *Let M be a Riemannian spin manifold with spinor bundle ΣM and spinor connection ∇^Σ. Then for all vector fields $X, Y \in C^\infty(M, TM)$ and all spinor fields $\phi \in C^\infty(M, \Sigma M)$, we have*

$$
\nabla^\Sigma_X (Y \cdot \phi) = (\nabla^{\mathrm{LC}}_X Y) \cdot \phi + Y \cdot \nabla^\Sigma_X \phi. \tag{2.23}
$$

Proof. Fix $x \in M$ and let U be a neighborhood of x. Let $H : U \to P^{\mathrm{Spin}}(M)$ be a local section and $h = \bar{\varrho} \circ H : U \to P^{\mathrm{SO}}(M)$ be the corresponding local section of $P^{\mathrm{SO}}(M)$. Then the vector fields $b_1 := h(e_1), \dots, b_n := h(e_n)$ form an oriented orthonormal local frame of TM.
Since the spinor connection is tensorial in the vector fields, it suffices to prove (2.23) for $X = b_i$. We thus write $Y = [\![h, Y']\!]$ and $\phi = [\![H, \varphi]\!]$ on U, where $Y' : U \to \mathbb{R}^n$ and $\varphi : U \to \Sigma_n$. Now we compute:

$$
\begin{aligned}
\nabla^\Sigma_{b_i}(Y \cdot \phi) &= \nabla^\Sigma_{b_i} [\![H, Y' \cdot \varphi]\!] \\
&\overset{(2.21)}{=} \Big[\!\!\Big[H, \partial_{b_i}(Y' \cdot \varphi) + \frac{1}{4} \sum_{j,k=1}^n \Gamma^k_{ij}\, e_j \cdot e_k \cdot Y' \cdot \varphi \Big]\!\!\Big] \\
&= \Big[\!\!\Big[H, (\partial_{b_i} Y') \cdot \varphi + Y' \cdot \partial_{b_i} \varphi - \frac{1}{4} \sum_{j,k=1}^n \Gamma^k_{ij}\, e_j \cdot Y' \cdot e_k \cdot \varphi - \frac{1}{2} \sum_{j,k=1}^n \Gamma^k_{ij}\, e_j \langle e_k, Y' \rangle \cdot \varphi \Big]\!\!\Big] \\
&= \Big[\!\!\Big[H, (\partial_{b_i} Y') \cdot \varphi + Y' \cdot \partial_{b_i} \varphi + \frac{1}{4} \sum_{j,k=1}^n \Gamma^k_{ij}\, Y' \cdot e_j \cdot e_k \cdot \varphi + \frac{1}{2} \sum_{j,k=1}^n \Gamma^k_{ij} \langle e_j, Y' \rangle e_k \cdot \varphi \\
&\qquad - \frac{1}{2} \sum_{j,k=1}^n \Gamma^k_{ij}\, e_j \langle e_k, Y' \rangle \cdot \varphi \Big]\!\!\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\!\!\left[H, Y' \cdot \Big(\partial_{b_i} \varphi + \frac{1}{4} \sum_{j,k=1}^{n} \Gamma_{ij}^k \, e_j \cdot e_k \cdot \varphi \Big) \right]\!\!\right] \\
&\qquad + \left[\!\!\left[H, \Big(\partial_{b_i} Y' + \sum_{j,k=1}^{n} \langle Y', e_j \rangle \Gamma_{ij}^k \, e_k \Big) \cdot \varphi \right]\!\!\right] \\
&= Y \cdot \nabla^{\Sigma}_{b_i} \phi + \nabla^{\mathrm{LC}}_{b_i} Y \cdot \phi.
\end{aligned}
$$

□

In (2.21) we gave an expression of the spinor connection ∇^{Σ} in terms of the Christoffel symbols of the Levi-Civita connection ∇^{LC}. Now we want to do the same for the corresponding curvatures.

Let R^{Σ} be the curvature for ∇^{Σ}, i.e. the endomorphism field on ΣM, defined by

$$
R^{\Sigma}(X,Y)\phi := \nabla^{\Sigma}_X \nabla^{\Sigma}_Y \phi - \nabla^{\Sigma}_Y \nabla^{\Sigma}_X \phi - \nabla^{\Sigma}_{[X,Y]} \phi.
$$

Lemma 2.4.13. *Let M be a Riemannian spin manifold with spinor bundle ΣM, and let ∇^{LC} and ∇^{Σ} be the Levi-Civita connection and the corresponding spinor connection, respectively. Then the curvatures R^{Σ} of ∇^{Σ} and R of ∇^{LC} are related by*

$$
R^{\Sigma}(X,Y)\phi = -\frac{1}{4} \sum_{i=1}^{n} \big(R(X,Y) b_i \big) \cdot b_i \cdot \phi \,. \tag{2.24}
$$

Here $b_1, \ldots, b_n$ denotes an orthonormal basis of $T_x M$.

Proof. Fix $x \in M$ and let U be a neighborhood of x. Let $H : U \to P^{\mathrm{Spin}}(M)$ be a local section and $h = \bar{\varrho} \circ H : U \to P^{\mathrm{SO}}(M)$ be the corresponding local section of $P^{\mathrm{SO}}(M)$. The vector fields $b_1 := h(e_1), \ldots, b_n := h(e_n)$ form an oriented orthonormal local frame of TM, which we assume to be synchronous in x, i.e. $(\nabla^{\mathrm{LC}}_{b_i} b_j)(x) = 0$ for $i,j = 1, \ldots, n$. In particular, $\Gamma_{ij}^k(x) = 0$ and $[b_i, b_j](x) = 0$ for $i,j,k = 1, \ldots, n$.
For a local section $[\![H, \varphi]\!] \in C^{\infty}(U, \Sigma M)$, we compute:

$$
\begin{aligned}
\big(\nabla^{\Sigma}_{b_i} \nabla^{\Sigma}_{b_j} [\![H,\varphi]\!] \big)_{(x)} &= \Big(\nabla^{\Sigma}_{b_i} \left[\!\!\left[H, \partial_{b_j} \varphi + \frac{1}{4} \sum_{\alpha,\beta} \Gamma_{j\alpha}^{\beta} \, e_\alpha \cdot e_\beta \cdot \varphi \right]\!\!\right] \Big)_{(x)} \\
&= \left[\!\!\left[H, \partial_{b_i} \big(\partial_{b_j} \varphi + \frac{1}{4} \sum_{\alpha,\beta} \Gamma_{j\alpha}^{\beta} \, e_\alpha \cdot e_\beta \cdot \varphi \big) \right]\!\!\right]_{(x)} \\
&= \left[\!\!\left[H, \partial_{b_i} \partial_{b_j} \varphi + \frac{1}{4} \sum_{\alpha,\beta} \big(\partial_{b_i} \Gamma_{j\alpha}^{\beta} \big) \, e_\alpha \cdot e_\beta \cdot \varphi \right]\!\!\right]_{(x)}
\end{aligned}
$$

This yields for the curvature at x:

$$\begin{aligned}
\left(R^{\Sigma}(b_i,b_j)\phi\right)_{(x)} &= \left[\!\left[H, \partial_{b_i}\partial_{b_j}\varphi + \frac{1}{4}\sum_{\alpha,\beta}\left(\partial_{b_i}\Gamma^{\beta}_{j\alpha}\right) e_\alpha\cdot e_\beta\cdot\varphi \right]\!\right]_{(x)} \\
&\qquad - \left[\!\left[H, \partial_{b_j}\partial_{b_i}\varphi + \frac{1}{4}\sum_{\alpha,\beta}(\partial_{b_j}\Gamma^{\beta}_{i\alpha})\, e_\alpha\cdot e_\beta\cdot\varphi \right]\!\right]_{(x)} \\
&= \left[\!\left[H, \underbrace{(\partial_{b_i}\partial_{b_j} - \partial_{b_j}\partial_{b_i})\varphi}_{=\partial_{[b_i,b_j](x)}\varphi=0} + \frac{1}{4}\sum_{\alpha,\beta}\left(\partial_{b_i}\Gamma^{\beta}_{j\alpha} - \partial_{b_j}\Gamma^{\beta}_{i\alpha}\right) e_\alpha\cdot e_\beta\cdot\varphi \right]\!\right]_{(x)} \\
&= -\frac{1}{4}\left[\!\left[H, \sum_{\alpha,\beta}\left(\partial_{b_i}\Gamma^{\beta}_{j\alpha} - \partial_{b_j}\Gamma^{\beta}_{i\alpha}\right) e_\beta\cdot e_\alpha\cdot\varphi \right]\!\right]_{(x)}.
\end{aligned}$$

On the other hand we have $\nabla^{\mathrm{LC}}_{b_j} b_\alpha = \sum_\beta \Gamma^{\beta}_{j\alpha} b_\beta$ and thus at x:

$$R(b_i,b_j)b_\alpha = \sum_\beta \left(\partial_{b_i}\Gamma^{\beta}_{j\alpha} - \partial_{b_j}\Gamma^{\beta}_{i\alpha}\right) b_\beta.$$

We conclude that $(R^{\Sigma}(b_i,b_j)\phi)(x) = -\frac{1}{4}\sum_\alpha (R(b_i,b_j)b_\alpha)\cdot b_\alpha\cdot\phi(x)$. □

2.5. The classical Dirac operator on spinors

Let M be an n-dimensional Riemannian spin manifold. Clifford multiplication in the spinor bundle ΣM defines a smooth section $A \in C^\infty(M, \mathrm{Hom}(T^*M \otimes \Sigma M, \Sigma M))$ by $A(\xi\otimes\phi) = \xi^\sharp\cdot\phi$.

Definition 2.5.1. Let M be a Riemannian spin manifold with spinor bundle ΣM. The **classical Dirac operator** is the first order operator

$$D := D_{A,\nabla^\Sigma} \in \mathscr{D}\!\mathit{iff}_1(\Sigma M, \Sigma M)$$

as defined in (1.28) for $E = F = \Sigma M$ and A given by Clifford multiplication.

Recall from equation (1.28) that for a local orthonormal frame $b_1,\dots,b_n \in T_xM$, the operator D_{A,∇^Σ} is defined as

$$D_{A,\nabla^\Sigma} = \sum_i A\big(b_i^*\otimes\nabla^\Sigma_{b_i}\phi\big).$$

Thus, for the classical Dirac operator, we have:

$$D = D_{A,\nabla^\Sigma} = \sum_i \left(b_i^*\right)^\sharp\cdot\nabla^\Sigma_{b_i}\phi = \sum_i b_i\cdot\nabla^\Sigma_{b_i}\phi. \tag{2.25}$$

Remark 2.5.2. The classical Dirac operator is an operator of Dirac-type, as defined in Definition 1.3.7. Its principal symbol is given by

$$\sigma(D,\xi)\phi = \sigma(D_{A,\nabla^\Sigma},\xi)\phi \overset{(1.31)}{=} A(\xi\otimes\phi) = \xi^\sharp\cdot\phi.$$

By Lemmas 2.3.9 and 2.3.14, Clifford multiplication by tangent vectors is a skew-symmetric endomorphism of ΣM. Together with the Clifford relation (2.1), we conclude that D satisfies Definition 1.3.7.

Remark 2.5.3. On an even dimensional Riemannian spin manifold, the classical Dirac operator D interchanges chirality. With respect to the splitting $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, the operator takes the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix},$$

where $D^+ \in \mathscr{Diff}_1(\Sigma^+ M, \Sigma^- M)$ and $D^- \in \mathscr{Diff}_1(\Sigma^- M, \Sigma^+ M)$.

Definition 2.5.4. Let M be a Riemannian spin manifold, let D be the classical Dirac operator, and let C be a vector bundle on M with connection ∇^C. The operator

$$D^{\nabla^C} \in \mathscr{Diff}_1(\Sigma M\otimes C, \Sigma M\otimes C). \tag{2.26}$$

as in Definition 1.3.21 is called **twisted Dirac operator**.

Lemma 2.5.5. *The classical Dirac operator is formally self-adjoint.*

Proof. Let $\phi,\psi \in C_c^\infty(M,\Sigma M)$ be compactly supported spinor fields. Define $X \in C_c^\infty(M, TM\otimes\mathbb{C})$ by

$$\langle X, Y\rangle = \langle Y\cdot\phi,\psi\rangle \qquad \text{for all } Y\in TM.$$

Let $b_1,\dots,b_n$ be a local orthonormal frame. Then we have:

$$\begin{aligned}
\operatorname{div} X &= \sum_{i=1}^n \langle \nabla_{b_i} X, b_i\rangle \\
&= \sum_{i=1}^n \partial_{b_i}\langle X, b_i\rangle - \sum_i \langle X, \nabla_{b_i} b_i\rangle \\
&= \sum_{i=1}^n \partial_{b_i}\langle b_i\cdot\phi,\psi\rangle - \langle \nabla_{b_i} b_i\cdot\phi,\psi\rangle \\
&\overset{(2.22)}{=} \sum_{i=1}^n \langle \nabla^\Sigma_{b_i}(b_i\cdot\phi),\psi\rangle + \langle b_i\cdot\phi, \nabla^\Sigma_{b_i}\psi\rangle - \langle \nabla_{b_i} b_i\cdot\phi,\psi\rangle
\end{aligned}$$

$$\overset{(2.23)}{=} \sum_{i=1}^{n} \langle \nabla_{b_i} b_i \cdot \phi + b_i \cdot \nabla^{\Sigma}_{b_i} \phi, \psi \rangle - \langle \phi, b_i \cdot \nabla^{\Sigma}_{b_i} \psi \rangle - \langle \nabla_{b_i} b_i \cdot \phi, \psi \rangle$$
$$\overset{(2.25)}{=} \langle D\phi, \psi \rangle - \langle \phi, D\psi \rangle.$$

Integration over M yields

$$(D\phi, \psi)_{L^2} - (\phi, D\psi)_{L^2} = \int_M \operatorname{div}(X)\, dvol = 0. \qquad \square$$

Corollary 2.5.6. *Let D^{∇^C} be a twisted Dirac operator. If ∇^C is a metric connection for a Hermitian metric on the vector bundle C then D^{∇^C} is formally self-adjoint with respect to the induced metric on $\Sigma M \otimes C$.*

Proof. This follows from Lemma 2.5.5 together with Corollary 1.3.25. $\square$

Remark 2.5.7. The proof of the Lemma 2.5.5 shows more than the formal self-adjointness of D: if M is a Riemannian spin manifold with boundary then for all compactly supported $\phi, \psi \in C^\infty_c(M, \Sigma M)$ we have

$$(D\phi, \psi)_{L^2} - (\phi, D\psi)_{L^2} = \int_{\partial M} \langle X, \nu \rangle\, dA. \tag{2.27}$$

Here ν denotes the exterior unit normal vector field on ∂M.

Example 2.5.8

(i) Consider $M = S^1$ with the trivial spin structure. Let $b_1 = \frac{\partial}{\partial t}$. Then we have

$$\nabla^{\Sigma}_{b_1} [\![H, \varphi]\!] = [\![H, \frac{d\varphi}{dt}]\!]$$

and

$$D[\![H, \varphi]\!] = b_1 \cdot \nabla^{\Sigma}_{b_1} [\![H, \varphi]\!] = \left[\!\!\left[H, -i\,\frac{d\varphi}{dt}\right]\!\!\right].$$

By Example 2.4.8, spinor fields $[\![H, \varphi]\!]$ correspond to $\mathbb{Z}$-periodic complex valued functions on $\mathbb{R}$. Under this identification, the Dirac operator D corresponds to the operator $-i\,\frac{d}{dt}$. Using the Fourier expansion

$$\varphi(t) = \sum_{k=-\infty}^{\infty} \alpha_k \cdot e^{2\pi i\, kt}.$$

of $\mathbb{Z}$-periodic complex functions, we find $-i\frac{d}{dt}e^{2\pi i\,kt} = 2\pi\,k\,e^{2\pi i\,kt}$.

Hence the spectrum of the Dirac operator D on S^1 for the trivial spin structure is $2\pi\,\mathbb{Z}$, and all eigenvalues of D have the multiplicity 1.

(ii) Consider $M = S^1$ with the non-trivial spin structure. As above, the Dirac operator D corresponds to the operator $-i\,\frac{d}{dt}$, acting on $\mathbb{Z}$-anti-periodic complex functions this time, see Example 2.4.8. Since a function $\varphi : \mathbb{R} \to \mathbb{C}$ is $\mathbb{Z}$-anti-periodic if and only if $e^{i\pi t}\,\varphi(t)$ is $\mathbb{Z}$-periodic, we have the Fourier expansion

$$e^{i\pi t}\,\varphi(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{i\,2\pi\,kt}\,,$$

which yields

$$\varphi(t) = \sum_{k=-\infty}^{\infty} \alpha_k\, e^{2\pi i(k-\frac{1}{2})t}.$$

We thus find $-i\,\frac{d}{dt}\,e^{2\pi\,i(k-\frac{1}{2})t} = 2\pi\left(k-\frac{1}{2}\right)e^{2\pi\,i(k-\frac{1}{2})t}$. Hence the spectrum of the Dirac operator D on S^1 for the non-trivial spin structure is $2\pi(\mathbb{Z}-\frac{1}{2})$, and all multiplicities are 1.

In particular, 0 is an eigenvalue for the trivial spin structure on S^1, but not for the non-trivial spin structure.

By Remark 2.5.2, the classical Dirac operator D is of Dirac-type, and by Lemma 2.5.5 it is formally self-adjoint. By Lemma 1.3.5, we have

$$D^2 = \widetilde{\nabla}^*\widetilde{\nabla} + B,$$

for a connection $\widetilde{\nabla}$ on ΣM and an endomorphism field $B \in C^\infty(M, \mathrm{End}(\Sigma M))$. We now want to determine $\widetilde{\nabla}$ and B.

Lemma 2.5.9. *Let M be a n-dimensional Riemannian spin manifold with spinor bundle ΣM. For any orthonormal basis $b_1, \ldots, b_n$ of T_pM and any $X \in T_pM$, we have*

$$\sum_{j=1}^{n} b_j \cdot R^\Sigma(b_j, X)\phi = \frac{1}{2}\,\mathrm{Ric}(X)\cdot\phi. \tag{2.28}$$

Proof. a) By the first Bianchi identity, we have

$$TBC := \sum_{j=1}^{n} b_j \cdot R^{\Sigma}(b_j, X)\phi \overset{(2.24)}{=} -\frac{1}{4}\sum_{i,j} b_j \cdot R(b_j, X)b_i \cdot b_i \cdot \phi \tag{2.29}$$

$$= \frac{1}{4}\sum_{i,j} b_j \cdot \big(R(b_i, b_j)X + R(X, b_i)b_j\big) \cdot b_i \cdot \phi. \tag{2.30}$$

Now we compute the two contributing terms separately:

b) For the first term on the right hand side of (2.30), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n} b_j \cdot R(b_i, b_j)X \cdot b_i \cdot \phi \\
&\quad = \sum_{i,j,k=1}^{n} b_j \cdot \langle R(b_i, b_j)X, b_k\rangle\, b_k \cdot b_i \cdot \phi \\
&\quad = \sum_{i,j,k=1}^{n} b_j \cdot \langle R(X, b_k)b_i, b_j\rangle\, b_k \cdot b_i \cdot \phi \\
&\quad = \sum_{i,k=1}^{n} R(X, b_k)b_i \cdot b_k \cdot b_i \cdot \phi \\
&\quad = -\sum_{i,k=1}^{n} R(b_k, X)b_i \cdot b_k \cdot b_i \cdot \phi \\
&\quad \overset{(2.1)}{=} \sum_{i,k=1}^{n} b_k \cdot R(b_k, X)b_i \cdot b_i \cdot \phi + 2\sum_{i,k=1}^{n} \langle R(b_k, X)b_i, b_k\rangle\, b_i \cdot \phi \\
&\quad \overset{(2.29)}{=} -4TBC + 2\,\mathrm{Ric}(X) \cdot \phi.
\end{aligned} \tag{2.31}$$

c) For the second term on the right hand side of (2.30), we obtain

$$\begin{aligned}
&\sum_{i,j=1}^{n} b_j \cdot R(X, b_i)b_j \cdot b_i \cdot \phi \\
&\quad \overset{(2.1)}{=} -\sum_{i,j=1}^{n} R(X, b_i)b_j \cdot b_j \cdot b_i \cdot \phi - 2\sum_{i,j=1}^{n} \underbrace{\langle b_j, R(X, b_i)b_j\rangle}_{=0}\, b_i \cdot \phi \\
&\quad = \sum_{i,j=1}^{n} R(X, b_i)b_j \cdot b_i \cdot b_j \cdot \phi + 2\underbrace{\sum_{i=1}^{n} R(X, b_i)b_i}_{=\mathrm{Ric}(X)} \cdot \phi \\
&\quad = -\sum_{i,j=1}^{n} b_i \cdot R(X, b_i)b_j \cdot b_j \cdot \phi + 2\underbrace{\sum_{i,j=1}^{n} \langle R(b_i, X)b_j, b_i\rangle\, b_j}_{=\mathrm{Ric}(X)} \cdot \phi + 2\mathrm{Ric}(X) \cdot \phi
\end{aligned}$$

$$\overset{(2.29)}{=} -4TBC + 4\,\mathrm{Ric}(X)\cdot\phi. \tag{2.32}$$

d) Inserting (2.31) and (2.32) into (2.30) yields

$$\mathrm{TBC} = -\mathrm{TBC} + \frac{1}{2}\mathrm{Ric}(X)\cdot\phi - \mathrm{TBC} + \mathrm{Ric}(X)\cdot\phi,$$

and hence

$$\mathrm{TBC} = \frac{1}{2}\mathrm{Ric}(X)\cdot\phi.$$ □

Lemma 2.5.10. *Let E be a Riemannian or Hermitian vector bundle over a Riemannian manifold M, and let ∇ be a metric connection on E.*
Then for any local orthonormal tangent frame $b_1,\dots,b_n$ we have

$$\nabla^*\nabla = -\sum_{i=1}^{n}\left(\nabla_{b_i}\nabla_{b_i} - \nabla_{\nabla^{\mathrm{LC}}_{b_i}b_i}\right). \tag{2.33}$$

Proof. Let $\phi,\psi \in C_c^\infty(M,E)$ be sections in E with support in the domain of definition of $b_1,\dots,b_n$. Let $X \in C_c^\infty(M, TM\otimes\mathbb{C})$ be the vector field defined by

$$\langle X, Y\rangle = \langle \phi, \nabla_Y\psi\rangle, \quad \text{for all } Y \in TM.$$

Since the Levi-Civita connection ∇^{LC} and the connection ∇ on E are metric, we get

$$\begin{aligned}
\operatorname{div} X &= \sum_i \langle \nabla^{\mathrm{LC}}_{b_i}X, b_i\rangle \\
&= \sum_i \Big(\partial_{b_i}\langle X, b_i\rangle - \langle X, \nabla^{\mathrm{LC}}_{b_i}b_i\rangle\Big) \\
&= \sum_i \Big(\partial_{b_i}\langle \phi, \nabla_{b_i}\psi\rangle - \langle \phi, \nabla_{\nabla^{\mathrm{LC}}_{b_i}b_i}\psi\rangle\Big) \\
&= \sum_i \Big(\langle \nabla_{b_i}\phi, \nabla_{b_i}\psi\rangle + \langle \phi, \nabla_{b_i}\nabla_{b_i}\psi\rangle - \langle \phi, \nabla_{\nabla^{\mathrm{LC}}_{b_i}b_i}\psi\rangle\Big) \\
&= \langle \nabla\phi, \nabla\psi\rangle + \langle \phi, \sum_i \big(\nabla_{b_i}\nabla_{b_i} - \nabla_{\nabla^{\mathrm{LC}}_{b_i}b_i}\big)\psi\rangle
\end{aligned}$$

By Gauß' divergence theorem, we obtain

$$\begin{aligned}
0 &= \int_M \operatorname{div}(X)\, dvol \\
&= (\nabla\phi, \nabla\psi)_{L^2} + \big(\phi, \sum_i \big(\nabla_{b_i}\nabla_{b_i} - \nabla_{\nabla^{\mathrm{LC}}_{b_i}b_i}\big)\psi\big)_{L^2}
\end{aligned}$$

and thus

$$(\phi, \nabla^*\nabla\psi)_{L^2} = -\big(\phi, \sum_i \big(\nabla_{b_i}\nabla_{b_i} - \nabla_{\nabla^{\mathrm{LC}}_{b_i} b_i}\big)\psi\big)_{L^2}$$

for any $\phi, \psi \in C^\infty_c(M, E)$ as above and hence (2.33) holds at any point $p \in M$. $\square$

Theorem 2.5.11 (Lichnerowicz (1963)). *Let M be a Riemannian spin manifold, and let D be the classical Dirac operator. Then we have*

$$D^2 = \big(\nabla^\Sigma\big)^* \nabla^\Sigma + \frac{\mathrm{scal}}{4} \cdot \mathrm{id}_{\Sigma M}\,. \tag{2.34}$$

Proof. Let $x \in M$, and $b_1, \ldots, b_n$ be a local orthonormal tangent frame, defined in a neighborhood of x such that $\nabla^{\mathrm{LC}}_{b_i} b_j(x) = 0$ and $\mathrm{Ric}(b_i)(x) = \lambda_i\, b_i(x)$ for all i, j.
Now for any spinor field $\phi \in C^\infty(M, \Sigma)$, we have:

$$\begin{aligned}
(D^2\phi)_{(x)} &\overset{(2.25)}{=} \Big(\sum_{i,j=1}^n b_j \cdot \nabla^\Sigma_{b_j}\big(b_i \cdot \nabla^\Sigma_{b_i}\phi\big) \Big)_{(x)} \\
&\overset{(2.23)}{=} \Big(\sum_{i,j=1}^n b_j \cdot \big(\underbrace{\nabla^{\mathrm{LC}}_{b_j} b_i}_{=0} \cdot \nabla^\Sigma_{b_i}\phi + b_i \cdot \nabla^\Sigma_{b_j}\nabla^\Sigma_{b_i}\phi\big)\Big)_{(x)} \\
&\overset{(2.1)}{=} -\Big(\sum_{i=1}^n \nabla^\Sigma_{b_i}\nabla^\Sigma_{b_i}\phi + \sum_{i<j} b_j \cdot b_i \cdot \big(\nabla^\Sigma_{b_j}\nabla^\Sigma_{b_i} - \nabla^\Sigma_{b_i}\nabla^\Sigma_{b_j}\big)\phi\Big)_{(x)} \\
&\overset{(2.33)}{=} \Big(\big(\nabla^\Sigma\big)^* \nabla^\Sigma\phi + \sum_{i<j} b_j \cdot b_i \cdot R^\Sigma(b_j, b_i)\phi\Big)_{(x)}.
\end{aligned}$$

By Lemma 2.5.9, we have:

$$\begin{aligned}
\sum_{i<j} b_j \cdot b_i \cdot R^\Sigma(b_j, b_i)\phi \overset{(2.1)}{=} \frac{1}{2}\sum_{i,j=1}^n b_j \cdot b_i \cdot R^\Sigma(b_j, b_i)\phi &\overset{(2.28)}{=} -\frac{1}{4}\sum_{j=1}^n b_j \cdot \mathrm{Ric}(b_j) \cdot \phi \\
&= -\frac{1}{4}\sum_{j=1}^n b_j \cdot \lambda_j b_j \cdot \phi = \frac{1}{4}\sum_{j=1}^n \lambda_j \cdot \phi \\
&= \frac{1}{4}\,\mathrm{tr}(\mathrm{Ric}) \cdot \phi = \frac{1}{4}\,\mathrm{scal} \cdot \phi.
\end{aligned}$$

$\square$

Corollary 2.5.12. *Let M be an n-dimensional connected compact Riemannian spin manifold with $\mathrm{scal} \geq 0$. Then the multiplicity of 0 in $\mathrm{spec}(D)$ is bounded by*

$$\mathrm{mult}(0) \leq \dim \Sigma_n = 2^{[\frac{n}{2}]}.$$

Moreover, any harmonic spinor field (i.e. eigenspinor of D for the eigenvalue 0) is ∇^Σ-parallel.

Proof. Let $\phi \in \ker(D)$. Then we have:

$$\begin{aligned}
0 &= (D^2\phi, \phi)_{L^2} \\
&\overset{(2.34)}{=} \Big((\nabla^\Sigma)^*\nabla^\Sigma\phi, \phi\Big)_{L^2} + \Big(\frac{\mathrm{scal}}{4}\cdot\phi, \phi\Big)_{L^2} \\
&= \big\|\nabla^\Sigma\phi\big\|^2_{L^2} + \underbrace{\int_M \underbrace{\frac{\mathrm{scal}}{4}}_{\geq 0} |\phi|^2\, dvol}_{\geq 0} \qquad (2.35) \\
&\geq \big\|\nabla^\Sigma\phi\big\|^2_{L^2} \geq 0
\end{aligned}$$

Thus, $\|\nabla^\Sigma\phi\|_{L^2} = 0$ and hence $\nabla^\Sigma\phi = 0$.

Let $x_0 \in M$. Since M is connected and $\nabla^\Sigma\phi = 0$, the value of ϕ at any point $x \in M$ is determined by $\phi(x_0)$. As in the proof of the Bochner Theorem 1.5.16 we conclude

$$\dim\ker(D) \leq \dim \Sigma_{x_0}M = \dim \Sigma_n. \qquad \square$$

Corollary 2.5.13. *Let M be a connected compact Riemannian spin manifold with $\mathrm{scal} \geq 0$ and $\mathrm{scal} > 0$ somewhere.*
Then there are no non-trivial harmonic spinor fields, i.e., $0 \notin \mathrm{spec}(D)$.

Proof. Let $\phi \in \ker(D)$. By Corollary 2.5.12 we have $\nabla^\Sigma\phi = 0$, which yields

$$\partial_X|\phi|^2 = \langle\nabla^\Sigma_X\phi, \phi\rangle + \langle\phi, \nabla^\Sigma_X\phi\rangle = 0.$$

Since M is connected, we conclude that $|\phi|$ is constant. Inserting back into equation (2.35), we obtain

$$\int_M \frac{\mathrm{scal}}{4}\cdot|\phi|^2\, dvol = |\phi|\cdot\int_M \frac{\mathrm{scal}}{4}\, dvol = 0.$$

Now let x be a point with $\mathrm{scal}(x) > 0$. By continuity, this also holds in a neighborhood of x. We thus have $\int_M \frac{\mathrm{scal}}{4}\, dvol > 0$, hence $|\phi| = 0$. $\square$

Remark 2.5.14. Let M be a compact Riemannian spin manifold with $\mathrm{scal} \geq s_0 > 0$. Let $\lambda \in \mathrm{spec}(D)$ and let ϕ be an eigenspinor. Then we have:

$$\lambda^2 \|\phi\|^2_{L^2} = (D^2\phi, \phi)_{L^2} \geq \left\|\nabla^\Sigma \phi\right\|^2_{L^2} + \frac{s_0}{4}\|\phi\|^2_{L^2} \geq \frac{s_0}{4}\|\phi\|^2_{L^2}$$

Hence

$$|\lambda| \geq \sqrt{\frac{s_0}{4}}. \tag{2.36}$$

That is, there is a spectral gap around 0.

But this estimate is not sharp. More precisely, equality can never be achieved in (2.36), as we will see in the following:

Theorem 2.5.15 (Friedrich's inequality, 1980). *Let M be a compact n-dimensional Riemannian spin manifold with* $\mathrm{scal} \geq s_0 > 0$. *Then for any $\lambda \in \mathrm{spec}(D)$, we have*

$$|\lambda| \geq \sqrt{\frac{n}{n-1}\frac{s_0}{4}}. \tag{2.37}$$

Proof. By Lemmas 2.3.9 and 2.3.14, Clifford multiplication is skew-symmetric. Thus for any $X \in T_xM$ and any $\phi \in \Sigma_x M$, we have

$$|X \cdot \phi|^2 = \langle X \cdot \phi, X \cdot \phi \rangle = -\langle \phi, X \cdot X \cdot \phi \rangle \overset{(2.1)}{=} -\langle \phi, -|X|^2 \phi \rangle = |X|^2 |\phi|^2$$

Hence

$$|X \cdot \phi| = |X| \cdot |\phi|. \tag{2.38}$$

Now let $\phi \in C^\infty(M, \Sigma M)$. Fix $x \in M$ and let $b_1, \ldots, b_n$ be an orthonormal tangent frame in a neighborhood of x. Using equation (2.38) and the Cauchy-Schwarz inequality, we find:

$$\begin{aligned}
\left(|D\phi|^2\right)_{(x)} &= \left|\sum_i b_i \cdot \nabla^\Sigma_{b_i}\phi\right|^2_{(x)} \leq \left(\sum_{i=1}^n \left|b_i \cdot \nabla^\Sigma_{b_i}\phi\right|\right)^2_{(x)} = \left(\sum_{i=1}^n |b_i| \cdot \left|\nabla^\Sigma_{b_i}\phi\right|\right)^2_{(x)} \\
&\leq \sum_{i=1}^n \underbrace{|b_i|^2_{(x)}}_{=1} \cdot \sum_{i=1}^n \left|\nabla^\Sigma_{b_i}\phi\right|^2_{(x)} = n \cdot \left|\nabla^\Sigma \phi\right|^2_{(x)}.
\end{aligned}$$

Thus for any $x \in M$, we have the estimate:

$$\left|\nabla^\Sigma \phi\right|^2_{(x)} \geq \frac{1}{n}|D\phi|^2_{(x)}. \tag{2.39}$$

By Lichnerowicz' Theorem 2.5.11 and equation (2.39), we find:

$$\begin{aligned}
\|D\phi\|_{L^2}^2 &= (D^2\phi,\phi)_{L^2} \\
&= \Big((\nabla^\Sigma)^* \nabla^\Sigma \phi, \phi \Big)_{L^2} + \Big(\frac{\mathrm{scal}}{4} \cdot \phi, \phi \Big)_{L^2} \\
&\geq \|\nabla^\Sigma \phi\|_{L^2}^2 + \frac{s_0}{4} \|\phi^2\|_{L^2} \\
&\overset{(2.39)}{\geq} \frac{1}{n} \|D\phi\|_{L^2}^2 + \frac{s_0}{4} \|\phi^2\|_{L^2}.
\end{aligned}$$

Thus

$$\|D\phi\|_{L^2}^2 \geq \frac{n}{n-1} \frac{s_0}{4} \cdot \|\phi\|_{L^2}^2.$$

Now if $D\phi = \lambda\phi$ then we obtain

$$\lambda^2 \cdot \|\phi\|_{L^2}^2 \geq \frac{n}{n-1} \frac{s_0}{4} \cdot \|\phi\|_{L^2}^2 .$$

Thus any eigenvalue λ of the classical Dirac operator D satisfies

$$\lambda^2 \geq \frac{n}{n-1} \frac{s_0}{4}.$$

□

Remark 2.5.16. Friedrich's estimate (2.37) is sharp: equality is achieved e.g. for S^n with metrics of constant curvature.

Theorem 2.5.17 (Bär, 1991). *Let $M = S^2$ with any Riemannian metric. Then all eigenvalues of the classical Dirac operator D satisfy*

$$\lambda^2 \geq \frac{4\pi}{\mathrm{area}(M)}. \tag{2.40}$$

In particular, by the estimate (2.40), 0 can not be an eigenvalue of D for any metric on S^2.

Remark 2.5.18

1) Equality in (2.40) (for the eigenvalue with the smallest absolute value) is achieved iff the curvature of M is constant.

2) Lott (1986) proved with different methods the estimate:

$$\exists\, C > 0 : \quad \lambda^2 \geq \frac{C}{\mathrm{area}(M)} . \tag{2.41}$$

Lott already conjectured that $C = 4\pi$ was the optimal constant.

3) Hersch (1970) has proved that for the first positive eigenvalue $\lambda_1(\Delta)$ of the Laplace-Beltrami operator on a manifold M diffeomorphic to S^2, the following estimate holds:

$$\lambda_1(\Delta) \leq \frac{8\pi}{\mathrm{area}(M)}.$$

4) Every compact orientable surface of genus ≥ 1 admits a spin structure and a Riemannian metric such that $0 \in \mathrm{spec}(D)$. Also, on S^3 there are Riemannian metrics with harmonic spinors, i.e. with $0 \in \mathrm{spec}(D)$. Thus, for these manifolds, there are no estimates like (2.40).

5) It is conjectured that every compact spin manifold of dimension $n \geq 3$ admits a Riemannian metric with harmonic spinor, i.e. with $0 \in \mathrm{spec}(D)$. If the conjecture holds then there are no estimates like (2.40) for $n \geq 3$. The conjecture has been proved for the cases $n = 0, 1, 7 \bmod 8$ by Hitchin (1974) and for the cases $n = 3 \bmod 4$ by Bär (1996).

6) Up to today, Theorem 2.5.17 is the only estimate for Dirac eigenvalues not involving any curvature assumptions.

Proof. [of Theorem 2.5.17]

a) Let M be an arbitrary 2-dimensional Riemannian manifold. Let $\lambda \in \mathbb{R}$ and $f \in C^\infty(M)$. Define a new connection $\tilde{\nabla}$ on ΣM by

$$\widetilde{\nabla}_X \phi := \nabla^\Sigma_X \phi + \frac{\lambda}{2} X \cdot \phi + X \cdot \operatorname{grad} f \cdot \phi.$$

Claim:

$$\begin{aligned}\widetilde{\nabla}^*\big(e^{-2f}\tilde{\nabla}\phi\big) = e^{-2f} \Big\{ D^2 - \lambda D - 2 \operatorname{grad} f \cdot D - 2 \Big[\nabla^\Sigma_{\operatorname{grad} f} + \big(\nabla^\Sigma_{\operatorname{grad} f}\big)^*\Big] \\ - \frac{\mathrm{scal}}{4} + \frac{\lambda^2}{2} + \Delta f + \lambda \cdot \operatorname{grad} f \cdot \Big\} \phi.\end{aligned} \tag{2.42}$$

The proof of the claim is an elementary but tedious computation. Notice that (2.42) is only valid for 2-dimensional manifolds.

b) We compute:

$$\begin{aligned}&\int_M e^{-2f} \Big\langle \Big[\nabla^\Sigma_{\operatorname{grad} f} + \big(\nabla^\Sigma_{\operatorname{grad} f}\big)^*\Big] \phi, \phi \Big\rangle \, dA \\ &= \Big(\nabla^\Sigma_{\operatorname{grad} f}\phi, e^{-2f}\phi\Big)_{L^2} + \Big(\phi, \nabla^\Sigma_{\operatorname{grad} f}\big(e^{-2f}\phi\big)\Big)_{L^2} \\ &= \int_M \partial_{\operatorname{grad} f} \langle \phi, e^{-2f}\phi\rangle \, dA \\ &= \int_M \Big\langle \operatorname{grad} f, \operatorname{grad} \langle \phi, e^{-2f}\phi\rangle \Big\rangle \, dA \\ &= \int_M \Delta f \cdot \langle \phi, e^{-2f}\phi\rangle \, dA \\ &= \int_M e^{-2f} \, \Delta f \, |\phi|^2 \, dA.\end{aligned}$$

c) Now let $\phi \in C^\infty(M, \Sigma M)$ be an eigenspinor for the eigenvalue λ. Using a) and b), we find:

$$\begin{aligned}0 &\le \int_M e^{-2f} \, \big|\widetilde{\nabla}\phi\big|^2 dA \\ &= \Big(\widetilde{\nabla}^*\big(e^{-2f}\tilde{\nabla}\phi\big), \phi\Big) \\ &\le \int_M e^{-2f} \Big\langle \Big\{ \lambda^2 - \lambda^2 - 2\lambda \operatorname{grad} f - 2\Delta f - \frac{\mathrm{scal}}{4} + \frac{\lambda^2}{2} + \Delta f + \lambda \operatorname{grad} f \Big\} \cdot \phi, \phi \Big\rangle \, dA \\ &= \int_M e^{-2f} \Big\langle \Big\{ \frac{\lambda^2}{2} - \frac{\mathrm{scal}}{4} - \Delta f \Big\} \phi, \phi \Big\rangle \, dA - \int_M e^{-2f} \lambda \langle \operatorname{grad} \; f \cdot \phi, \phi\rangle \, dA.\end{aligned} \tag{2.43}$$

By Lemmas 2.3.9 and 2.3.14, Clifford multiplication is skew symmetric, i.e.

$$\langle \mathrm{grad}\ f\cdot\phi,\phi\rangle = -\langle\phi,\mathrm{grad}\ f\cdot\phi\rangle = -\overline{\langle \mathrm{grad}\ f\cdot\phi,\phi\rangle},$$

and hence $\langle \mathrm{grad}\ f\cdot\phi,\phi\rangle \in i\mathbb{R}$. Since all the other terms in (2.43) are real, we conclude

$$\int_M e^{-2f}\lambda\langle \mathrm{grad}\ f\cdot\phi,\phi\rangle\, dA = 0.$$

This yields the estimate

$$0 \le \int_M e^{-2f}\Big(\frac{\lambda^2}{2} - \frac{\mathrm{scal}}{4} - \Delta f\Big)\cdot|\phi|^2\, dA. \tag{2.44}$$

d) Since the Laplace-Beltrami operator is self-adjoint, any $h \in C^\infty(M)$ perpendicular to $\ker\Delta$ is of the form $h = \Delta f$ for some $f \in C^\infty(M)$.
Choose $-h := \frac{\mathrm{scal}}{4} - \frac{1}{4\,\mathrm{area}(M)}\int_M \mathrm{scal}(y)\, dA(y) \in C^\infty$. Then we have

$$0 = \int_M h(y)\, dA(y) = (h,1)_{L^2},$$

and thus $h \perp \ker(\Delta)$. So let $f \in C^\infty(M)$ with $\Delta f = h$. Inserting this particular choice of f into (2.44) yields:

$$\begin{aligned}
0 &\le \int_M e^{-2f}\Big(\frac{\lambda^2}{2} - \frac{\mathrm{scal}}{4} - h\Big)\cdot|\phi|^2\, dA\\
&\le \int_M e^{-2f}\Big(\frac{\lambda^2}{2} - \frac{1}{4\,\mathrm{area}(M)}\int_M \mathrm{scal}(y)\, dA(y)\Big)\cdot|\phi|^2\, dA\\
&= \Big(\frac{\lambda^2}{2} - \frac{1}{4\,\mathrm{area}(M)}\int_M \mathrm{scal}(y)\, dA(y)\Big)\cdot\underbrace{\int_M e^{-2f}|\phi|^2\, dA}_{\ge 0}.
\end{aligned}$$

We thus have the estimate

$$\frac{\lambda^2}{2} \ge \frac{1}{4\,\mathrm{area}(M)}\int_M \underbrace{\mathrm{scal}(y)}_{=2K}\, dA(y),$$

where K denotes the Gauß curvature. By the Gauß-Bonnet Theorem, we end up with

$$\frac{\lambda^2}{2} \ge \frac{1}{4\,\mathrm{area}(M)}\cdot 4\pi\cdot\chi(M) = \frac{2\pi}{\mathrm{area}(M)}.$$

Observe that only in this last step, we used the fact that $M = S^2$. $\square$

2.6. Hypersurfaces

Let M be an $(n+1)$-dimensional Riemannian spin manifold, and let $N \subset M$ be an oriented hypersurface. We want to construct a spin structure on N and relate the spinor bundles ΣM and ΣN and the Dirac operators D^M and D^N.

Let ν be the unit normal vector field along N such that $(b_1, \ldots, b_n)$ is a positively oriented basis of T_xN if and only if $(b_1, \ldots, b_n, \nu(x))$ is a positively oriented basis of T_xM. Using the canonical embedding

$$\begin{aligned} \mathrm{SO}(n) &\hookrightarrow \mathrm{SO}(n+1) \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

the action of $\mathrm{SO}(n)$ on $(b_1, \ldots, b_n, \nu(x))$ preserves the normal $\nu(x)$.
Consider the map $\mathbb{C}\mathrm{l}_n^0 \subset \mathbb{C}\mathrm{l}_n \xrightarrow{\cong} \mathbb{C}\mathrm{l}_{n+1}^0$, induced by $\mathbb{R}^n \ni X \mapsto X \cdot e_{n+1}$. Since $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n^0$ and $\mathrm{Spin}(n+1) \subset \mathbb{C}\mathrm{l}_{n+1}^0$, we obtain a map

$$\begin{aligned} \mathrm{Spin}(n) &\hookrightarrow \mathrm{Spin}(n+1) \\ a = v_1 \cdot v_2 \cdot \ldots \cdot v_{2m} &\mapsto v_1 \cdot e_{n+1} \cdot \ldots \cdot v_{2m} \cdot e_{n+1} = v_1 \cdot \ldots \cdot v_{2m}. \end{aligned}$$

With this embedding we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, hook] \arrow[d, "\varrho"'] & \mathrm{Spin}(n+1) \arrow[d, "\varrho"'] \\ \mathrm{SO}(n) \arrow[r, hook] & \mathrm{SO}(n+1) \end{tikzcd}.$$

Moreover, we have a canonical embedding of frame bundles

$$\begin{aligned} P^{\mathrm{SO}}(N) &\hookrightarrow P^{\mathrm{SO}}(M), \\ \big(h : \mathbb{R}^n \to T_pN\big) &\mapsto \big(h' : \mathbb{R}^{n+1} \to T_pM\big), \end{aligned}$$

where $h'(x_1, \ldots, x_n, 0) = h(x_1, \ldots, x_n)$ and $h'(0, \ldots, 0, 1) := \nu(p)$. This embedding is compatible with the embedding $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(n+1)$ defined above. Thus, the diagram

$$\begin{tikzcd} P^{\mathrm{SO}}(N) \times \mathrm{SO}(n) \arrow[r] \arrow[d, hook] & P^{\mathrm{SO}}(N) \arrow[d, hook] \\ P^{\mathrm{SO}}(M) \times \mathrm{SO}(n+1) \arrow[r] & P^{\mathrm{SO}}(M) \end{tikzcd}.$$

commutes.

Now let $\bar{\varrho}$: $P^{\mathrm{Spin}}(M) \to P^{\mathrm{SO}}(M)$ be a spin structure on M. We set

$$P^{\mathrm{Spin}}(N) := \bar{\varrho}^{-1}\big(P^{\mathrm{SO}}(N)\big) \tag{2.45}$$

This defines a spin structure on N:

- The action of $\mathrm{Spin}(n+1)$ on $P^{\mathrm{Spin}}(M)$ restricts to an action of $\mathrm{Spin}(n)$ on $P^{\mathrm{Spin}}(N)$: For $H \in P^{\mathrm{Spin}}(N)$ and $a \in \mathrm{Spin}(n)$ we have $H \cdot a \in P^{\mathrm{Spin}}(M)$ and

$$\bar{\varrho}(H \cdot a) = \underbrace{\bar{\varrho}(H)}_{\in P^{\mathrm{SO}}(N)} \cdot \underbrace{\varrho(a)}_{\in \mathrm{SO}(n)} \in P^{\mathrm{SO}}(N).$$

  Thus $H \cdot a \in P^{\mathrm{Spin}}(N)$.

- Obviously, the action of $\mathrm{Spin}(n)$ on $P^{\mathrm{Spin}}(N)$ is compatible with the action of $\mathrm{SO}(n)$ on $P^{\mathrm{SO}}(N)$, hence $\bar{\varrho} : P^{\mathrm{Spin}}(N) \to P^{\mathrm{SO}}(N)$ is a spin structure on N.

In particular, orientable hypersurfaces of spinnable manifolds are again spinnable.

Spinor bundles

We study how the spinor bundles of N and M are related to one another.

Case 1: $\mathbf{n+1}$ is even

In this case, $\Sigma_n = \Sigma_{n+1}^+$. For any $x \in N$, we have[1]

$$\Sigma_x N = P_x^{\mathrm{Spin}}(N) \times_{\sigma_n} \Sigma_n = P_x^{\mathrm{Spin}}(N) \times_{\sigma_{n+1}^+|_{\mathrm{Spin}(n)}} \Sigma_{n+1}^+ \cong P_x^{\mathrm{Spin}}(M) \times_{\sigma_{n+1}^+} \Sigma_{n+1}^+$$

Thus, $\Sigma N = \Sigma^+ M|_N$.
The Clifford multiplication of $\mathbb{R}^n$ on $\Sigma_n = \Sigma_{n+1}^+$ is given by

$$X \cdot \varphi = X \cdot e_{n+1} \cdot \varphi,$$

where the $\cdot$ on the left hand side is the Clifford multiplication in $\mathbb{C}\mathrm{l}_n$, while the $\cdot$ on the right hand side is the Clifford multiplication in $\mathbb{C}\mathrm{l}_{n+1}$. Thus, the Clifford multiplication in ΣN is given by

$$X \cdot \varphi = X \cdot \nu \cdot \varphi,$$

where $X \in T_x N$ and $\varphi \in \Sigma_x N$.

Case 2: $\mathbf{n+1}$ is odd

The inclusion of Clifford algebras

$$\mathbb{C}\mathrm{l}_n \xrightarrow{\cong} \mathbb{C}\mathrm{l}_{n+1}^0 \hookrightarrow \mathbb{C}\mathrm{l}_{n+1} \xrightarrow{\cong} \mathbb{C}\mathrm{l}_{n+2}^0 \hookrightarrow \mathbb{C}\mathrm{l}_{n+2}$$

[1] Let $X \subset Y$ be sets and $H \subset G$ be groups. Let G act simply transitively from the right on Y such that the action restricts to a simply transitive right action of H on X. Then for any representation of G on Σ the inclusions induce a bijection $X \times_H \Sigma \cong Y \times_G \Sigma$.

together with the inclusions $\Sigma_n \subset \mathbb{C}\mathrm{l}_n$ and $\Sigma_{n+1} \cong \Sigma^+_{n+2} \subset \mathbb{C}\mathrm{l}_{n+2}$ induces an isomorphism $\Xi_n : \Sigma_n \to \Sigma_{n+1}$ such that the diagram

$$\begin{tikzcd}
\Sigma_n \arrow[r, "\Xi_n"] \arrow[d, "X\cdot()"'] & \Sigma_{n+1} = \Sigma^+_{n+2} \arrow[d, "X\cdot e_{n+1}\cdot() = X\cdot e_{n+2}\cdot e_{n+1}\cdot e_{n+2}\cdot()"] \\
\Sigma_n \arrow[r, "\Xi_n"] & \Sigma_{n+1} = \Sigma^+_{n+2}
\end{tikzcd}$$

of Clifford multiplications with $X \in \mathbb{R}^n$ commutes.
As in case 1 we obtain the canonical isomorphism $\Sigma N \cong \Sigma M|_N$ such that again

$$X \cdot \varphi = X \cdot \nu \cdot \varphi.$$

for $X \in T_xN$, $\varphi \in \Sigma_x N$.
In the following we treat both cases simultaneously using the notation

$$\Sigma^{(+)}M := \begin{cases} \Sigma^+ M & \text{if } n+1 \text{ is even,} \\ \Sigma M & \text{if } n+1 \text{ is odd.} \end{cases}$$

Spinor connections

The Levi-Civita connections on TM and TN are related by the **Gauß equation**

$$\underbrace{\nabla^M_X Y}_{\in T_xM} = \underbrace{\nabla^N_X Y}_{\in T_xN} + \underbrace{\mathrm{II}(X,Y)}_{\in (T_xN)^\perp}, \tag{2.46}$$

where $X \in T_xN$ and $Y \in C^\infty(N, TN)$. The second fundamental form is a symmetric bilinear map $\mathrm{II} : T_xN \times T_xN \to (T_xN)^\perp$, given by the orthogonal projection of $\nabla^M_X Y$ to $(T_xN)^\perp$. The Weingarten map is the corresponding symmetric endomorphism $B : T_xN \to T_xN$ such that for all $X, Y \in T_xN$

$$\mathrm{II}(X,Y) = g(B(X),Y)\,\nu =: \big\langle B(X), Y\big\rangle\,\nu\,.$$

The mean curvature field $\mathcal{H} \in C^\infty(N, TN^\perp)$ is defined by

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^{n} \big\langle B(b_i), b_i\big\rangle\,\nu = \frac{1}{n}\,\mathrm{tr}(B)\,\nu = H\,\nu,$$

where $b_1, \ldots, b_n$ is a local orthonormal tangent frame for N and $H : N \to \mathbb{R}$ is the mean curvature of the hypersurface $N \subset M$.

The spinor connections of M and N are related by the Weingarten map. Let $(b_1, \ldots, b_n)$ be a local oriented orthonormal tangent frame for N. Then $(b_1, \ldots, b_n, b_{n+1} = \nu)$ is a local orthonormal tangent frame for M along N. The Christoffel symbols for the Levi-Civita connections ∇^M and ∇^N are defined by

$$\nabla^M_{b_i} b_j = \sum_{k=1}^{n+1} {}^M\Gamma^k_{ij}\, b_k \qquad \text{and} \qquad \nabla^N_{b_i} b_j = \sum_{k=1}^{n} {}^N\Gamma^k_{ij}\, b_k.$$

By the Gauß equation (2.46), we have for $i, j \in \{1, \dots, n\}$:

$$\nabla^M_{b_i} b_j = \nabla^N_{b_i} b_j + \big\langle B(b_i), b_j \big\rangle\, \nu = \sum_{k=1}^{n} {}^N\Gamma^k_{ij}\, b_k + \big\langle B(b_i), b_j \big\rangle\, b_{n+1}.$$

Comparing coefficients yields

$$\begin{aligned}
{}^M\Gamma^k_{ij} &= {}^N\Gamma^k_{ij} && \forall\, i, j, k = \{1, \dots, n\},\\
{}^M\Gamma^{n+1}_{ij} &= -\,{}^M\Gamma^j_{i,n+1} = \big\langle B(b_i), b_j \big\rangle && \forall\, i, j = \{1, \dots, n\}.
\end{aligned}$$

For the covariant derivative a section of $\Sigma^{(+)}M$, we compute for $i \in \{1, \dots, n\}$:

$$\begin{aligned}
{}^M\nabla^\Sigma_{b_i} [\![H, \varphi]\!] &\overset{(2.21)}{=} \Big[\!\!\Big[H, \partial_{b_i}\varphi + \frac{1}{4} \sum_{j,k=1}^{n+1} {}^M\Gamma^k_{ij}\, e_j \cdot e_k \cdot \varphi \Big]\!\!\Big]\\
&= \Big[\!\!\Big[H, \partial_{b_i}\varphi + \frac{1}{4} \sum_{j,k=1}^{n} {}^N\Gamma^k_{ij}\, e_j \cdot e_k \cdot \varphi + \frac{1}{4} \sum_{j=1}^{n} \big\langle B(b_i), b_j \big\rangle\, e_j \cdot e_{n+1} \cdot \varphi\\
&\qquad - \frac{1}{4} \sum_{k=1}^{n} \big\langle B(b_i), b_k \big\rangle\, e_{n+1} \cdot e_k \cdot \varphi \Big]\!\!\Big]\\
&= {}^N\nabla^\Sigma_{b_i} [\![H, \varphi]\!] + \frac{1}{2} \sum_{j=1}^{n} \big\langle B(b_i), b_j \big\rangle\, b_j \cdot b_{n+1} \cdot [\![H, \varphi]\!]\\
&= {}^N\nabla^\Sigma_{b_i} [\![H, \varphi]\!] + \frac{1}{2}\, B(b_i) \cdot \nu \cdot [\![H, \varphi]\!].
\end{aligned}$$

Hence for all $\phi \in C^\infty(M, \Sigma^{(+)}M)$ and for all $X \in TN$, we have along N:

$${}^M\nabla^\Sigma_X \phi = {}^N\nabla^\Sigma_X \phi + \frac{1}{2}\, B(X) \cdot \nu \cdot \phi.$$

Dirac operators

For a spinor field $\phi \in C^\infty(M, \Sigma^{(+)}M)$ we have along the hypersurface N:

$$\begin{aligned}
D^M \phi &= \sum_{j=1}^{n} b_j \cdot {}^M\nabla^\Sigma_{b_j} \phi + \nu \cdot {}^M\nabla^\Sigma_\nu \phi\\
&\overset{(2.1)}{=} \sum_{j=1}^{n} \nu \cdot b_j \cdot \nu \cdot \Big({}^N\nabla^\Sigma_{b_j} \phi + \frac{1}{2}\, B(b_j) \cdot \nu \cdot \phi \Big) + \nu \cdot {}^M\nabla^\Sigma_\nu \phi
\end{aligned}$$

$$= \nu \cdot \Big(D^N \phi + \frac{1}{2} \sum_{j=1}^{n} b_j \cdot \nu \cdot B(b_j) \cdot \nu \cdot \phi \Big) + \nu \cdot {}^M\nabla^{\Sigma}_{\nu} \phi$$
$$= \nu \cdot D^N \phi + \frac{1}{2} \nu \cdot \sum_{j=1}^{n} b_j \cdot B(b_j) \cdot \phi + \nu \cdot {}^M\nabla^{\Sigma}_{\nu} \phi.$$

Since B is a symmetric endomorphism, we may choose $b_1, \ldots, b_n$ as an eigenbasis at $x \in M$, thus $B(b_j) = \kappa_j \cdot b_j$ for $j = 1, \ldots, n$. Then we have

$$D^M \phi = \nu \cdot D^N \phi - \frac{1}{2} \nu \cdot \mathrm{tr}(B)\, \phi + \nu \cdot {}^M\nabla^{\Sigma}_{\nu} \phi$$
$$= \nu \cdot D^N \phi - \frac{n}{2} \nu \cdot H\, \phi + \nu \cdot {}^M\nabla^{\Sigma}_{\nu} \phi.$$

Hence

$$-\nu \cdot D^M \phi = D^N \phi - \frac{n}{2} H \phi + {}^M\nabla^{\Sigma}_{\nu} \phi. \tag{2.47}$$

Theorem 2.6.1 (Bär, 1998). *Let $N \subset \mathbb{R}^{n+1}$ be an oriented compact hypersurface with induced spin structure.*
Then there are at least $2^{[\frac{n}{2}]}$ eigenvalues λ of D^N (counted with multiplicity) satisfying

$$\lambda^2 \le \frac{n^2}{4} \frac{1}{\mathrm{vol}(N)} \int_N H^2 \, dvol.$$

Here $H : N \to \mathbb{R}$ is the mean curvature of $N \subset \mathbb{R}^{n+1}$.

For the proof of this theorem we need the following variational characterization of eigenvalues:

Lemma 2.6.2 (minimax principle). *Let $\mathscr{H}$ be a Hilbert space, let A be a self-adjoint operator on $\mathscr{H}$. Assume that $\mathscr{H}$ has an orthonormal basis consisting of eigenvectors of A and let $\mu_1 \le \mu_2 \le \mu_3 \le \ldots$ be the eigenvalues, each one repeated according to its multiplicity. Then the eigenvalues are characterized as:*

$$\mu_k = \min_{\substack{V \subset \mathrm{dom}(A) \\ \dim V = k}} \max_{f \in V \setminus \{0\}} \frac{(Af, f)}{\|f\|^2}.$$

Proof. Let $V \subset \mathrm{dom}(A)$ be a vector subspace of dimension k. Let $f_1, f_2, \ldots$ be an orthonormal basis of $\mathscr{H}$ with $A f_j = \mu_j f_j$. Let $l_j : V \to \mathbb{C}$ be the linear functional defined by

$$l_j(f) = (f, f_j).$$

Since $\dim V = k$, there is an $f \in V \setminus \{0\}$ such that $l_1(f) = \ldots = l_{k-1}(f) = 0$ and hence $f = \sum_{j=k}^{\infty} \alpha_j f_j$. This yields the estimate

$$(Af, f) = \left(\sum_{j=k}^{\infty} \alpha_j \, \mu_j \, f_j, \sum_{i=k}^{\infty} \alpha_i \, f_i \right) = \sum_{j=k}^{\infty} \mu_j \, |\alpha_j|^2 \geq \mu_k \sum_{j=k}^{\infty} |\alpha_j|^2 = \mu_k \, \|f\|^2,$$

that is, $\frac{(A\,f,f)}{\|f\|^2} \geq \mu_k$ and in particular, $\max\limits_{f \in V \setminus \{0\}} \frac{(Af,f)}{\|f\|^2} \geq \mu_k$. Hence

$$\inf_{\substack{V \subset \mathrm{dom}(A) \\ \dim V = k}} \max_{f \in V \setminus \{0\}} \frac{(Af, f)}{\|f\|^2} \geq \mu_k.$$

Equality is attained for $V = \mathbb{C}\, f_1 \oplus \ldots \oplus \mathbb{C}\, f_k$ and $f = f_k$. □

Proof. [of Theorem 2.6.1] The spinor bundle of the hypersurface $\Sigma N = \Sigma^{(+)}\mathbb{R}^{n+1}|_N$ has rank $2^{[\frac{n}{2}]}$. The spinor bundle $\Sigma^{(+)}\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ can be trivialized by parallel sections $\psi_1, \ldots, \psi_{2^{[\frac{n}{2}]}}$. In particular, $\psi_1, \ldots, \psi_{2^{[\frac{n}{2}]}}$ are linearly independent at each point. Thus $\psi_1|_N, \ldots, \psi_{2^{[\frac{n}{2}]}}|_N$ are still linearly independent. We define

$$V := \mathbb{C} \cdot \psi_1|_N \oplus \ldots \oplus \mathbb{C} \cdot \psi_{2^{[\frac{n}{2}]}}|_N \subset C^\infty(N, \Sigma N) \subset \mathrm{dom}\big((D^N)^2\big).$$

Since any $\psi \in V$ is parallel with respect to the connection ${}^{\mathbb{R}^{n+1}}\nabla^\Sigma$, we have:

$$\begin{aligned}
\big((D^N)^2\psi, \psi\big)_{L^2(N)} &= \|D^N \psi\|^2_{L^2(N)} \\
&\overset{(2.47)}{=} \Big\| -\nu \cdot \underbrace{D^{\mathbb{R}^{n+1}}\psi}_{=0} + \frac{n}{2}\, H\psi - \underbrace{{}^{\mathbb{R}^{n+1}}\nabla^\Sigma_\nu \psi}_{=0} \Big\|^2_{L^2(N)} \\
&= \Big\| \frac{n}{2}\, H\psi \Big\|^2_{L^2(N)} \\
&= \frac{n^2}{4} \int_N H^2 \cdot |\psi|^2 \, dvol.
\end{aligned}$$

Since ∇^Σ is metric, ∇^Σ-parallel sections have constant length. Thus, $|\psi(x)| = |\psi(x_0)|$ for an arbitrary $x_0 \in N$ and we obtain

$$\begin{aligned}
\big((D^N)^2\psi, \psi\big)_{L^2(N)} &= \frac{n^2}{4}\, |\psi(x_0)|^2 \int_N H^2 \, dvol \\
&= \frac{n^2}{4} \frac{\|\psi\|^2_{L^2(N)}}{\mathrm{vol}(N)} \int_N H^2 \, dvol.
\end{aligned}$$

Hence

$$\frac{\big((D^N)^2\psi,\psi\big)_{L^2(N)}}{\|\psi\|^2_{L^2(N)}} = \frac{n^2}{4}\frac{1}{\mathrm{vol}(N)}\int\limits_N H^2\,dvol.$$

The statement now follows from Lemma 2.6.2. ☐

Example 2.6.3. We consider the n-sphere $S^n \subset \mathbb{R}^{n+1}$ of radius 1. Then $H^2 \equiv 1$ and by Theorem 2.6.1, the first $2^{[\frac{n}{2}]}$ Dirac eigenvalues of S^n satisfy $\lambda^2 \leq \frac{n^2}{4}$.
The scalar curvature of S^n is $n(n-1)$. Thus, by Friedrich's inequality (2.37), we have

$$\lambda^2 \geq \frac{n}{n-1}\frac{n(n-1)}{4} = \frac{n^2}{4}.$$

Hence, for the sphere S^n, the first $2^{[\frac{n}{2}]}$ Dirac eigenvalues of S^n satisfy $\lambda^2 = \frac{n^2}{4}$. In particular, Friedrich's inequality cannot be improved in general.

Remark 2.6.4. For $n = 2$, the integral $\int\limits_N H^2\,dvol$ is called **Willmore energy**.

- If $N \cong S^2$, Theorem 2.5.17 and 2.6.1 yield

$$\frac{4\pi}{\mathrm{area}(N)} \leq \lambda^2 \leq \frac{1}{\mathrm{area}(N)}\int\limits_N H^2\,dA.$$

  Hence, the Willmore energy is bounded from below by 4π. Equality is attained if and only if the curvature of N is constant.

- For a torus $N \cong T^2$, the **Willmore conjecture** states that the Willmore energy is bounded from below by $2\pi^2$. This famous conjecture was open for a long time and finally proved by Marques and Neves in 2014, see [8].

3. The heat equation and index theory

3.1. The heat kernel

Throughout this chapter, let Δ be a formally self-adjoint Laplace-type operator, acting on sections of a Euclidean or Hermitian vector bundle E over a compact Riemannian manifold M. Similarly as in Theorem 1.4.18 one can show that the eigenspaces of Δ are finite-dimensional and that there exists an orthonormal basis of $L^2(M, E)$ consisting of eigensections of Δ. One can also show that an analogue of the elliptic estimates from Proposition 1.4.2 holds. For later purposes we need a lower estimate for the growth of the eigenvalues of Δ.

Proposition 3.1.1. *Let Δ be a self-adjoint Laplace-type operator on a compact Riemannian manifold M. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \nearrow +\infty$ be the eigenvalues of Δ, where each eigenvalue is repeated according to multiplicity. Then there exists a positive constant $c = c(M, \Delta)$ such that for all $k \in \mathbb{N}$, the following estimate holds:*

$$\lambda_k \geq c \cdot k^{\frac{4}{n(n+6)}} + \lambda_1 - 1. \tag{3.1}$$

Proof. a) Replacing Δ by $\Delta - \lambda_1 \cdot \mathrm{id}$ shifts the spectrum of Δ by λ_1. Hence we can assume w.l.o.g. that $\lambda_1 = 0$. Moreover, it suffices to prove the estimate for sufficiently large k. Then there are only finitely many values of λ_k left, for which the estimate (3.1) may not hold. This can be corrected by making the constant c smaller. In the limit $c \to 0$, the right hand side of (3.1) tends to -1, but $\lambda_k \geq 0$ for any k. Hence the right hand side can be made sufficiently small such that the estimate holds for all $k \in \mathbb{N}$.

Now let $\varepsilon > 0$. Choose a maximal $\frac{\varepsilon}{2}$-net in M, i.e., a set of points $\{p_1, \ldots, p_N\} \subset M$, such that

$$B\left(p_i, \frac{\varepsilon}{2}\right) \cap B\left(p_j, \frac{\varepsilon}{2}\right) = \emptyset, \quad \forall\, i \neq j,$$

and the number N of points satisfying this property is maximal. Then we have:

$$\bigcup_{i=1}^{N} B(p_i, \varepsilon) = M.$$

In fact, for any $x \in M$, there is an $i \in \{1, \ldots, N\}$ such that $B(x, \frac{\varepsilon}{2}) \cap B(p_i, \frac{\varepsilon}{2}) \neq \emptyset$. Otherwise, the set $\{p_1, \ldots, p_N, x\}$ would be another $\frac{\varepsilon}{2}$-net, in contradiction to the maximality of $\{p_1, \dots, p_N\}$. Now let $y \in B(x, \frac{\varepsilon}{2}) \cap B(p_i, \frac{\varepsilon}{2})$. By the triangle inequality, we have

$$\mathrm{dist}(p_i, x) \le \mathrm{dist}(p_i, y) + \mathrm{dist}(y, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $x \in B(p_i, \varepsilon)$.

For small radii, the Riemannian volume of balls in M can be estimated against the volume of the Euclidean balls: There is a constant $c_0 > 0$ such that $\mathrm{vol} B(p, r) \ge c_0 \cdot r^n$ holds for any point $p \in M$ and any radius $r > 0$ sufficiently small. We thus obtain a lower bound for the volume of M im terms of the maximal $\frac{\varepsilon}{2}$-net:

$$\mathrm{vol}(M) \ge \sum_{i=1}^{N} \mathrm{vol}\Big(B\Big(p_i, \frac{\varepsilon}{2}\Big)\Big) \ge \sum_{i=1}^{N} c_0 \cdot \Big(\frac{\varepsilon}{2}\Big)^n = N \cdot c_0 \cdot \Big(\frac{\varepsilon}{2}\Big)^n.$$

Hence, there is a constant $c_1 = c_1(M)$ such that for all $\varepsilon > 0$, we have:

$$N = N(\varepsilon) \le c_1 \cdot \varepsilon^{-n}.$$

b) Let $V \subset L^2(M, E)$ be the subspace spanned by the first k eigensections $\varphi_1, \dots, \varphi_k$ of Δ. Let $\varphi = \sum_{i=1}^{k} \alpha_i \varphi_i \in V$ with $\varphi(p_i) = 0$ for all $i = 1, \dots, N$. We want to estimate several norms of φ:

Given $x \in M$ choose p_i such that $x \in B(p_i, \varepsilon)$. Differentiation along a shortest geodesic from p_i to x yields

$$\begin{aligned}
|\varphi(x)| &= |\varphi(x)| - |\varphi(p_i)| \\
&= \int_0^1 \frac{d}{dt} \big|\varphi(\gamma(t))\big| \, dt \\
&= \int_0^1 \frac{d}{dt} \sqrt{\big\langle \varphi(\gamma(t)), \varphi(\gamma(t)) \big\rangle} \, dt \\
&= \int_0^1 \frac{\big\langle \nabla_{\dot\gamma}\varphi, \varphi(\gamma(t)) \big\rangle + \big\langle \varphi(\gamma(t)), \nabla_{\dot\gamma}\varphi \big\rangle}{2\big|\varphi(\gamma(t))\big|} \, dt \\
&\le \int_0^1 \frac{|\nabla_{\dot\gamma}\varphi| \cdot |\varphi(\gamma(t))|}{|\varphi(\gamma(t))|} \, dt \\
&\le \int_0^1 |\dot\gamma| \cdot |\nabla\varphi| \, dt \\
&\le \varepsilon \cdot \|\nabla\varphi\|_{C^0}.
\end{aligned}$$

In the last equality, we used the fact that $\int_0^1 |\dot\gamma|\,dt = L[\gamma] < \varepsilon$. Integration over M yields

$$\|\varphi\|_{L^2}^2 = \int_M |\varphi(x)|^2\, dvol \le \varepsilon^2 \cdot \|\nabla\varphi\|_{C^0}^2 \cdot \ \mathrm{vol}(M)$$

and thus

$$\|\varphi\|_{L^2} \le \varepsilon\sqrt{\mathrm{vol}(M)} \cdot \|\nabla\varphi\|_{C^0} \le \varepsilon\sqrt{\mathrm{vol}(M)} \cdot \|\varphi\|_{C^1}.$$

Let $\ell := \left[\frac{n}{2}\right] + 2$. By the Sobolev embedding theorem 1.2.13 there exists a constant $c_2 > 0$ such that

$$\|\varphi\|_{C^1} \le c_2 \cdot \|\varphi\|_{H^\ell}.$$

By the elliptic estimates (1.37), we have:

$$\begin{aligned}
\|\varphi\|_{H^\ell} &\le c_3 \cdot \left(\|\varphi\|_{L^2} + \left\|\Delta^{\left[\frac{\ell+1}{2}\right]}\varphi\right\|_{L^2}\right) \\
&\le c_3 \cdot \left(1 + \lambda_k^{\left[\frac{\ell+1}{2}\right]}\right) \cdot \|\varphi\|_{L^2} \\
&\le c_3 \cdot (1+\lambda_k)^{\left[\frac{\ell+1}{2}\right]} \cdot \|\varphi\|_{L^2} \\
&\le c_3 \cdot (1+\lambda_k)^{\frac{n+6}{4}} \cdot \|\varphi\|_{L^2}.
\end{aligned}$$

In the last inequality, we used the estimate $\left[\frac{\ell+1}{2}\right] = \left[\frac{1}{2}\left[\frac{n}{2}\right] + 1 + \frac{1}{2}\right] \le \frac{n}{4} + \frac{3}{2}$. Combining the above estimates we obtain

$$\begin{aligned}
\|\varphi\|_{L^2} &\le \varepsilon \cdot \sqrt{\mathrm{vol}(M)} \cdot c_2 \cdot c_3 \cdot (1+\lambda_k)^{\frac{n+6}{4}} \cdot \|\varphi\|_{L^2} \\
&= \varepsilon \cdot c_4 \cdot (1+\lambda_k)^{\frac{n+6}{4}} \cdot \|\varphi\|_{L^2}.
\end{aligned}$$

For $\varepsilon = \frac{1}{2c_4} \cdot (1+\lambda_k)^{-\frac{n+6}{4}}$ we conclude $\|\varphi\|_{L^2} \le \frac{1}{2}\|\varphi\|_{L^2}$, hence $\varphi \equiv 0$. Thus for such an ε the only section $\varphi \in V$ satisfying $\varphi(p_i) = 0$ is the section $\varphi \equiv 0$. Hence the linear mapping

$$\begin{array}{rcl}
V & \to & E_{p_1} \oplus \cdots \oplus E_{p_N} \\
\varphi & \mapsto & \big(\varphi(p_1), \ldots, \varphi(p_N)\big)
\end{array}$$

is injective. We thus obtain the estimate

$$\begin{aligned}
k = \dim V &\le \dim(E_{p_1} \oplus \ldots \oplus E_{p_N}) = N \cdot \mathrm{rk}\, E \\
&\le c_1 \cdot \varepsilon^{-n} \cdot \mathrm{rk}\, E \le c_5 \cdot (1+\lambda_k)^{\frac{n(n+6)}{4}}.
\end{aligned}$$

Hence, for the eigenvalues, we obtain the lower bound:

$$1 + \lambda_k \ge \left(\frac{k}{c_5}\right)^{\frac{4}{n(n+6)}} = c_6 \cdot k^{\frac{4}{n(n+6)}}. \qquad \square$$

In the following, we study the **heat equation**

$$\left(\frac{\partial}{\partial t} + \Delta\right)\varphi_t = 0,$$

where φ_t is a smooth section in E for each $t \geq 0$ and φ_t depends smoothly on t.

Remark 3.1.2.
Let Δ be a self-adjoint operator of Laplace-type. By Lemma 1.3.5 we may write $\Delta = (\nabla^E)^*\nabla^E + K$, where ∇^E is a metric connection and K is a symmetric endomorphism field.
The connection ∇^E on E induces a connection ∇^{E^*} on the dual bundle E^* via the requirement that the Leibniz rule

$$\partial_X\left(l(\varphi)\right) = \left(\nabla^{E^*}_X l\right)(\varphi) + l\left(\nabla^E_X\varphi\right) \tag{3.2}$$

holds for any section $l \in C^\infty(M, E^*)$ and any section $\varphi \in C^\infty(M, E)$. The endomorphism field K of E yields an endomorphism field K^* of E^*. Hence we obtain a Laplace-type operator

$$\Delta^{E^*} := \left(\nabla^{E^*}\right)^* \nabla^{E^*} + K^*$$

on E^*.
Let g denote the Riemannian metric on M and equip $M \times M$ with the product metric $g \oplus g$. It follows that the operator

$$\widetilde{\Delta} := \Delta \otimes \mathrm{id}_{E^*} + \mathrm{id}_E \otimes \Delta^{E^*}$$

is a formally self-adjoint Laplace-type operator on the Riemannian manifold $M \times M$.

For a smooth section $\varphi \in C^\infty(M, E)$ we define the section $\varphi^* \in C^\infty(M, E^*)$ by the requirement

$$\varphi^*(\psi) := \langle \psi, \varphi\rangle \qquad \forall\, \psi \in E.$$

Now, if $\varphi \in C^\infty(M, E)$ satisfies $\Delta\varphi = \lambda\varphi$ then the section φ^* satisfies $\Delta^{E^*}\varphi^* = \lambda\varphi^*$. Hence any orthonormal basis of $L^2(M, E)$ consisting of eigensections of Δ yields an orthonormal eigenbasis of $L^2(M, E^*)$ of eigensections of Δ^{E^*} with the same eigenvalues.
If $\varphi \in C^\infty(M, E)$ satisfies $\Delta\varphi = \lambda\varphi$ and $\psi \in C^\infty(M, E^*)$ satisfies $\Delta^{E^*}\psi = \mu\psi$ then we have:

$$\widetilde{\Delta}(\varphi \otimes \psi) = \Delta\varphi \otimes \psi + \varphi \otimes \Delta^{E^*}\psi = (\lambda + \mu)\varphi \otimes \psi.$$

Thus, an orthonormal basis $\{\varphi_k \,|\, k \in \mathbb{N}\}$ of $L^2(M, E)$ consisting of eigensections of Δ yields an orthonormal basis of $L^2(M \times M, E \boxtimes E^*)$, consisting of the eigensections

$$(x, y) \mapsto \left(\varphi_k(x) \otimes \varphi_l^*(y)\right), \quad (k, l) \in \mathbb{N} \times \mathbb{N},$$

of $\widetilde{\Delta}$.

Definition 3.1.3. Let M be a compact Riemannian manifold, let $E \to M$ be a Riemannian or Hermitian vector bundle, and let Δ be a self-adjoint Laplace-type operator on E. Let $\{\varphi_j \,|\, j \in \mathbb{N}\}$ be an orthonormal basis of $L^2(M,E)$ consisting of eigensections of Δ. The series

$$k_t(x,y) := \sum_{j=1}^{\infty} e^{-t\lambda_j}\varphi_j(x) \otimes \varphi_j^*(y), \tag{3.3}$$

where $x, y \in M$, $t > 0$, is called the **(true) heat kernel** of Δ on M.

Proposition 3.1.4. *Let $t_0 > 0$. Then the heat kernel and all its t-derivatives converge uniformly in $t \geq t_0$ in all H^k-norms and all C^k-norms. In particular, $k_t(x,y)$ is smooth in t, x, and y, and we can differentiate the series termwise.*

Proof.
a) In view of the Sobolev embedding theorem 1.2.13 it is sufficient to prove the proposition for the H^k-norms. All but finitely many λ_j satisfy $\lambda_j \geq 1$. Thus by splitting a finite part from the series if necessary, we may assume that $\lambda_j \geq 1$.
By the elliptic estimates (1.37) for $\widetilde{\Delta}$, we then have:

$$\begin{aligned}
\left\| e^{-t\lambda_j}\varphi_j \otimes \varphi_j^* \right\|_{H^{2k}} &\leq c_1 \cdot e^{-t\lambda_j} \cdot \left(\|\varphi_j \otimes \varphi_j^*\|_{L^2} + \|\widetilde{\Delta}^k(\varphi_j \otimes \varphi_j^*)\|_{L^2} \right) \\
&= c_1 \cdot e^{-t\lambda_j} \cdot \left(1 + (2\lambda_j)^k\right) \\
&\leq c_2 \cdot e^{-t\lambda_j} \cdot \lambda_j^k \qquad\qquad \text{(since } \lambda_j \geq 1) \\
&\leq c_2 \cdot e^{-t_0\lambda_j} \cdot \lambda_j^k .
\end{aligned}$$

For λ sufficiently large, we have $e^{-t_0\frac{\lambda}{2}} \cdot \lambda^k \leq 1$. Thus for $j \gg 0$, we obtain:

$$\left\| e^{-t\lambda_j}\varphi_j \otimes \varphi_j^* \right\|_{H^{2k}} \leq c_2 \cdot e^{-t_0\frac{\lambda_j}{2}} .$$

b) By Proposition 3.1.1 we have $\lambda_j \geq c_3 \cdot j^\alpha + c_4$, where $\alpha = \frac{4}{n(n+6)}$, and therefore

$$\left\| e^{-t\lambda_j}\varphi_j \otimes \varphi_j^* \right\|_{H^{2k}} \leq c_2 \cdot e^{-\frac{t_0 c_3 \cdot j^\alpha}{2} - \frac{t_0 c_4}{2}} \leq c_5 \cdot e^{-c_6 \cdot j^\alpha} .$$

The series $\sum\limits_{j=1}^{\infty} e^{-c_6 \cdot j^\alpha}$ converges, since we have:

$$\sum_{j=1}^{\infty} e^{-c_6 \cdot j^\alpha} \leq \int_0^\infty e^{-c_6 \cdot t^\alpha} dt = c_7 \cdot \int_0^\infty e^{-s} \cdot s^{\frac{1-\alpha}{\alpha}} ds = c_7 \cdot \Gamma\left(\frac{1}{\alpha}\right) < \infty .$$

Thus, we have shown that the series

$$\sum_{j=1}^{\infty} e^{-t\lambda_j} \varphi_j \otimes \varphi_j^*$$

converges in each H^k-norm, uniformly in $t \geq t_0$.

c) The same argument applies to the t-derivatives

$$\left(\frac{d}{dt}\right)^m \left(e^{-t\lambda_j} \varphi_j \otimes \varphi_j^*\right) = (-\lambda_j)^m e^{-t\lambda_j} \varphi_j \otimes \varphi_j^*.$$

Hence the series (3.3), together with all its t-dervivatives, converges in any H^k-norm and consequently in any C^k-norm. The series thus defines a family of smooth sections $k_t \in C^\infty(M \times M, E \boxtimes E^*)$. The family is also smooth in t, and all derivatives can be computed termwise. □

Since the heat kernel can be differentiated termwise, we compute for a fixed $y \in M$:

$$\begin{aligned}
\frac{\partial}{\partial t} k_t(x,y) &= \frac{\partial}{\partial t} \sum_{j=1}^{\infty} e^{-t\lambda_j}\, \varphi_j(x) \otimes \varphi_j^*(y) \\
&= \sum_{j=1}^{\infty} \frac{\partial}{\partial t} e^{-t\lambda_j}\, \varphi_j(x) \otimes \varphi_j^*(y) \\
&= \sum_{j=1}^{\infty} (-\lambda_j)\, e^{-t\lambda_j}\, \varphi_j(x) \otimes \varphi_j^*(y) \\
&= -\sum_{j=1}^{\infty} e^{-t\lambda_j}\, (\Delta\varphi_j)\,(x) \otimes \varphi_j^*(y) \\
&= -(\Delta_x k_t)(x,y).
\end{aligned}$$

Thus the heat kernel satisfies the heat equation $\left(\frac{\partial}{\partial t} + \Delta_x\right) k_t(x,y) = 0$.

For any $\varphi \in L^2(M,E)$, we have:

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) \int_M k_t(x,y)\, \varphi(y)\, dvol(y) = \int_M \left(\frac{\partial}{\partial t} + \Delta_x\right) k_t(x,y)\, \varphi(y)\, dvol(y) = 0.$$

Thus the section

$$x \mapsto \int_M k_t(x,y)\, \varphi(y)\, dvol(y)$$

solves the heat equation. Moreover, the map

$$\varphi \mapsto \int_M k_t(\cdot, y)\, \varphi(y)\, dvol(y)$$

is a bounded operator on $L^2(M, E)$. Applying this operator to an eigensection φ_k from the orthonormal basis, we obtain:

$$\begin{aligned}
\int_M k_t(x,y)\, \varphi_k(y)\, dvol(y) &= \int_M \sum_{j=1}^{\infty} e^{-t\lambda_j} \left(\varphi_j(x) \otimes \varphi_j^*(y)\right) \cdot \varphi_k(y)\, dvol(y) \\
&= \sum_{j=1}^{\infty} e^{-t\lambda_j}\, \varphi_j(x) \cdot \left(\varphi_j, \varphi_k\right)_{L^2} \\
&= e^{-t\lambda_k}\, \varphi_k(x).
\end{aligned}$$

Thus, the operator $\varphi \mapsto \int_M k_t(\cdot, y)\, \varphi(y)\, dvol(y)$ coincides with the operator $e^{-t\Delta}$, defined by the functional calculus. In other words, the heat kernel $k_t(x, y)$ is the integral kernel of the operator $e^{-t\Delta}$.
As $t \searrow 0$ the heat kernel becomes singular. Indeed, since $e^{-0\cdot\Delta} = \mathrm{id}$, we expect the heat kernel to concentrate along the diagonal $\{(y, y) \in M \times M \mid y \in M\}$.

Next we want to examine the asymptotic behavior of $k_t(x, y)$ for $t \searrow 0$.

3.2. The formal heat kernel

Definition 3.2.1. We define

$$M \bowtie M := \{(x, y) \in M \times M \,|\, y \text{ is not a cut point of } x\}.$$

Remark 3.2.2. $M \bowtie M$ is an open dense subset of $M \times M$, containing the diagonal $\{(x, x) \in M \times M \,|\, x \in M\}$.

Definition 3.2.3. Let M be a connected Riemannian manifold of dimension n. The **Euclidean heat kernel** of M is the function $q_t : M \times M \to \mathbb{R}$, defined by

$$q_t(x, y) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{\mathrm{dist}(x, y)^2}{4t}\right).$$

Remark 3.2.4. The map

$$\begin{aligned} M \times M \times (0,\infty) &\to \mathbb{R} \\ (x,y,t) &\mapsto q_t(x,y) \end{aligned}$$

is continuous but it is smooth only on $M \bowtie M \times (0,\infty)$.

Lemma 3.2.5. *Let M be a connected Riemannian manifold, and let Δ_0 be the Laplace-Beltrami operator on functions. Then the Euclidean heat kernel satisfies*

$$\left(\frac{\partial}{\partial t} + \Delta_{0,x}\right) q_t(x,y) = \frac{a(x,y)}{t} \cdot q_t(x,y),$$

where a is smooth on $(M \bowtie M)$ and it vanishes along the diagonal, i.e., $a(x,x) = 0$ for all $x \in M$.
Moreover, in geodesic polar coordinates around y we have

$$a(x,y) = \frac{r}{2}\frac{d}{dr}\Big(\ln\det\big(d\exp_y(rX)\big)\Big),$$

where $\exp_y : T_yM \to M$ denotes the Riemannian exponential map, $x = \exp_y(rX)$ and $X \in T_yM$ with $\|X\| = 1$.

Thus the function a in the Lemma is essentially given by the radial logarithmic derivative of the volume distortion of the exponential map.

Proof. Fix a point $y \in M$. We express the operator Δ_0 in geodesic polar coordinates about y:

$$\Delta_0 = \Delta^{S_r} - \frac{\partial^2}{\partial r^2} + (n-1)\cdot H \cdot \frac{\partial}{\partial r}\,.$$

Here $S_r := \{x \in M \mid \operatorname{dist}(x,y) = r\}$ denotes the distance sphere of radius r, and H denotes its mean curvature with respect to the unit normal $\frac{\partial}{\partial r}$.
A direct calculation yields

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_{0,x}\right) q_t &= \underbrace{\left(\frac{\partial}{\partial t} + \Delta^{S_r} - \frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\cdot\frac{\partial}{\partial r}\right) q_t}_{=0} + (n-1)\left(\frac{1}{r} + H\right)\frac{\partial q_t}{\partial r} \\ &= (n-1)\left(\frac{1}{r} + H\right)\frac{\partial}{\partial r}\left((4\pi t)^{-\frac{n}{2}}\exp\left(-\frac{r^2}{4t}\right)\right) \\ &= (n-1)\left(\frac{1}{r} + H\right)\left(-\frac{r}{2t}\right) q_t \\ &= -(n-1)\,\frac{1+Hr}{2t}\cdot q_t. \end{aligned}$$

Hence $a(x,y) = -\frac{n-1}{2}(1+Hr)$.

In order to compute this term we fix $X \in T_yM$ with $\| X \|= 1$, and consider the unit speed geodesic $c(r) = \exp_y(rX)$ emanating from y in direction X. Let $e_1 = X, e_2, \ldots, e_n$ be an orthonormal basis of T_yM. Let V_i be the Jacobi field along c determined by the initial conditions $V_i(0) = 0$ and $\frac{\nabla}{dr}V_i(0) = e_i$ for $i = 1, \ldots, n$. It is well-known that the differential of the exponential map at the point rX is given by

$$d\exp_y(rX)(e_i) = \frac{1}{r}\, V_i(r).$$

(see e. g. Proposition 3.4.13 in [1]). It follows that

$$\left(\frac{\nabla}{dr}\, d\exp_y(rX)\right)(e_i) = -\frac{1}{r^2}V_i(r) + \frac{1}{r}\frac{\nabla}{dr}V_i(r).$$

Since $V_1(r) = rc'(r)$, we have

$$\left(\frac{\nabla}{dr}\, d\exp_y(rX)\right)(e_1) = 0.$$

For $i = 2, \ldots, n$, we have $\frac{\nabla}{dr}V_i(r) = -B(V_i(r))$, where B denotes the Weingarten map of S_r. We thus obtain

$$\begin{aligned}\left(\frac{\nabla}{dr}\, d\exp_y(rX)\right)(e_i) &= \left(-\frac{1}{r^2}\,\mathrm{id} - \frac{1}{r}B\right) V_i(r)\\ &= \left(-\frac{1}{r}\,\mathrm{id} - B\right) d\exp_y(rX)(e_i), \qquad i = 2, \ldots, n.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dr}\det\Big(d\exp_y(rX)\Big) &= \det\Big(d\exp_y(rX)\Big)\,\mathrm{tr}\left(\left(\frac{\nabla}{dr}\, d\exp_y(rX)\right)\cdot\Big(d\exp_y(rX)\Big)^{-1}\right)\\ &= \det\Big(d\exp_y(rX)\Big)\,\mathrm{tr}\left(-\frac{1}{r}\,\mathrm{id}_{X^\perp} - B\right)\\ &= \det\Big(d\exp_y(rX)\Big)\left(-\frac{n-1}{r} - (n-1)H\right)\\ &= \frac{2}{r}\cdot\det\Big(d\exp_y(rX)\Big)\cdot a.\end{aligned}$$

Hence

$$\begin{aligned}a(r) &= \frac{r}{2}\,\det\Big(d\exp_y(rX)\Big)^{-1}\cdot\frac{d}{dr}\det\Big(d\exp_y(rX)\Big)\\ &= \frac{r}{2}\frac{d}{dr}\ln\det\big(d\exp_y(rX)\big). \qquad (3.4)\end{aligned}$$

$\square$

Definition 3.2.6. Let M be a Riemannian manifold with Euclidean heat kernel q_t. Let Δ be a formally self-adjoint Laplace-type operator, acting on sections of a vector bundle E over M. A formal series of the form

$$\widetilde{k}_t(x,y) = q_t(x,y) \cdot \sum_{j=0}^{\infty} t^j \cdot \Phi_j(x,y)$$

with $\Phi_j \in C^\infty(M \bowtie M, E \boxtimes E^*)$ is called a **formal heat kernel** for Δ if for each $N \in \mathbb{N}$ there exists an m_0 such that for all $m \geq m_0$ we have as $t \searrow 0$:

$$\left(\frac{\partial}{\partial t} + \Delta_x\right)\left\{q_t(x,y) \cdot \sum_{j=0}^{m} t^j \cdot \Phi_j(x,y)\right\} = q_t(x,y) \cdot O\big(t^N\big).$$

Proposition 3.2.7. *Let M be a connected Riemannian manifold, and let Δ be a formally self-adjoint Laplace-type operator, acting on sections of a vector bundle E over M. Then there exists a unique formal heat kernel $\widetilde{k}_t$ for Δ, satisfying*

$$\Phi_0(x,x) = \mathrm{id}_{E_x}, \qquad \forall x \in M.$$

Proof. a) We first show uniqueness of the Φ_j. To do this we differentiate the formal series $\widetilde{k}_t(x,y)$ term by term, order the result by powers of t and equate the resulting coefficients to zero. We use the Leibniz rule for the Laplacian

$$\Delta(f \cdot \varphi) = (\Delta_0 f) \cdot \varphi - 2\nabla_{\mathrm{grad} f}\varphi + f\Delta\varphi \tag{3.5}$$

where f is a function, φ a section in E and $\Delta = \nabla^*\nabla + K$ for some endomorphism field K. Now we fix $y \in M$ and set $r(x) := \mathrm{dist}(x,y)$, as before. We compute:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + \Delta_x\right)\widetilde{k}_t \\
&= \left[\left(\frac{\partial}{\partial t} + \Delta_{0,x}\right) q_t\right] \cdot \sum_{j=0}^{\infty} t^j \Phi_j - 2\sum_{j=0}^{\infty} t^j\, \nabla_{\mathrm{grad}_x q_t} \Phi_j + q_t \sum_{j=0}^{\infty} \left(\frac{\partial}{\partial t} + \Delta_x\right) t^j \Phi_j \\
&= \frac{a}{t}\, q_t \cdot \sum_{j=0}^{\infty} t^j \Phi_j + \frac{r}{t} \cdot q_t \sum_{j=0}^{\infty} t^j \left(\nabla_{\frac{\partial}{\partial r}} \Phi_j\right) + q_t \cdot \sum_{j=0}^{\infty} j\, t^{j-1} \Phi_j + q_t \cdot \sum_{j=0}^{\infty} t^j (\Delta_x \Phi_j) \\
&= q_t \cdot \sum_{j=-1}^{\infty} t^j \cdot \left\{a \cdot \Phi_{j+1} + (j+1)\Phi_{j+1} + r\, \nabla_{\frac{\partial}{\partial r}} \Phi_{j+1} + \Delta_x \Phi_j\right\}
\end{aligned}$$

The last equation holds with the convention that $\Phi_{-1} := 0$.

Hence, $\widetilde{k}_t$ is a formal heat kernel, if and only if

$$a \cdot \Phi_{j+1} + (j+1)\Phi_{j+1} + r\,\nabla_{\frac{\partial}{\partial r}}\Phi_{j+1} + \Delta_x \Phi_j = 0.$$

Thus, along any unit speed geodesic $c(r) = \exp_y(rX)$ emanating from y we obtain the following singular ordinary differential equations $\big(\Phi_j(r) := \Phi_j(\exp_y(rX), y)\big)$:

$$r\,\frac{\nabla}{dr}\Phi_{j+1}(r) + \big(a(r) + j + 1\big)\Phi_{j+1}(r) = -\left(\Delta_x \Phi_j\right)(r). \tag{3.6}$$

This equation is called a **transport equation**. To solve it we introduce the **integrating factor**

$$R_j(r) := r^{j+1} \cdot \exp\left(\int_0^r \frac{a(\rho)}{\rho}\,d\rho\right).$$

We rewrite (3.6) as

$$\begin{aligned}
-\left(\Delta_x \Phi_j\right)(r) &= \frac{r}{R_j(r)} \cdot \Bigg\{ R_j(r)\,\frac{\nabla}{dr}\Phi_{j+1}(r) + R_j(r) \cdot \frac{a(r)}{r} \cdot \Phi_{j+1}(r) \\
&\qquad\qquad + \frac{j+1}{r}\,R_j(r)\,\Phi_{j+1}(r) \Bigg\} \\
&= \frac{r}{R_j(r)} \cdot \frac{\nabla}{dr}\Big(R_j(r)\,\Phi_{j+1}(r)\Big)
\end{aligned}$$

We denote the parallel translation along c from $c(r)$ to $c(0) = y$ by π_r. Then we obtain the ordinary differential equation:

$$\frac{d}{dr}\big(R_j \cdot \pi_r \circ \Phi_{j+1}\big) = -\frac{R_j}{r} \cdot \pi_r \circ \Delta_x \Phi_j$$

This equation can integrated to obtain

$$R_j(r) \cdot \pi_r \circ \Phi_{j+1}(r) - R_j(0)\Phi_{j+1}(0) = -\int_0^r \frac{R_j(\rho)}{\rho} \cdot \pi_\rho \left(\Delta_x \Phi_j\right)(\rho)\,d\rho. \tag{3.7}$$

Evaluating this equation for $j = -1$ yields $R_{-1}(r) \cdot \pi_r \circ \Phi_0(r) - \Phi_0(0) = 0$, i.e.

$$\Phi_0(r) = \exp\left(-\int_0^r \frac{a(\rho)}{\rho}\,d\rho\right)\pi_r^{-1} \circ \mathrm{id}_{E_y}$$

i.e.

$$\Phi_0(x,y) = \exp\left(-\int_0^r \frac{a(\rho)}{\rho}\,d\rho\right)\pi_{y,x}\,, \qquad r = d(x,y), \tag{3.8}$$

where $\pi_{y,x}$ denotes parallel translation from y to x (along the unique shortest geodesic connecting y and x).

For $j \geq 0$, equation (3.7) yields:

$$R_j(r) \cdot \pi_r \circ \Phi_{j+1}(r) = -\int_0^r \frac{R_j(\rho)}{\rho} \cdot \pi_\rho \circ (\Delta_x \Phi_j)(\rho)\, d\rho.$$

Hence,

$$\Phi_{j+1}(r) = -\frac{1}{R_j(r)} \cdot \pi_r^{-1} \circ \int_0^r \frac{R_j(\rho)}{\rho} \cdot \pi_\rho \circ (\Delta_x \Phi_j)(\rho)\, d\rho.$$

This way we can recursively determine the Φ_j and uniqueness is proven.

b) For the existence part simply use the above equations to define the Φ_j recursively. □

Now we compute the coefficient $\Phi_1(x,x)$ by use of the transport equation (3.6). For $j = 0$ and $x = y$, we obtain

$$\big(\underbrace{a(x,x)}_{=0} + 1\big)\Phi_1(x,x) = -(\Delta_x \Phi_0)(x,x).$$

By assumption we have $\Phi_0(x,x) = \mathrm{id}_{E_x}$. Using equations (3.8) and (3.4), we find:

$$\begin{aligned}
\Phi_0(x,y) &\overset{(3.8)}{=} \exp\left(-\int_0^r \frac{a(\rho)}{\rho}\, d\rho\right) \pi_{y,x}\,, \qquad \text{where } r = d(x,y). \\
&\overset{(3.4)}{=} \det\Big(d(\exp_y^{-1})(x)\Big)^{\frac{1}{2}} \cdot \pi_{y,x} \\
&= \Big(\det g_{ij}^{(y)}(x)\Big)^{-\frac{1}{4}} \pi_{y,x} \\
&=: \mu_y(x)\, \pi_{y,x},
\end{aligned}$$

where $g_{ij}^{(y)}$ denotes the coefficients of the metric g, expressed in Riemannian normal coordinates around the point y.

Thus,

$$\begin{aligned}
\Phi_1(x,x) &= -(\Delta_x \Phi_0)(x,x) \\
&= -\Delta_x \big(\mu_y(x)\, \pi_{y,x}\big)\big|_{y=x}.
\end{aligned}$$

We use the Leibniz rule (3.5) for $\Delta = \nabla^* \nabla + K$ to obtain

$$\begin{aligned}
\Phi_1(x,x) &= -\Big(\Delta_{0,x}\big(\mu_y(x)\big)\pi_{y,x} + \underbrace{\mu_y(x)}_{\substack{=1 \\ \text{for } y=x}} \cdot K \circ \pi_{y,x}\Big)\Big|_{y=x} \\
&= -\Delta_{0,x}\big(\mu_y(x)\big) \cdot \mathrm{id}_{E_x} - K(x)
\end{aligned}$$

In Riemannian normal coordinates around $y = 0$, the Riemannian metric has the Taylor expansion

$$g_{ij}(x) = \delta_{ij} + \frac{1}{3}\sum_{k,l=1}^{n} R_{ikjl}(0)\, x^k x^l + O\big(\|x\|^3\big).$$

This yields

$$\begin{aligned}
\det\big(g_{ij}(x)\big)^{-\frac{1}{4}} &= \left(1 + \operatorname{tr}\left[\frac{1}{3}\sum_{k,l=1}^{n} R_{ikjl}(0)\, x^k x^l + O\big(\|x\|^3\big)\right] + O\big(\|x\|^4\big)\right)^{-\frac{1}{4}} \\
&= \left(1 + \frac{1}{3}\sum_{j,k,l=1}^{n} R_{jkjl}(0)\, x^k x^l + O\big(\|x\|^3\big)\right)^{-\frac{1}{4}} \\
&= \left(1 - \frac{1}{3}\sum_{k,l=1}^{n} \mathrm{ric}_{kl}(0)\, x^k x^l + O\big(\|x\|^3\big)\right)^{-\frac{1}{4}} \\
&= 1 + \frac{1}{12}\sum_{k,l=1}^{n} \mathrm{ric}_{kl}(0) x^k x^l + O\big(\|x\|^3\big)
\end{aligned}$$

and therefore

$$-\Delta_{0,x}(\mu_y)|_{x=y} = \frac{1}{6}\sum_{k} \mathrm{ric}_{kk}(0) = \frac{1}{6}\,\mathrm{scal}(0).$$

Thus,

$$\Phi_1(x,x) = \frac{1}{6}\,\mathrm{scal}(x)\cdot \mathrm{id}_{E_x} - K(x). \tag{3.9}$$

Examples 3.2.8

1) Let E be the trivial real line bundle and consider $\Delta = \Delta_{0,x}$. Then $K = 0$ and

$$\Phi_1(x,x) = \frac{1}{6}\,\mathrm{scal}(x).$$

2) Consider $E = T^*M$ and the Hodge-Laplacian $\Delta = \Delta_1$. By the Bochner formula (1.17), we have $K = \mathrm{Ric}$ and thus

$$\Phi_1(x,x) = \frac{1}{6}\,\mathrm{scal}(x)\cdot \mathrm{id}_{T_x^*M} - \mathrm{Ric}_x.$$

Moreover, we have

$$\operatorname{tr}\Phi_1(x,x) = \frac{n}{6}\,\mathrm{scal}(x) - \mathrm{scal}(x) = \frac{n-6}{6}\,\mathrm{scal}(x).$$

3) Let M be an n-dimensional oriented Riemannian manifold. Let $\Delta = \Delta_n$ be the Laplacian on n-forms and denote by $*$ the Hodge operator. Then we have

$$\Delta_n = dd^* + d^*d = d(-*d*) = -*^2 (d*d*) = *(\underbrace{-*d*}_{=d^*}d*) = *(d^*d)*$$
$$= *\Delta_0* = *\Delta_0*^{-1},$$

the Laplacians Δ_n and Δ_0 are conjugate operators. Thus,

$$\widetilde{k}_t^{\Delta_n} = * \circ \widetilde{k}_t^{\Delta_0} \circ *^{-1}$$

and for Φ_1 of Δ_n, we have:

$$\Phi_1(x,x) = \frac{1}{6}\operatorname{scal}(x) \cdot \mathrm{id}_{\Lambda^n T_x^* M}\,.$$

4) In a similar way, one can show that Δ_{n-k} and Δ_k are conjugate operators, hence

$$\Phi_j^{\Delta_{n-k}}(x,y) = * \circ \Phi_j^{\Delta_k} \circ *^{-1}$$

and thus

$$\operatorname{tr}\left(\Phi_j^{\Delta_{n-k}}(x,x)\right) = \operatorname{tr}\left(\Phi_j^{\Delta_k}(x,x)\right).$$

5) Let $\Delta = D^2 = \left(\nabla^\Sigma\right)^* \nabla^\Sigma + \frac{\operatorname{scal}}{4} \cdot \mathrm{id}_{\Sigma M}$ be the square of the classical Dirac operator. Then we have

$$\begin{aligned}\Phi_1(x,x) &= \frac{1}{6}\operatorname{scal}(x) \cdot \mathrm{id}_{\Sigma_x M} - \frac{1}{4}\operatorname{scal}(x) \cdot \mathrm{id}_{\Sigma_x M} \\ &= -\frac{1}{12}\operatorname{scal}(x) \cdot \mathrm{id}_{\Sigma_x M}\,.\end{aligned}$$

Now we discuss the relation of the formal heat kernel $\widetilde{k}_t$ to the true heat kernel k_t: Let M be a compact Riemannian manifold. There exists $\varepsilon_0 > 0$ such that

$$\{(x,y) \in M \times M \,|\, \operatorname{dist}(x,y) \le \varepsilon_0\} \subset M \bowtie M,$$

for example we can take ε_0 to be the injectivity radius of M. Pick a smooth cut-off function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(t) = \begin{cases} 1 & \text{for } t \le \frac{\varepsilon_0}{3}, \\ 0 & \text{for } t > \frac{2\varepsilon_0}{3}, \end{cases}$$

and $0 \le \chi \le 1$ everywhere. We define

$$\widehat{k}_t(x,y) := \chi\big(\operatorname{dist}(x,y)\big) \cdot \widetilde{k}_t(x,y) = \chi\big(\operatorname{dist}(x,y)\big) \cdot q_t(x,y) \cdot \sum_{j=0}^{\infty} t^j\, \Phi_j(x,y).$$

Hence $\widehat{k}_t$ coincides with the formal heat kernel $\widetilde{k}_t$ on a neighborhood of the diagonal. Moreover, the finite partial sums

$$\widehat{k}_t^{(m)}(x,y) := \chi(\mathrm{dist}(x,y)) \cdot q_t(x,y) \cdot \sum_{j=0}^{m} t^j \, \Phi_j(x,y)$$

are defined and smooth on all of $M \times M$.
We show that $\widehat{k}_t$ is asymptotic to the true heat kernel k_t as $t \searrow 0$:

Proposition 3.2.9. *For every $N \in \mathbb{N}$ and every $t_0 > 0$ there exist an $m_0 \in \mathbb{N}$ and a constant $C = C(N, m_0) > 0$, such that for all $m \geq m_0$, we have:*

$$\left| k_t(x,y) - \widehat{k}_t^{(m)}(x,y) \right| \leq C \cdot t^N, \qquad \forall\, t \in (0,t_0), \forall\, x,y \in M.$$

Proof. a) Denote by ε_0 the injectivity radius of M. Let $\varphi \in C^0(M,E)$ supported in a ball of radius $\frac{2\varepsilon_0}{3}$ around $x \in M$, so that we can use Riemannian normal coordinates around x. Let $\chi : \mathbb{R} \to \mathbb{R}$ be defined as above. The Euclidean heat kernel q_t on Euclidean $\mathbb{R}^n$ satisfies for all $f \in C^0(\mathbb{R}^n \times \mathbb{R}^n)$ and all $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} q_t(x,y) f(x,y)\, dy = f(x,x).$$

It follows that

$$\begin{aligned}
&\lim_{t \searrow 0} \int_M q_t(x,y)\, \Phi_0(x,y)\, \chi\big(\mathrm{dist}(x,y)\big) \varphi(y)\, dvol(y) \\
&\quad = \lim_{t \searrow 0} \int_{B\left(0, \frac{2\varepsilon_0}{3}\right)} (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{r^2}{4t}}\, (\det g_{ij}^{(y)}(x))^{-\frac{1}{4}} \cdot \big(\pi_{y,x}\varphi(y)\big) \cdot (\det g_{ij}^{(x)}(y))^{\frac{1}{2}} \chi\big(\mathrm{dist}(x,y)\big)\, dy_1 \dots dy_n \\
&\quad = \lim_{t \searrow 0} \int_{\mathbb{R}^n} (4\pi t)^{-\frac{n}{2}}\, e^{-\frac{r^2}{4t}} \cdot \big(\pi_{y,x}\varphi(y)\big) \cdot (\det g_{ij}^{(y)}(x))^{-\frac{1}{4}} (\det g_{ij}^{(x)}(y))^{\frac{1}{2}} \chi\big(\mathrm{dist}(x,y)\big)\, dy_1 \dots dy_n \\
&\quad = \pi_{x,x}\, \varphi(x) \\
&\quad = \varphi(x),
\end{aligned}$$

where $r(y) = \mathrm{dist}(x,y)$. For arbitrary $\varphi \in C^0(M,E)$ write

$$\varphi(y) = \chi\big(\mathrm{dist}(x,y)\big) \cdot \varphi(y) + \big(1 - \chi\big(\mathrm{dist}(x,y)\big)\big) \cdot \varphi(y).$$

Since $1-\chi(\mathrm{dist}(x,y))=0$ on $B(x,\frac{\varepsilon_0}{3})$ we have

$$\begin{aligned}
&\lim_{t\searrow 0}\int_M q_t(x,y)\,\Phi_0(x,y)\,\chi\big(\mathrm{dist}(x,y)\big)\varphi(y)\,dvol(y)\\
&\quad=\lim_{t\searrow 0}\int_M q_t(x,y)\,\Phi_0(x,y)\,\chi\big(\mathrm{dist}(x,y)\big)^2\varphi(y)\,dvol(y)\\
&\qquad+\lim_{t\searrow 0}\int_{M\setminus B\left(x,\frac{\varepsilon_0}{3}\right)} q_t(x,y)\,\Phi_0(x,y)\,\chi\big(\mathrm{dist}(x,y)\big)\big(1-\chi\big(\mathrm{dist}(x,y)\big)\big)\varphi(y)\,dvol(y)\\
&\quad=\varphi(x)
\end{aligned}$$

since as $t\searrow 0$ we have $q_t(x,y)\to 0$ on $M\setminus B(x,\frac{\varepsilon_0}{3})$ uniformly in y. Thus we get for all $m\in\mathbb{N}$ and $\varphi\in C^0(M,E)$

$$\lim_{t\searrow 0}\int_M \widehat{k}_t^{(m)}(x,y)\varphi(y)\,dvol(y)=\varphi(x).$$

On the other hand, since $e^{-t\Delta}\varphi\to\varphi$ in $L^2(M,E)$ as $t\searrow 0$, we also have

$$\lim_{t\searrow 0}\int_M k_t(x,y)\varphi(y)\,dvol(y)=\varphi(x).$$

Hence, for all $\varphi\in C^0(M,E)$ and for all $m\in\mathbb{N}$

$$\lim_{t\searrow 0}\int_M \Big(k_t-\widehat{k}_t^{(m)}\Big)(x,y)\,\varphi(y)\,dvol(y)=0.$$

b) Define $\delta_t^{(m)}:=k_t-\widehat{k}_t^{(m)}$. Then we have

$$\begin{aligned}
\eta_t^{(m)}&:=\left(\frac{\partial}{\partial t}+\Delta_x\right)\delta_t^{(m)}\\
&=-\left(\frac{\partial}{\partial t}+\Delta_x\right)\widehat{k}_t^{(m)}\\
&=-\chi\cdot\left(\frac{\partial}{\partial t}+\Delta_x\right)\widetilde{k}_t^{(m)}+\underbrace{2\nabla_{\mathrm{grad}_x\chi}\widetilde{k}_t^{(m)}-(\Delta_{0,x}\chi)\cdot\widetilde{k}_t^{(m)}}_{=:R_t^{(m)}}.
\end{aligned}$$

By Definition 3.2.6 of the formal heat kernel, we have:

$$\left(\frac{\partial}{\partial t}+\Delta_x\right)\widetilde{k}_t^{(m)}=q_t\cdot O\big(t^N\big).$$

The term $R_t^{(m)}$ is of the form q_t times a smooth section vanishing for $r < \frac{\varepsilon_0}{3}$. For $r \geq \frac{\varepsilon_0}{3}$ we have

$$\frac{q_t(x,y)}{q_{2t}(x,y)} = \frac{(4\pi\, t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{4t}\right)}{(8\pi\, t)^{-\frac{n}{2}} \exp\left(-\frac{r^2}{8t}\right)} = C \cdot \exp\left(-\frac{r^2}{8t}\right) \leq C \cdot \exp\left(-\frac{C'}{t}\right).$$

This yields the estimate

$$q_t(x,y) \leq C \cdot \exp\left(-\frac{C'}{t}\right) \cdot q_{2t}(x,y) \qquad (\text{for } r \geq \tfrac{\varepsilon_0}{3})$$

for suitable constants $C, C' > 0$. For the remainder terms we thus find

$$R_t^{(m)} = q_{2t} \cdot O\big(t^N\big).$$

Hence we get

$$\eta_t^{(m)} = q_t \cdot O\big(t^N\big) + q_{2t} \cdot O\big(t^N\big). \tag{3.10}$$

c) Now define $\widetilde{\delta}_t^{(m)} := \int\limits_0^t e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\, d\tau$. Then we have:

$$\frac{\partial}{\partial t}\, \widetilde{\delta}_t^{(m)} = e^{-(t-t)\Delta_x}\, \eta_t^{(m)} + \int\limits_0^t -\Delta_x e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\, d\tau = \eta_t^{(m)} - \Delta_x \widetilde{\delta}_t^{(m)}.$$

Therefore

$$\left(\frac{\partial}{\partial t} + \Delta_x\right) \widetilde{\delta}_t^{(m)} = \eta_t^{(m)} \quad \text{and} \quad \left(\frac{\partial}{\partial t} + \Delta_x\right) \left(\widetilde{\delta}_t^{(m)} - \delta_t^{(m)}\right) = 0.$$

Since $\widetilde{\delta}_t^{(m)} - \delta_t^{(m)} \xrightarrow{t \searrow 0} 0$, the Duhamel principle implies:

$$\delta_t^{(m)} = \widetilde{\delta}_t^{(m)} = \int\limits_0^t e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\, d\tau.$$

Hence

$$
\begin{aligned}
\big\|\delta_t^{(m)}\big\|_{H^{2k}} &= \Big\| \int_0^t e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\, d\tau \Big\|_{H^{2k}} \\
&\le t \cdot \sup_{\tau\in[0,t]} \big\|e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\big\|_{H^{2k}} \\
&\overset{(1.38)}{\le} t \cdot \sup_{\tau\in[0,t]} \Big\{ c \Big(\big\|e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\big\|_{L^2} + \big\|\Delta^k\, e^{-(t-\tau)\Delta_x}\, \eta_\tau^{(m)}\big\|_{L^2} \Big) \Big\} \\
&\overset{(1.45)}{\le} t \cdot c \cdot \sup_{\tau\in[0,t]} \Big\{ \big\|\eta_\tau^{(m)}\big\|_{L^2} + \big\|\Delta^k \eta_\tau^{(m)}\big\|_{L^2} \Big\} \\
&\le t \cdot c \cdot \sup_{\tau\in[0,t]} \big\|\eta_\tau^{(m)}\big\|_{H^{2k}} \\
&\overset{(3.10)}{=} O\left(t^{N+1}\right).
\end{aligned}
$$

Now applying the Sobolev embedding theorem 1.2.13, we find:

$$\big\|\delta_t^{(m)}\big\|_{C^0} \le O\big(t^{N+1}\big). \qquad \square$$

Corollary 3.2.10. *Let Δ be a self-adjoint Laplace-type operator acting on sections of a Riemannian or Hermitian vector bundle E over a compact Riemannian manifold M. Then we have the following short time asymptotics of the heat kernel:*

$$k_t(x,x) \overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j\, \Phi_j(x,x)$$

uniformly in $x \in M$.

Integrating over M, we obtain:

Corollary 3.2.11. *Let Δ be a self-adjoint Laplace-type operator acting on sections of a Riemannian or Hermitian vector bundle E over a compact Riemannian manifold M. Then we have the following short time asymptotics of the heat trace:*

$$
\begin{aligned}
\operatorname{tr}\big(e^{-t\,\Delta}\big) &= \sum_{j=1}^{\infty} e^{-t\,\lambda_j} \\
&\overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j \int_M \operatorname{tr} \Phi_j(x,x)\, dvol(x)
\end{aligned}
$$

$$= (4\pi\, t)^{-\frac{n}{2}} \left(\mathrm{rk}(E)\, \mathrm{vol}(M) + t \left[\frac{\mathrm{rk}(E)}{6} \int\limits_M \mathrm{scal}(x)\, dx - \int\limits_M \mathrm{tr} K_x\, dx \right] + \mathrm{O}(t^2) \right). \tag{3.11}$$

Proof. By Corollary 3.2.10, we have

$$\mathrm{tr}\big(k_t(x,x)\big) \sim (4\pi\, t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j\, \mathrm{tr}\, \Phi_j(x,x)$$

and thus

$$\begin{aligned}
\mathrm{tr}\big(e^{-t\,\Delta}\big) &= \sum_{j=1}^{\infty} e^{-t\,\lambda_j} \\
&= \int\limits_M \mathrm{tr}\, k_t(x,x)\, dvol(x) \\
&\overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j \int\limits_M \mathrm{tr}\, \Phi_j(x,x)\, dvol(x) \\
&= (4\pi\, t)^{-\frac{n}{2}} \left(\mathrm{rk}(E)\, \mathrm{vol}(M) + t \left[\frac{\mathrm{rk}(E)}{6} \int\limits_M \mathrm{scal}(x)\, dx - \int\limits_M \mathrm{tr} K_x\, dx \right] + \mathrm{O}(t^2) \right)
\end{aligned}$$

where we have used the equations (3.8) and (3.9). ☐

The short time asymptotics of the heat trace implies that the dimension $n = \dim(M)$ and the coefficients of powers of t on the right hand side of (3.11) are determined by the spectrum of the operator Δ: In particular,

$$\int\limits_M \mathrm{tr}\, \underbrace{\Phi_0(x,x)}_{=\,\mathrm{id}\,|_{E_x}}\, dvol(x) = \mathrm{rk}(E) \cdot \mathrm{vol}(M)$$

and

$$\begin{aligned}
\int\limits_M \mathrm{tr}\, \Phi_1(x,x)\, dvol(x) &\overset{(3.9)}{=} \int\limits_M \mathrm{tr} \left(\frac{1}{6} \mathrm{scal}(x)\ \mathrm{id}_{E_x} - K_x \right) dvol(x) \\
&= \frac{1}{6}\, \mathrm{rk}(E) \int\limits_M \mathrm{scal}(x)\, dvol(x) - \int\limits_M \mathrm{tr}(K_x)\, dvol(x).
\end{aligned}$$

are determined by the spectrum of Δ.

Example 3.2.12. Consider $\Delta = \Delta_0$ over a compact surface M. Then we have

$$\operatorname{tr}\Phi_1(x,x) \overset{(3.9)}{=} \frac{1}{6}\operatorname{scal}(x) = \frac{1}{3}K(x).$$

Hence the Euler characteristic $\chi(M) = \frac{1}{2\pi}\int_M K(x)\,dvol(x)$ is a spectral invariant of the Laplacian, i.e. it is determined by the spectrum of Δ.

3.3. Growth of eigenvalues

In Proposition 3.1.1, we derived an estimate for the k-th eigenvalue of a self-adjoint Laplace-type operator on a Riemannian manifold in terms of its first eigenvalue. Now we show the following improvement of this estimate, which goes under the name **Weyl asymptotics**.

Theorem 3.3.1 (Weyl). *Let Δ be a self-adjoint Laplace-type operator, acting on sections of a Riemannian or Hermitian vector bundle E over a compact Riemannian manifold M. For any $\lambda \in \mathbb{R}$ let $N(\lambda)$ be the total number (counted with multiplicities) of eigenvalues of Δ that are less than or equal to λ. Then we have:*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\operatorname{rk}(E)\cdot\operatorname{vol}(M)}{(4\pi)^{\frac{n}{2}}\cdot\Gamma\left(\frac{n}{2}+1\right)}. \tag{3.12}$$

For $\lambda = \lambda_k$, we have $N(\lambda) = k$ and the Weyl asymptotics (3.12) implies:

$$\frac{k}{\lambda^{\frac{n}{2}}} \xrightarrow{k\to\infty} \frac{\operatorname{rk}(E)\cdot\operatorname{vol}(M)}{(4\pi)^{\frac{n}{2}}\cdot\Gamma\left(\frac{n}{2}+1\right)} =: C,$$

i.e., $k \sim C\cdot\lambda_k^{\frac{n}{2}}$.
For the proof of Theorem 3.3.1 we need the following tool:

Lemma 3.3.2 (Karamata). *Let μ be a Borel measure on $(0,\infty)$, satisfying*

$$\int_0^\infty e^{-t\lambda}\,d\mu(\lambda) < \infty$$

for all $t > 0$. Let $\alpha > 0$ and $C > 0$ be positive constants such that

$$\lim_{t\searrow 0} t^\alpha \int_0^\infty e^{-t\lambda}\,d\mu(\lambda) = C.$$

Then for all $f \in C^0([0,1],\mathbb{R})$ *we have*

$$\lim_{t\searrow 0} t^\alpha \int_0^\infty f\big(e^{-t\lambda}\big)\, e^{-t\lambda}\, d\mu(\lambda) = \frac{C}{\Gamma(\alpha)} \int_0^\infty f\big(e^{-t}\big)\, t^{\alpha-1} e^{-t}\, dt. \tag{3.13}$$

Proof. By the Weierstrass' approximation theorem, polynomials are dense in $C^0([0,1],\mathbb{R})$ with respect to the C^0-norm. Hence it suffices to prove Lemma 3.3.2 for polynomials f instead of arbitrary continuous functions. Assume $f(x) = x^k$. For the left hand side of (3.13) we get:

$$\begin{aligned}
\lim_{t\searrow 0} t^\alpha \int_0^\infty f\big(e^{-t\lambda}\big)\, e^{-t\lambda}\, d\mu(\lambda) &= \lim_{t\searrow 0} t^\alpha \int_0^\infty e^{-(k+1)t\lambda}\, d\mu(\lambda) \\
&= \lim_{s\searrow 0} \left(\frac{s}{k+1}\right)^\alpha \int_0^\infty e^{-s\lambda}\, d\mu(\lambda) \\
&= \frac{C}{(k+1)^\alpha}.
\end{aligned}$$

For the right hand side of (3.13) we get:

$$\begin{aligned}
\frac{C}{\Gamma(\alpha)} \int_0^\infty f\left(e^{-t}\right) t^{\alpha-1} e^{-t}\, dt &= \frac{C}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(k+1)t}\, dt \\
&= \frac{C}{\Gamma(\alpha)} \int_0^\infty \left(\frac{s}{k+1}\right)^{\alpha-1} \cdot e^{-s} \cdot \frac{ds}{k+1} \\
&= \frac{C}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{(k+1)^\alpha}.
\end{aligned}$$

□

Proof. [of Theorem 3.3.1]

a) Replacing Δ by $\Delta + c \cdot \mathrm{id}$ if necessary, we may assume all eigenvalues λ_i of Δ to be positive. Such a shift of course does not affect the claimed asymptotics. We apply the Karamata Lemma 3.3.2 with $\alpha = \frac{n}{2}$, with $C = (4\pi)^{-\frac{n}{2}} \operatorname{rk}(E) \cdot \operatorname{vol}(M)$ and $d\mu = \sum_{j=0}^{\infty} \delta_{\lambda_j}$.

By Corollary 3.2.11, we have

$$\int_0^\infty e^{-t\lambda}\, d\mu(\lambda) = \sum_{j=0}^{\infty} e^{-t\lambda_j} = \operatorname{tr}\left(e^{-t\Delta}\right) < \infty$$

and

$$\lim_{t\searrow 0} t^{\alpha}\cdot\int_0^\infty e^{-t\lambda}\,d\mu(\lambda)=\lim_{t\searrow 0} t^{\frac{n}{2}}\cdot\mathrm{tr}\left(e^{-t\Delta}\right)=C.$$

Thus the assumptions in Lemma 3.3.2 are satisfied.

b) Let $\varepsilon>0$ and pick a continuous function $f:[0,1]\to\mathbb{R}$ such that $f(x)=0$ for $x\le e^{-(1+\varepsilon)}$ and $f(x)=x^{-1}$ for $x\ge e^{-1}$ and $0\le f(x)\le x^{-1}$ everywhere. For the left hand side of (3.13) we get

$$\begin{aligned}\lim_{t\searrow 0} t^{\alpha}\int_0^\infty f\left(e^{-t\lambda}\right)e^{-t\lambda}\,d\mu(\lambda) &= \lim_{t\searrow 0} t^{\alpha}\int_0^{\frac{1+\varepsilon}{t}} f\left(e^{-t\lambda}\right)e^{-t\lambda}\,d\mu(\lambda)\\ &\ge \limsup_{t\searrow 0} t^{\alpha}\int_0^{\frac{1}{t}} d\mu(\lambda)\\ &= \limsup_{t\searrow 0} t^{\alpha}\cdot N\left(\frac{1}{t}\right)\\ &= \limsup_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\frac{n}{2}}}.\end{aligned}$$

c) For the right hand side of (3.13) we obtain

$$\begin{aligned}\frac{C}{\Gamma(\alpha)}\int_0^\infty f\left(e^{-t}\right)t^{\alpha-1}e^{-t}\,dt &= \frac{C}{\Gamma(\alpha)}\int_0^{1+\varepsilon} f\left(e^{-t}\right)t^{\alpha-1}e^{-t}\,dt\\ &\le \frac{C}{\Gamma(\alpha)}\int_0^{1+\varepsilon} t^{\alpha-1}dt\\ &= \frac{C\cdot(1+\varepsilon)^{\alpha}}{\Gamma(\alpha)\cdot\alpha}=\frac{C\cdot(1+\varepsilon)^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

Thus

$$\limsup_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\frac{n}{2}}}\le\frac{C\cdot(1+\varepsilon)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$$

and $\varepsilon\searrow 0$ yields

$$\limsup_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\frac{n}{2}}}\le\frac{C}{\Gamma\left(\frac{n}{2}+1\right)}=\frac{\mathrm{rk}(E)\cdot\mathrm{vol}(M)}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}.$$

d) Using in b) and c) a continuous functions $f:[0,1]\to\mathbb{R}$ satisfying $f(x)=0$ for $x\le e^{-1}$ and $f(x)=x^{-1}$ for $x\ge e^{-1+\varepsilon}$ and $0\le f(x)\le x^{-1}$ everywhere yields $\liminf_{\lambda\to\infty}\frac{N(\lambda)}{\lambda^{\frac{n}{2}}}\ge\frac{C}{\Gamma\left(\frac{n}{2}+1\right)}$. □

3.4. The index of Dirac-type operators

Throughout this section, let E, F be Riemannian or Hermitian vector bundles over a compact Riemannian manifold M, and let $D \in \mathscr{D}\!\mathit{iff}_1(E,F)$ be a Dirac-type operator. Thus

$$\begin{aligned} \Delta^+ &:= D^*D \in \mathscr{D}\!\mathit{iff}_2(E,E) \quad \text{and} \\ \Delta^- &:= DD^* \in \mathscr{D}\!\mathit{iff}_2(F,F) \end{aligned}$$

are formally self-adjoint Laplace-type operators.

If $\varphi \in \ker(D)$ then $\varphi \in \ker(\Delta^+)$. Conversely, if $\varphi \in \ker(\Delta^+)$ then we have

$$0 = (\Delta^+\varphi, \varphi)_{L^2} = (D^*D\varphi, \varphi)_{L^2} = (D\varphi, D\varphi)_{L^2} = \|D\varphi\|^2_{L^2}.$$

Hence $D\varphi = 0$ that is, $\varphi \in \ker(D)$. We thus conclude that $\ker(D) = \ker(\Delta^+)$.

Similarly, we may conclude $\ker(D^*) = \ker(\Delta^-)$. In particular, by the Hodge Theorem 1.5.9 both $\ker(D)$ and $\ker(D^*)$ are finite dimensional.

Definition 3.4.1. Let $D \in \mathscr{D}\!\mathit{iff}_1(E,F)$ be a Dirac-type operator, where E, F are Riemannian or Hermitian vector bundles over a compact Riemannian manifold M. Then

$$\operatorname{ind}(D) := \dim\ker(D) - \dim\ker(D^*)$$

is called the **index** of D.

Remark 3.4.2. If $D \in \mathscr{D}\!\mathit{iff}_1(E,E)$ is a formally self-adjoint Dirac-type operator then $\operatorname{ind}(D) = 0$, since $D = D^*$.

Example 3.4.3
1) For $E = \Lambda^{\mathrm{even}}T^*M$ and $F = \Lambda^{\mathrm{odd}}T^*M$, the Euler operator

$$D = d + d^* \in \mathscr{D}\!\mathit{iff}_1(E,F).$$

is of Dirac-type (see Example 1.3.10). We have

$$\Delta^+ = D^*D = \bigoplus_{k\,\mathrm{even}} \Delta_k \qquad \text{and} \qquad \Delta^- = DD^* = \bigoplus_{k\,\mathrm{odd}} \Delta_k$$

where Δ_k denotes the Hodge-Laplacian on k-forms. By the Hodge Theorem 1.5.9, we have:

$$\ker(\Delta^+) = \bigoplus_{k\,\mathrm{even}} \ker(\Delta_k) \cong \bigoplus_{k\,\mathrm{even}} H^k_{dR}(M)$$

hence

$$\dim\ker(\Delta^+) = \sum_{k\,\text{even}} \dim H^k_{dR}(M) = \sum_{k\,\text{even}} b_k(M).$$

Similarly,

$$\dim\ker(\Delta^-) = \sum_{k\,\text{odd}} b_k(M).$$

Thus, the index of the Euler operator is the Euler characteristic of M (hence the name of the operator).

$$\operatorname{ind}(D) = \sum_{k=0}^{n} (-1)^k\, b_k(M) = \chi(M). \tag{3.14}$$

2) Let M be a compact oriented Riemannian manifold of even dimension $n = 2m$, and for $k \in \{0, \dots, n\}$ let

$$\tau = i^{k(k-1)+m} * : \Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{n-k} T^*M \otimes_{\mathbb{R}} \mathbb{C}.$$

Consider the signature operator $D = d + d^* \in \mathscr{Diff}_1(E^+, E^-)$, introduced in Example 1.3.19.

For $k \in \{0, \dots, m-1\}$, we define

$$E_k^{\pm} := E^{\pm} \cap (\Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \oplus \Lambda^{n-k} T^*M \otimes_{\mathbb{R}} \mathbb{C}).$$

We have

$$\begin{aligned} \tau(\omega \oplus \eta) = \pm\omega \oplus \eta &\Leftrightarrow \tau\omega = \pm\eta \quad \text{and} \quad \tau\eta = \pm\omega \\ &\Leftrightarrow \tau\omega = \pm\eta. \end{aligned}$$

Thus

$$E_k^{\pm} = \big\{(\omega, \pm\tau\omega) \,|\, \omega \in \Lambda^k T^*M \otimes \mathbb{C}\big\}.$$

Since τ maps harmonic forms to harmonic forms, we have

$$\ker(\Delta^{\pm}) \cap C^\infty(M, E_k^{\pm}) = \{(\omega, \pm\tau\omega) \,|\, \omega \in \ker(\Delta_k)\},$$

hence we get for all $k \in \{0, \dots, m-1\}$

$$\dim\ker(\Delta^{\pm}|_{C^\infty(M, E_k^{\pm})}) = b_k(M).$$

For the index of the signature operator, we thus obtain:

$$\operatorname{ind}(D) = \dim\ker(\Delta^+) - \dim\ker(\Delta^-) = \dim\ker\big(\Delta_{\frac{n}{2}}|_{E^+}\big) - \dim\ker\big(\Delta_{\frac{n}{2}}|_{E^-}\big).$$

Assume now that $\dim M = 4k$. On $\Lambda^{2k}T^*M\otimes_{\mathbb{R}}\mathbb{C}$ we have $\tau = *$ and thus $\ker(\Delta_{2k}|_{E^\pm})$ coincides with the space $\mathcal{H}^\pm$ of (anti-)self-dual $2k$-forms. The index of the signature operator is then equal to the signature of M:

$$\operatorname{ind}(D) = b^+(M) - b^-(M) = \operatorname{sign}(M). \tag{3.15}$$

3) Let M be a compact complex Hermitian manifold of complex dimension m. Consider the Dolbeault operator $D_{\bar\partial} = \sqrt{2}(\partial + \bar\partial) \in \mathscr{Diff}_1(\Lambda^{p,\mathrm{even}}T^*M, \Lambda^{p,\mathrm{odd}}T^*M)$ for a fixed $p \in \{0,\ldots,m\}$. The same reasoning as for the Euler operator yields the index of the Dolbeault operator:

$$\operatorname{ind}(D_{\bar\partial}) = \sum_{q=0}^{m} (-1)^q\, h^{p,q}(M). \tag{3.16}$$

Here $h^{p,q}(M)$ denote the Hodge numbers of M.

Now we come back to the general situation of a Dirac-type operator $D \in \mathscr{Diff}_1(E,F)$ on a compact Riemannian manifold M, with the associated Laplace-type operators $\Delta^+ = D^*D$ and $\Delta^- = DD^*$. Let $\lambda \neq 0$ be an eigenvalue of Δ^+, and let φ be a corresponding eigensection. Then we have:

$$\Delta^-\, D\varphi = DD^*\, D\varphi = D\,\Delta^+\varphi = \lambda\, D\varphi.$$

Hence D maps the eigenspace $E(\lambda,\Delta^+)$ to $E(\lambda,\Delta^-)$. Similarly, D^* maps $E(\lambda,\Delta^-)$ to $E(\lambda,\Delta^+)$. Since we have

$$D^*D|_{E(\lambda,\Delta^+)} = \Delta^+|_{E(\lambda,\Delta^+)} = \lambda\cdot\mathrm{id}_{E(\lambda,\Delta^+)}$$

we see that D induces an isomorphism $E(\lambda,\Delta^+) \to E(\lambda,\Delta^-)$ with inverse $\frac{1}{\lambda}\, D^*$. Hence except possibly for $\lambda = 0$, the operators Δ^+ and Δ^- have equal spectra. In particular, we have:

$$\begin{aligned} \operatorname{tr}\left(e^{-t\,\Delta^+}\right) - \operatorname{tr}\left(e^{-t\,\Delta^-}\right) &= \sum_{j=1}^{\infty} e^{-t\,\lambda_j(\Delta^+)} - \sum_{j=1}^{\infty} e^{-t\,\lambda_j(\Delta^-)} \\ &= \dim\ker(\Delta^+) - \dim\ker(\Delta^-) \\ &= \operatorname{ind}(D). \end{aligned} \tag{3.17}$$

Applying the short time asymptotics (3.11) for the heat trace, we thus obtain:

$$\operatorname{ind}(D) \overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=0}^{\infty} t^j \int_M \underbrace{\left[\operatorname{tr}\Phi_j^{\Delta^+}(x,x) - \operatorname{tr}\Phi_j^{\Delta^-}(x,x)\right]}_{=:\, a_j(x)} dvol(x) \tag{3.18}$$

and

$$(4\pi\, t)^{\frac{n}{2}} \cdot \mathrm{ind}(D) \overset{t\searrow 0}{\sim} \sum_{j=0}^{\infty} t^j \int\limits_M a_j(x)\, dvol(x).$$

Evaluating at $t = 0$ yields

$$0 = \int\limits_M a_0(x)\, dvol(x)\,.$$

Inserting back into (3.18) yields

$$\mathrm{ind}(D) \overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=1}^{\infty} t^j \int\limits_M a_j(x)\, dvol(x)$$

and

$$t^{\frac{n}{2}-1}\mathrm{ind}(D) \overset{t\searrow 0}{\sim} (4\pi)^{-\frac{n}{2}} \sum_{j=1}^{\infty} t^{j-1} \int\limits_M a_j(x)\, dvol(x).$$

If $\frac{n}{2} > 1$, we may put $t = 0$ to obtain

$$0 = (4\pi)^{-\frac{n}{2}} \int\limits_M a_1(x)\, dvol(x)\,.$$

Repeating this argument yields

$$0 = \int\limits_M a_j(x)\, dvol(x) \qquad \text{for all } j < \frac{n}{2}.$$

Thus, we end up with:

$$\mathrm{ind}(D) \overset{t\searrow 0}{\sim} (4\pi\, t)^{-\frac{n}{2}} \sum_{j=[\frac{n+1}{2}]}^{\infty} t^j \int\limits_M a_j(x)\, dvol(x). \tag{3.19}$$

Case 1: dim(M) odd
In this case, we have:

$$\mathrm{ind}(D) \overset{t\searrow 0}{\sim} (4\pi)^{-\frac{n}{2}} \sum_{j=\frac{n+1}{2}}^{\infty} t^{j-\frac{n}{2}} \int\limits_M a_j(x)\, dvol(x).$$

Thus all terms in the short time asymptotics of the heat trace have order at least $\frac{1}{2}$ in t. In the limit $t \searrow 0$, we obtain $\operatorname{ind}(D) = 0$.

Case 2: $\dim(M)$ even
In this case, we have:

$$\operatorname{ind}(D) \overset{t \searrow 0}{\sim} (4\pi)^{-\frac{n}{2}} \sum_{j=\frac{n}{2}}^{\infty} t^{j-\frac{n}{2}} \int_M a_j(x)\, dvol(x).$$

In the limit $t \searrow 0$, we thus obtain:

$$\operatorname{ind}(D) = (4\pi)^{-\frac{n}{2}} \int_M a_{\frac{n}{2}}(x)\, dvol(x).$$

Summarizing the above discussion, we conclude:

Theorem 3.4.4 (Atiyah-Singer index theorem, preliminary version). *Let $D \in \mathscr{Diff}_1(E,F)$ be a Dirac-type operator, where E, F are Riemannian or Hermitian vector bundles over an n-dimensional compact Riemannian manifold M. Then we have:*

- *If n is odd then* $\operatorname{ind}(D) = 0$.
- *If n is even then*
$$\operatorname{ind}(D) = (4\pi)^{-\frac{n}{2}} \int_M a_{\frac{n}{2}}(x)\, dvol(x).$$

Remark 3.4.5. As explained in Section 3.2, the coefficients $\Phi_j^{\Delta^\pm}$ of the formal heat kernel can be computed recursively by solving the transport equation (3.6).
In local coordinates, the coefficient $\Phi_j^{\Delta^\pm}(x,x)$ is some universal algebraic expression in the coefficients (together with their derivatives) of the Riemannian metric and of the operators $\Delta^\pm$.

Example 3.4.6. Let D be the Euler operator on an oriented Riemannian 2-dimensional manifold. By Example 3.2.8 we have:

$$\operatorname{tr} \Phi_1^{\Delta_0}(x,x) = \operatorname{tr} \phi_1^{\Delta_2}(x,x) = \frac{1}{6}\operatorname{scal}(x)$$
$$\text{and} \qquad \operatorname{tr} \Phi_1^{\Delta_1}(x,x) = \frac{2-6}{6}\operatorname{scal}(x).$$

Moreover, $\Delta^+ = \Delta_0 + \Delta_2$ and $\Delta^- = \Delta_1$. We thus obtain

$$\operatorname{tr}\Phi_1^{\Delta^+}(x,x) = \frac{2}{6}\operatorname{scal}(x) \quad \text{and}$$
$$\operatorname{tr}\Phi_1^{\Delta^-}(x,x) = \frac{2-6}{6}\operatorname{scal}(x).$$

Hence, $a_1(x) = \operatorname{scal}(x)$ and

$$\chi(M) = \operatorname{ind}(D) = (4\pi)^{-1}\int\limits_M \operatorname{scal}(x)\,dA(x) = \frac{1}{2\pi}\int\limits_M K(x)\,dA(x).$$

Thus we have proved the Gauß-Bonnet Theorem.

$$\int\limits_M K(x)\,dA(x) = 2\pi\,\chi(M).$$

Corollary 3.4.7 (Homotopy invariance of the index). *Let E, F be Riemannian or Hermitian vector bundles over a compact manifold M. Let g_t, $t \in I \subset \mathbb{R}$ be a smooth family of Riemannian metrics on M, and let D_t be Dirac-type operators for g_t, varying smoothly with $t \in I$.*
Then $\operatorname{ind}(D_t)$ is constant in t.

Proof. The functions $a_j(x,t)$, defined in equation (3.18) depend smoothly on t. This follows from the fact that $a_j(x,t)$ are built from the coefficients $\Phi_j(x,t)$ of the formal heat kernel, which are solutions of transport equations. The coefficients of the transport equations depend smoothly on t, and so do their solutions.
Hence the integer valued function

$$\operatorname{ind}(D_t) = (4\pi)^{-\frac{n}{2}}\int\limits_M a_{\frac{n}{2}}(x,t)\,dvol_t(x)$$

depends smoothly on t and is thus constant in t. $\square$

Corollary 3.4.8 (Multiplicity of index for coverings). *Let $\widetilde{M} \xrightarrow{\pi} M$ be a Riemannian covering of compact Riemannian manifolds of degree k. Let E and F be Riemannian or Hermitian vector bundles over M, and let $D \in \mathscr{Diff}_1(E,F)$ be a Dirac-type operator. Let $\widetilde{D} \in \mathscr{Diff}_1(\pi^*E, \pi^*F)$ be the Dirac-type operator obtained by pull-back.*
Then we have:

$$\operatorname{ind}\left(\widetilde{D}\right) = k \cdot \operatorname{ind}(D).$$

Proof. A direct computation yields

$$\begin{aligned}
\operatorname{ind}(\widetilde{D}) &= (4\pi)^{-\frac{n}{2}} \int\limits_{\widetilde{M}} \widetilde{a}_{\frac{n}{2}}\, \widetilde{dvol}(x) = (4\pi)^{-\frac{n}{2}} \int\limits_{\widetilde{M}} (a_{\frac{n}{2}} \circ \pi)(x)\, \widetilde{dvol}(x) \\
&= k \cdot (4\pi)^{-\frac{n}{2}} \int\limits_{M} a_{\frac{n}{2}}(x)\, dvol = k \cdot \operatorname{ind}(D).
\end{aligned}$$

□

4. Characteristic Classes

4.1. Chern Classes

Let G be the Lie group $G := \mathrm{GL}(N, \mathbb{C})$ and $\mathfrak{g} = \mathrm{Mat}(N \times N, \mathbb{C})$ its Lie algebra.

Definition 4.1.1. A polynomial map $P : \mathfrak{g} \to \mathbb{C}$ is called **invariant**, iff

$$P\big(T\,X\,T^{-1}\big) = P(X). \tag{4.1}$$

holds for all $T \in G$ and all $X \in \mathfrak{g}$.

Example 4.1.2. It is well known from linear algebra that $P = \det$ and $P = \mathrm{tr}$ are invariant polynomial maps.

Remark 4.1.3. The condition (4.1) is equivalent to the following:

$$P(X\,Y) = P(Y\,X) \qquad \text{for all } X, Y \in \mathfrak{g}. \tag{4.2}$$

If (4.2) holds then we have for all $X \in \mathfrak{g}$ and for all $T \in G$:

$$P\big((TX)T^{-1}\big) \overset{(4.2)}{=} P\big(T^{-1}TX\big) = P(X).$$

Thus, we have (4.1).
Conversely, if (4.1) holds then we have for all $X \in G$ and for all $Y \in \mathfrak{g}$:

$$P(X\,Y) \overset{(4.1)}{=} P\big(X^{-1}XYX\big) = P(Y\,X).$$

Since $G \subset \mathfrak{g}$ is dense and P is continuous, this equation also holds for all $X, Y \in \mathfrak{g}$, thus we have (4.2).

Remark 4.1.4. If $P : \mathfrak{g} \to \mathbb{C}$ is a polynomial map and $\mathcal{A}$ is a commutative $\mathbb{C}$-algebra then P induces a map

$$\mathrm{Mat}(N \times N, \mathcal{A}) \to \mathcal{A}.$$

In the following, let $E \to M$ be a complex vector bundle of rank N with connection ∇. The corresponding curvature tensor is defined by

$$R(X, Y)e = \nabla_X \nabla_Y e - \nabla_Y \nabla_X e - \nabla_{[X,Y]} e.$$

Here $X, Y \in T_xM$ and $e \in E_x$.
Now let $U \subset M$ be an open set with local sections $s_1, \ldots, s_N \in C^\infty(U, E|_U)$, linearly independent at each point $x \in U$. Then the **connection 1-form** ω of ∇ is defined by

$$\nabla_X s_i = \sum_{j=1}^{N} \omega_i^j(X) s_j \,. \tag{4.3}$$

Here $\omega_i^j \in \Omega^1(U)$ are 1-forms on U, and $\omega = (\omega_i^j) \in \mathrm{Mat}(N \times N, \Omega^1(U))$ is a matrix of 1-forms on U. The **curvature 2-form** Ω of ∇ is defined by

$$R(X,Y)s_i = \sum_{j=1}^{N} \Omega_i^j(X,Y)s_j. \tag{4.4}$$

Here $\Omega = (\Omega_i^j) \in \mathrm{Mat}(N \times N, \Omega^2(U))$ is a matrix of 2-forms on U. Now, $\mathcal{A} := \bigoplus_{k\in\mathbb{N}} \Omega^{2k}(U)$ is a commutative $\mathbb{C}$-algebra and we consider $\Omega \in \mathrm{Mat}(N \times N, \mathcal{A})$.

Lemma 4.1.5. *Let $E \to M$ be a complex vector bundle with connection ∇. Let $s_1, \ldots, s_N$ and $\widetilde{s}_1, \ldots, \widetilde{s}_N$ be two local frames on $U \subset M$. Let $\Omega, \widetilde{\Omega}$ be the corresponding curvature 2-forms. Then for any invariant polynomial map $P : \mathfrak{g} \to \mathbb{C}$, we have:*

$$P(\Omega) = P\big(\widetilde{\Omega}\big).$$

Proof. Let $T : U \to G$ be the linear transformation that maps the frame $s_1, \ldots, s_N$ to the frame $\widetilde{s}_1, \ldots, \widetilde{s}_N$. From equation (4.4), we obtain $\widetilde{\Omega}(X,Y) = T \cdot \Omega(X,Y) \cdot T^{-1}$. By invariance of P, this yields $P\big(\widetilde{\Omega}\big) = P\big(T\,\Omega\,T^{-1}\big) = P(\Omega)$. □

Corollary 4.1.6. *Let $E \to M$ be a complex vector bundle with connection ∇, and let $P : \mathfrak{g} \to \mathbb{C}$ be an invariant polynomial. Then $P(\Omega)$ is defined globally on M, i.e., $P(\Omega) \in \bigoplus_{k\in\mathbb{N}} \Omega^{2k}(M)$.*

The connection 1-form and the curvature 2-form are related as follows:

Lemma 4.1.7. *Let $E \to M$ be a complex vector bundle with connection ∇. Let $s_1, \ldots, s_N : U \to E$ be a local frame and let $\omega \in \mathrm{Mat}(N \times N, \Omega^1(U))$ and $\Omega \in \mathrm{Mat}(N \times N, \Omega^2(U))$ be the corresponding connection and curvature forms. Then we* *have:*

$$\Omega = d\omega + \omega \wedge \omega \tag{4.5}$$
$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega \tag{4.6}$$

Proof. Let $X|_p$ and $Y|_p$ be tangent vectors at the point $p \in M$ and extend them to vector fields X and Y which are synchronous at p, i.e.$\nabla X|_p = \nabla Y|_p = 0$ and thus $[X,Y]|_p = 0$. Then, at p we have:

$$\begin{aligned}
\sum_j \Omega_i^j(X,Y)s_j &= R(X,Y)s_i \\
&= \nabla_X \nabla_Y s_i - \nabla_Y \nabla_X s_i \\
&= \nabla_X \Big(\sum_k \omega_i^k(Y)s_k \Big) - \nabla_Y \Big(\sum_k \omega_i^k(X)s_k \Big) \\
&= \sum_{k,l} \Big(\omega_i^k(Y)\omega_k^l(X)s_l - \omega_i^k(X)\omega_k^l(Y)s_l \Big) \\
&\qquad + \sum_k \Big(\partial_X \omega_i^k(Y)s_k - \partial_Y \omega_i^k(X)s_k \Big) \\
&= \sum_j \Big(\partial_X \omega_i^j(Y) - \partial_Y \omega_i^j(X) + \sum_k (\omega_i^k(Y)\omega_k^j(X) - \omega_i^k(X)\omega_k^j(Y)) \Big) s_j \\
&= \sum_j \Big(d\omega_i^j(X,Y) + \sum_k (\omega_k^j \wedge \omega_i^k)(X,Y) \Big) s_j.
\end{aligned}$$

Thus, we have

$$\Omega_i^j = d\omega_i^j + \sum_k \omega_k^j \wedge \omega_i^k.$$

For (4.6), we compute, using (4.5):

$$\begin{aligned}
d\Omega &= d^2\omega + d\omega \wedge \omega - \omega \wedge d\omega \\
&= 0 + (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega) \\
&= \Omega \wedge \omega - \omega \wedge \Omega.
\end{aligned}$$

□

Lemma 4.1.8. *Let $P(\Omega)$ be as in Corollary 4.1.6. Then $P(\Omega)$ is closed, i.e. we have $dP(\Omega) = 0$.*

Proof.

a) We denote by A^i_j the entry in the i-th row and the j-th column of the matrix $A \in \mathfrak{g}$. For $P : \mathfrak{g} \to \mathbb{C}$, we set $P'(A)^j_i := \frac{\partial P}{\partial A^i_j}(A)$ and we define $P'(A) := (P'(A)^j_i) \in \mathfrak{g}$.

We first show that $P'(A)$ commutes with A, i.e. $[P'(A), A] = 0$:

Let $E^j_i \in \mathfrak{g}$ be the matrix with all entries equal to zero except the entry in the j-th row and i-th column which is equal to 1. By the invariance of P, we have for all i, j and all $t \in \mathbb{R}$

$$P\big((\mathbb{1}_N + t\,E^j_i)A\big) = P\big(A(\mathbb{1}_N + t\,E^j_i)\big).$$

We differentiate both sides of this equation with respect to t. For the left hand side we get

$$\begin{aligned}
\frac{d}{dt}P\big((\mathbb{1}_N + tE^j_i)A\big)|_{t=0} &= \sum_{k,l} \frac{\partial P}{\partial A^l_k}(A) \cdot (E^j_i \cdot A)^l_k = \sum_k \frac{\partial P}{\partial A^j_k}(A) \cdot A^i_k \\
&= \sum_k A^i_k \cdot P'(A)^k_j = \big(A \cdot P'(A)\big)^i_j
\end{aligned}$$

and similarly we get for the right hand side

$$\frac{d}{dt}P\big(A(\mathbb{1}_N + t\,E^j_i)\big)|_{t=0} = \big(P'(A) \cdot A\big)^i_j.$$

We conclude that $P'(A) \cdot A = A \cdot P'(A)$.

b) By Lemma 4.1.7 and part a), we have:

$$\begin{aligned}
dP(\Omega) &= \sum_{i,j=1}^{N} \frac{\partial P}{\partial A^i_j}(\Omega) \wedge (d\Omega)^i_j \\
&= \mathrm{tr}\big(P'(\Omega) \wedge d\Omega\big) \\
&= \mathrm{tr}\big(P'(\Omega) \wedge \Omega \wedge \omega - P'(\Omega) \wedge \omega \wedge \Omega\big) \\
&= \mathrm{tr}\big(\Omega \wedge P'(\Omega) \wedge \omega - P'(\Omega) \wedge \omega \wedge \Omega\big).
\end{aligned}$$

We put $X := P'(\Omega) \wedge \omega = (X^i_j)_{i,j}$. Since Ω^j_i is a 2-form we have $\Omega^j_i \wedge X^i_j = X^i_j \wedge \Omega^j_i$ and thus

$$\begin{aligned}
dP(\Omega) &= \mathrm{tr}(\Omega \wedge X - X \wedge \Omega) \\
&= \sum_{i,j} (\Omega^i_j \wedge X^j_i - X^i_j \wedge \Omega^j_i) \\
&= \sum_{i,j} (\Omega^i_j \wedge X^j_i - \Omega^j_i \wedge X^i_j) \\
&= 0.
\end{aligned}$$

□

Since $dP(\Omega) = 0$, the differential form $P(\Omega) \in \bigoplus_k \Omega^{2k}(M;\mathbb{C})$ represents a de Rham cohomology class

$$[P(\Omega)] \in \bigoplus_{k\geq 0} H^{2k}_{\mathrm{dR}}(M;\mathbb{C}).$$

The construction of the closed form $P(\Omega)$ and the de Rham cohomology class $[P(\Omega)]$ is called the **Chern-Weil construction**. In the case of homogeneous polynomial maps, we set:

Definition 4.1.9. Let $E \to M$ be a complex vector bundle of rank N with connection ∇, and let $P : \mathfrak{g} \to \mathbb{C}$ be an invariant polynomial map, homogeneous of degree k. The differential form $P(\Omega) \in \Omega^{2k}(M;\mathbb{C})$ is called the **Chern-Weil form** associated with P. The de Rham cohomology class $[P(\Omega)] \in H^{2k}_{\mathrm{dR}}(M;\mathbb{C})$ is called the **Chern-Weil class** associated with P.

Remark 4.1.10
For a complex vector bundle $E \to M$ and a smooth map $f : N \to M$, we have the pull-back bundle f^*E and the commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

For a connection ∇ on E, we have the pull-back connection $f^*\nabla$ on f^*E, characterized by the following property: Let $s_1, \dots, s_N$ be a local frame of E over $U \subset M$ and let $f^*s_1, \dots, f^*s_N$ be the pull-back frame of f^*E over $f^{-1}(U) \subset N$, defined by

$$f^*s_j(x) := F^{-1}\big(s_j(f(x))\big).$$

For the connection 1-forms we have $\omega^{f^*\nabla} = f^*(\omega^\nabla)$. Thus, we compute for the curvature 2-forms:

$$\Omega^{f^*\nabla} = d(f^*\omega) - f^*\omega \wedge f^*\omega = f^*(d\omega - \omega \wedge \omega) = f^*\Omega^\nabla.$$

Thus

$$P\big(\Omega^{f^*\nabla}\big) = P\big(f^*\Omega^\nabla\big) = f^*P(\Omega^\nabla) \tag{4.7}$$

and hence

$$\big[P\big(\Omega^{f^*\nabla}\big)\big] = f^*\big[P(\Omega^\nabla)\big]. \tag{4.8}$$

Lemma 4.1.11. *Let $E \to M$ be a complex vector bundle, and let $P : \mathfrak{g} \to \mathbb{C}$ be an invariant polynomial map. Then the de Rham cohomology class*

$$[P(\Omega)] \in H^{\mathrm{even}}_{\mathrm{dR}}(M;\mathbb{C})$$

does not depend on the choice of connection on E.

Proof. Let ∇^0 and ∇^1 be two connections on E. Put $X := M \times \mathbb{R}$, and let $\pi : X \to M$ be the projection on the first factor. On the pull-back bundle $\widetilde{E} := \pi^* E \to X$ we have the pull-back connections $\widetilde{\nabla}^0 := \pi^*\nabla^0$ and $\widetilde{\nabla}^1 := \pi^*\nabla^1$. We define a connection $\widetilde{\nabla}$ on $\widetilde{E}$ by putting for $v \in T_{(m,\lambda)}X$:

$$\widetilde{\nabla}_v s := (1-\lambda)\,\widetilde{\nabla}^0_v s + \lambda\,\widetilde{\nabla}^1_v s.$$

For $\lambda \in \mathbb{R}$ let $i_\lambda : M \to X$, $m \mapsto (m, \lambda)$ be the inclusion. Then we have

$$i_\lambda^* \widetilde{\nabla} = (1-\lambda)\nabla^0 + \lambda \nabla^1.$$

From equation (4.7), we obtain $P(\Omega^{\nabla^0}) = i_0^* P(\widetilde{\Omega})$ and $P(\Omega^{\nabla^1}) = i_1^* P(\widetilde{\Omega})$. Since the inclusions i_0 and i_1 are homotopic they induce the same map on cohomology: $i_0^* = i_1^*$. Thus we get

$$\big[P\big(\Omega^{\nabla^1}\big)\big] = i_1^*\big[P\big(\widetilde{\Omega}\big)\big] = i_0^*\big[P\big(\widetilde{\Omega}\big)\big] = \big[P\big(\Omega^{\nabla^0}\big)\big].$$ □

As a consequence, for any complex vector bundle $E \to M$ and any invariant polynomial map $P : \mathfrak{g} \to \mathbb{C}$ we obtain a de Rham cohomology class

$$P(E) := \big[P(\Omega)\big] \in H^{\mathrm{even}}_{\mathrm{dR}}(M; \mathbb{C}).$$

Moreover, the Chern-Weil construction is natural with respect to pull-back diagrams: for any complex vector bundle $E \to M$ and any smooth map $f : N \to M$, we have

$$P(f^*E) = f^*P(E). \tag{4.9}$$

Definition 4.1.12. Let $E \to M$ be a complex vector bundle of rank N. Set

$$P(A) := \det\left(\mathbb{1}_N + \frac{1}{2\pi i}\,A\right).$$

Then

$$c(E) := P(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(M; \mathbb{C})$$

is called the **(total) Chern class** of E.

If $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \in \mathfrak{g}$ is a diagonal matrix then we have

$$\begin{aligned} P(A) &= \det \begin{pmatrix} 1+\frac{\lambda_1}{2\pi i} & & \\ & \ddots & \\ & & 1+\frac{\lambda_N}{2\pi i} \end{pmatrix} \\ &= \prod_{j=1}^{N} \left(1+\frac{\lambda_j}{2\pi i}\right) \\ &= \sum_{k=0}^{N} \sigma_k \left(\frac{\lambda_1}{2\pi i}, \ldots, \frac{\lambda_N}{2\pi i}\right), \end{aligned} \tag{4.10}$$

where σ_k is the k-th elementary-symmetric function. In particular, we have

$$\begin{aligned} \sigma_1 \left(\frac{\lambda_1}{2\pi i}, \ldots, \frac{\lambda_N}{2\pi i}\right) &= \sum_{j=1}^{N} \frac{\lambda_j}{2\pi i} = \frac{1}{2\pi i} \cdot \mathrm{tr}(A) \\ \sigma_N \left(\frac{\lambda_1}{2\pi i}, \ldots, \frac{\lambda_N}{2\pi i}\right) &= \prod_{j=1}^{N} \frac{\lambda_j}{2\pi i} = \left(\frac{1}{2\pi i}\right)^N \cdot \det(A). \end{aligned}$$

By the invariance of P, the formula (4.10) also holds for all diagonalizable matrices A. Since these are dense[1] in $\mathfrak{g}$ and P is continuous, (4.10) holds for all $A \in \mathfrak{g}$.

For $k \in \{0, \ldots, N\}$ we put

$$P_k(A) := \sigma_k \left(\frac{\lambda_1}{2\pi i}, \ldots, \frac{\lambda_N}{2\pi i}\right) = \left(\frac{1}{2\pi i}\right)^k \sigma_k(\lambda_1, \ldots, \lambda_N).$$

Definition 4.1.13. Let $E \to M$ be a complex vector bundle of rank N with curvature 2-form Ω. Then

$$c_k(E) := [P_k(\Omega)] \in H^{2k}_{\mathrm{dR}}(M; \mathbb{C})$$

is called the ***k*-th Chern class** of E.

We have $c(E) = c_0(E) + \ldots + c_N(E)$.

[1] In fact, the matrices with pairwise distinct eigenvalues are dense, and they are diagonalizable by the theorem on the Jordan normal form.

Proposition 4.1.14. *Let $E, E_1, E_2 \to M$ be complex vector bundles, and let $E^* \to M$ be the dual bundle of E. Then we have:*

$$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2) \tag{4.11}$$

$$c_k(E^*) = (-1)^k \, c_k(E) \tag{4.12}$$

If $E_1 \cong E_2$ then we have

$$c(E_1) = c(E_2). \tag{4.13}$$

Proof. 1) Let ∇^i be a connection on E_i and let $\nabla := \nabla^1 \oplus \nabla^2$, i.e. for sections s_i of E_i we define

$$\nabla_X(s_1 \oplus s_2) := (\nabla^1_X s_1) \oplus (\nabla^2_X s_2).$$

Then, the curvature form of $E_1 \oplus E_2$ with the connection ∇ is given by:

$$\Omega^\nabla = \begin{pmatrix} \Omega^{\nabla^1} & 0 \\ 0 & \Omega^{\nabla^2} \end{pmatrix}$$

A direct computation yields:

$$\begin{aligned} \det\left(\mathbb{1}_{N_1+N_2} + \frac{1}{2\pi i}\,\Omega^\nabla\right) &= \det\begin{pmatrix} \mathbb{1}_{N_1} + \frac{1}{2\pi i}\,\Omega^{\nabla^1} & 0 \\ 0 & \mathbb{1}_{N_2} + \frac{1}{2\pi i}\,\Omega^{\nabla^2}. \end{pmatrix} \\ &= \det\left(\mathbb{1}_{N_1} + \frac{1}{2\pi i}\,\Omega^{\nabla^1}\right) \cdot \det\left(\mathbb{1}_{N_2} + \frac{1}{2\pi i}\,\Omega^{\nabla^2}\right). \end{aligned}$$

2) Let ∇ be a connection on E and let $\widetilde{\nabla}$ be the dual connection on E^* induced by ∇, characterized by the condition (3.2). Let $s_1, \ldots, s_N$ be a local frame for E and $s_1^*, \ldots, s_N^*$ the dual frame for E^*. Then we have $\Omega^{\widetilde{\nabla}} = -(\Omega^\nabla)^\top$ and thus

$$c_k(E^*) = [P_k(\Omega^{\widetilde{\nabla}})] = [P_k(-\Omega^\nabla)] = (-1)^k\,[P_k(\Omega^\nabla)] = (-1)^k\,c_k(E).$$

3) Let $\phi : E_1 \to E_2$ be a vector bundle isomorphism. Let ∇ be a connection on E_2. Then

$$\widetilde{\nabla}_X := \phi^{-1} \circ \nabla_X \circ \phi$$

defines a connection on E_1. For any local frame $s_1, \ldots, s_N$ of E_2, we define a local frame $\widetilde{s}_1, \ldots, \widetilde{s}_N$ of E_1 by $\widetilde{s}_j := \phi^{-1} \circ s_j$. With respect to these local frames we get $\Omega^{\widetilde{\nabla}} = \Omega^\nabla$. Hence $c(E_1) = c(E_2)$. $\square$

Lemma 4.1.15. *Let $E \to M$ be a complex vector bundle. Then the total Chern class $c(E)$ is a class in the real de Rham cohomology:*

$$c(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(M) \subset H^{\mathrm{even}}_{\mathrm{dR}}(M;\mathbb{C}).$$

Proof. Choose a Hermitian metric and a metric connection ∇ on E. For a local orthonormal frame $s_1, \dots, s_N$ we have

$$\omega^\nabla(X), \Omega^\nabla(X,Y) \in \mathfrak{u}(N) = \{A \in \mathfrak{g} \mid A^* = -A\}.$$

Hence

$$\begin{aligned}\overline{\det\left(1 + \frac{1}{2\pi i}\,\Omega^\nabla\right)} &= \det\left(\overline{1 + \frac{1}{2\pi i}\,\Omega^\nabla}\right) = \det\left(1 - \frac{1}{2\pi i}\,\overline{\Omega^\nabla}\right) = \det\left(1 - \frac{1}{2\pi i}\,\overline{\Omega^\nabla}\right)^\top \\ &= \det\left(1 - \frac{1}{2\pi i}\,(\Omega^\nabla)^*\right) = \det\left(1 + \frac{1}{2\pi i}\,\Omega^\nabla\right).\end{aligned}$$

Thus $c(E) = \det\left(1 + \frac{1}{2\pi i}\,\Omega^\nabla\right)$ is real. □

Remark 4.1.16. The total Chern class $c(E)$ of a complex vector bundle $E \to M$ is not only a real cohomology class, it also has integral periods, i.e. for any smooth singular cycle γ in M, we have

$$\int_\gamma c(E) \in \mathbb{Z}.$$

Thus $c(E)$ lies in the image of $H^{\mathrm{even}}(M;\mathbb{Z})$ in the de Rham cohomology under the change of coefficients map $H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{R})$ composed with the de Rham isomorphism $H^*(M;\mathbb{R}) \to H^*_{\mathrm{dR}}(M)$.

Proposition 4.1.17. *a) If $E \to M$ is trivial then $c(E) = 1 \in H^0_{\mathrm{dR}}(M)$.*

b) If $E \to M$ has rank N and admits global sections $s_1, \dots, s_k \in C^\infty(M,E)$ linearly independent at each point then $c_j(E) = 0$ for all $j > N - k$.

Proof. a) If $E \to M$ is trivial then it has a flat connection ∇, i.e. $\Omega^\nabla = 0$, and thus $\det\left(\mathbb{1} + \frac{1}{2\pi i}\,\Omega^\nabla\right) = 1$.

b) Let $E_1 \subset E$ be the sub-bundle spanned by $s_1, \dots, s_k$. Let E_2 be a complementary bundle, i.e. $E = E_1 \oplus E_2$. Since E_1 is trivial by construction, $c(E_1) = 1$ and thus

$$c(E) = c(E_1) \cdot c(E_2) = c(E_2).$$

Hence $c_j(E) = c_j(E_2) = 0$ for $j > \mathrm{rk}(E_2) = N - k$. □

Example 4.1.18. Consider the n-sphere $M = S^n$ and its complexified tangent bundle $E := TS^n \otimes \mathbb{C}$. Let ν denote the normal bundle of S^n in $\mathbb{R}^{n+1}$. Since the normal field is globally defined on $S^n \subset \mathbb{R}^{n+1}$, the normal bundle ν is trivial. We also have $TS^n \oplus \nu = T\mathbb{R}^{n+1}|_{S^n}$, which is also trivial.
To compute the total Chern class of E, we denote by $\mathcal{E}^k$ be the trivial complex vector bundle of rank k on S^n. Then we have $\nu \otimes \mathbb{C} = \mathcal{E}^1$ and $E \oplus \mathcal{E}^1 = \mathcal{E}^{n+1}$. By the multiplicativity (4.11) of the total Chern class, we obtain

$$c(E) \cdot \underbrace{c(\mathcal{E}^1)}_{=1} = c(E \oplus \mathcal{E}^1) = c(\mathcal{E}^{n+1}) = 1.$$

Thus, $c(E) = 1$, although $E = TS^n \otimes \mathbb{C} \to S^n$ is not trivial.

Example 4.1.19. Consider the tautological line bundle $\gamma_m \to \mathbb{C}P^m$ on the complex projective space, defined by

$$\gamma_m = \big\{(\ell, v) \in \mathbb{C}P^m \times \mathbb{C}^{m+1} \,|\, v \in \ell\big\}.$$

Define $a := c_1(\gamma_m) \neq 0$. Then we have

$$H^k_{\mathrm{dR}}(\mathbb{C}P^m) = \begin{cases} \mathbb{R} \cdot a^j, & \text{for } k = 2j,\ j = 0, \ldots, m \\ 0 & \text{otherwise} \end{cases}.$$

The computation is spelled out e.g. in [2].

4.2. Additive and multiplicative classes

Let $R = R^0 \oplus R^1 \oplus R^2 \oplus \ldots$ be a commutative[2] graded real algebra with unit $1 \in R^0$. The term "graded" means that the R^j are linear subspaces of R satisfying $R^j \cdot R^k \subset R^{j+k}$. In the application we have in mind, R will be the algebra of even de Rham cohomology, i.e. $R^j = H^{2j}_{\mathrm{dR}}(M)$.

Definition 4.2.1. Let R be a commutative graded real algebra. Let $g(x) = g_0 + g_1 \cdot x + g_2 \cdot x^2 + \cdots \in \mathbb{R}[\![x]\!]$ be a formal power series. We define an associated vector space endomorphism $\Lambda_g : R \to R$ by

$$\Lambda_g|_{R^j} = (-1)^{j+1} \cdot j \cdot g_j \,\mathrm{id}_{R^j}\,. \tag{4.14}$$

Hence Λ_g preserves the grading.

[2] By **commutative** graded algebra we understand that it is a commutative algebra in the usual sense, i.e. $a \cdot b = b \cdot a$ holds for all $a, b \in R$. It does not mean that the algebra is **graded** commutative!

Additive classes

For $n \in \mathbb{N}$ and $c = 1 + c_1 + c_2 + \dots \in R = R^0 \oplus R^1 \oplus R^2 \oplus \dots$ we obtain another element $g_c \in R$ by setting

$$g_c := N \cdot g_0 \cdot 1 + \Lambda_g(\log c) \in R.$$

Here $\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} \mp \dots$ Notice that for fixed degree j only finitely many terms occur in $\log c$.
Now let $R^j = H^{2j}_{\mathrm{dR}}(M)$ and $c = c(E)$ be the total Chern class of a complex vector bundle $E \to M$ of rank N.

Definition 4.2.2. Let $c = c(E)$ be the total Chern class of a complex vector bundle E over M of rank N. Let $g(x) = g_0 + g_1 \cdot x + g_2 \cdot x^2 + \dots \in \mathbb{R}[\![x]\!]$ be a formal power series. Then

$$g_c(E) := N \cdot g_0 \cdot 1 + \Lambda_g\big(\log c(E)\big) \tag{4.15}$$

is called the **additive characteristic class** of E associated with the formal power series g.

By equation (4.8), the Chern classes are natural with respect to pull-back diagrams. Obviously, the same holds for any additive characteristic class g_c: for any smooth map $f : N \to M$ and any complex vector bundle $E \to M$, we have

$$g_c\big(f^*E\big) = f^* g_c(E).$$

Moreover, additive classes are additive with respect to the direct sum of bundles:

$$g_c(E_1 \oplus E_2) = g_c(E_1) + g_c(E_2).$$

Hence the name "additive class".

Example 4.2.3. The additive class with respect to the exponential function $g(x) = e^x$ is called **Chern character** of E. We write

$$g_c(E) =: \mathrm{ch}(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(M).$$

By definition the component $\mathrm{ch}_0(E) \in H^0_{\mathrm{dR}}(M)$ is the rank of E. We now compute $\mathrm{ch}_1(E) \in H^2_{\mathrm{dR}}(M)$ and $\mathrm{ch}_2(E) \in H^4_{\mathrm{dR}}(M)$. For this purpose we compute

$$\begin{aligned}
\log c(E) &= \log\big(1 + c_1(E) + c_2(E) + \dots\big)\\
&= \big(c_1(E) + c_2(E) + \dots\big) - \frac{1}{2}\big(c_1(E) + c_2(E) + \dots\big)^2 + \dots\\
&= c_1(E) + c_2(E) - \frac{1}{2}\, c_1(E)^2 + \text{ higher degree terms}
\end{aligned}$$

Thus

$$\Lambda_{e^x}\big(\log c(E)\big) \overset{(4.15)}{=} (-1)^{1+1}\cdot c_1(E) + (-1)^{2+1}\cdot 2\cdot\frac{1}{2}\cdot\Big(c_2(E)-\frac{1}{2}\,c_1(E)^2\Big)$$
$$+ \text{ higher degree terms.}$$

and hence

$$\begin{aligned}
\mathrm{ch}_0(E) &= \mathrm{rk}(E),\\
\mathrm{ch}_1(E) &= c_1(E),\\
\mathrm{ch}_2(E) &= \frac{1}{2}\,c_1(E)^2 - c_2(E).
\end{aligned}$$

Now for any additive character g_c, we consider the special case where $E = L_1 \oplus \ldots \oplus L_N$ is the direct sum of line bundles L_j, we have:

$$c(E) \overset{(4.11)}{=} c(L_1)\cdot\ldots\cdot c(L_N) = \big(1+c_1(L_1)\big)\cdot\ldots\cdot\big(1+c_1(L_N)\big).$$

Setting $x_j := c_1(L_j)$, we obtain for any additive class:

$$\begin{aligned}
g_c(E) &= N\cdot g_0 + \Lambda_g\big(\log c(E)\big)\\
&= N\cdot g_0 + \Lambda_g\Big(\log\left(1+x_1\right)+\ldots+\log\left(1+x_N\right)\Big)\\
&= N\cdot g_0 + \Lambda_g\Big(x_1+\ldots+x_N - \frac{1}{2}\left(x_1^2+\ldots+x_N^2\right) + \frac{1}{3}\left(x_1^3+\ldots+x_N^3\right) - \ldots\Big)\\
&= g(x_1)+\ldots+g(x_N).
\end{aligned}$$

Multiplicative classes

Definition 4.2.4. Let $f(x) \in \mathbb{R}[\![x]\!]$ be a formal power series of the form

$$f(x) = 1 + f_1\cdot x + f_2\cdot x^2 + \ldots \in \mathbb{R}[\![x]\!].$$

Then

$$F_c(E) := \exp\Big(\Lambda_{\log f}\big(\log c(E)\big)\Big) \in H^{\mathrm{even}}_{\mathrm{dR}}(M)$$

is called the **multiplicative characteristic class** of E associated with the formal power series f.

As for the additive classes, it follows from the naturality (4.9) of the total Chern class that any additive class F_c is natural with respect to pull-back diagrams: For any complex vector bundle $E \to M$ and any smooth map $f : N \to M$, we have

$$F_c\big(f^*E\big) = f^*F_c(E).$$

Moreover, multiplicative classes are multiplicative with respect to the direct sum of bundles:

$$F_c(E_1 \oplus E_2) = F_c(E_1) \cdot F_c(E_2).$$

Hence the name "multiplicative class".

Example 4.2.5. The multiplicative class associated with the formal power series

$$f(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} + \dots$$

is called **Todd class**. We write

$$F_c(E) =: \mathrm{Td}(E) \in H^{\mathrm{even}}_{\mathrm{dR}}(M).$$

A direct computation (see [2]) yields

$$\begin{aligned}\mathrm{Td}_1(E) &= \frac{c_1(E)}{2}\\ \mathrm{Td}_2(E) &= \frac{c_2(E) + c_1(E)^2}{12}.\end{aligned}$$

Additive and multiplicative characteristic classes are important, since they show up in index theorems. For example, the Atiyah-Singer index theorem applied to the Dolbeault Dirac operator reads:

Theorem 4.2.6 (Riemann-Roch-Hirzebruch). *Let $E \to M$ be a holomorphic vector bundle on a compact complex manifold. Then the index of the Dolbeault operator is given by:*

$$\mathrm{ind}(\bar\partial) \overset{(3.16)}{=} \sum_q (-1)^q\, h^{0,q}(M) = \int_M \mathrm{Td}(TM).$$

4.3. Pontryagin Classes

Let $V \to M$ be a real vector bundle and let $E = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Since V is a real bundle, we have $V \cong V^*$ and thus $E \cong E^*$. Hence

$$c_k(E) = c_k(E^*) = (-1)^k\, c_k(E)$$

and thus $c_k(E) = 0$ for all odd k.

Definition 4.3.1. Let $V \to M$ be a real vector bundle, and let $E = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. The cohomology class

$$p_k(V) := (-1)^k\, c_{2k}(E) \quad \in H^{4k}_{\mathrm{dR}}(M)$$

is called ***k*-th Pontryagin class** of V and

$$p(V) = 1 + \sum_k p_k(V) \quad \in H^{4*}_{\mathrm{dR}}(M)$$

is called **total Pontryagin class** of V.

Proposition 4.3.2. *a) The Pontryagin classes are natural with respect to pull-back diagrams, i.e. for any smooth map $f : N \to M$, we have*

$$p(f^*V) = f^*p(V). \tag{4.16}$$

b) For direct sums, we have

$$p(V_1 \oplus V_2) = p(V_1) \cdot p(V_2). \tag{4.17}$$

c) If $V_1 \cong V_2$ then we have $p(V_1) = p(V_2)$.

d) If V is trivial then we have $p(V) = 1 \in H^0_{\mathrm{dR}}(M)$.

Proof. The statements a), c) and d) follow from the corresponding statements for the Chern classes (see Proposition 4.1.17 and equation (4.9)).
To prove b) we write $E_1 = V_1 \otimes_{\mathbb{R}} \mathbb{C}$ and $E_2 = V_2 \otimes_{\mathbb{R}} \mathbb{C}$. Then we have

$$c_k(E_1 \oplus E_2) = \sum_{i+j=k} c_i(E_1) \cdot c_j(E_2),$$

and hence

$$\begin{aligned} p_j(V_1 \oplus V_2) &= (-1)^j\, c_{2j}(E_1 \oplus E_2) \\ &= (-1)^j \sum_{n+m=2j} c_n(E_1) \cdot c_m(E_2) \\ &= (-1)^j \sum_{\mu+\nu=j} c_{2\mu}(E_1) \cdot c_{2\nu}(E_2) \\ &= (-1)^j \sum_{\mu+\nu=j} (-1)^\mu\, p_\mu(V_1) \cdot (-1)^\nu\, p_\nu(V_2) \end{aligned}$$

$$= \sum_{\mu+\nu=j} p_\mu(V_1) \cdot p_\nu(V_2). \qquad \square$$

Example 4.3.3. Since the total Pontryagin class is multiplicative and vanishes for trivial bundles, we have $p(TS^n) = 1$: Similar to Example 4.1.18, we denote by $\mathcal{V}^k$ the trivial real bundle of rank k on S^n and by ν the normal bundle of $S^n \subset \mathbb{R}^{n+1}$. Then we have $\mathcal{V}^{n+1} = T\mathbb{R}^{n+1}|_{S^n} = TS^n \oplus \nu = TS^n \oplus \mathcal{V}^1$. Hence

$$1 = p\big(\mathcal{V}^{n+1}\big) = p\big(TS^n\big) \cdot \underbrace{p\big(\mathcal{V}^1\big)}_{=1} = p\big(TS^n\big).$$

Example 4.3.4. We compute the total Pontryagin class of the complex-projective space $p(T\mathbb{C}P^m) \in H^{4*}(\mathbb{C};\mathbb{R})$. As in Example 4.1.19, we use the fact that the cohomology ring of $\mathbb{C}P^m$ is generated by $c_1(\gamma_m)$, where

$$\gamma_m = \big\{(\ell, v) \in \mathbb{C}P^m \times \mathbb{C}^{m+1} \,|\, v \in \ell\big\} \to \mathbb{C}P^m$$

is the tautological line bundle.
We now consider the vector bundle $E \to \mathbb{C}P^m$ with total space

$$E = \big\{(\ell, v) \in \mathbb{C}P^m \times \mathbb{C}^{m+1} \,|\, v \perp \ell\big\}.$$

Here $\perp$ denotes orthogonality with respect to the usual Hermitian scalar product on $\mathbb{C}^{m+1}$. Then we have $\gamma_m \oplus E = \mathcal{E}^{m+1}$. Hence

$$1 = c\big(\mathcal{E}^{m+1}\big) = c(\gamma_m) \cdot c(E) = (1 + a) \cdot c(E)$$

and thus

$$c(E) = \frac{1}{1+a} = 1 - a + a^2 - a^3 \pm \ldots + (-1)^m a^m.$$

Claim: $T\mathbb{C}P^m \cong \mathrm{Hom}(\gamma_m, E)$ as complex vector bundles.

The Hopf fibration is a submersion $\pi : S^{2m+1} \subset \mathbb{C}^{m+1} \to \mathbb{C}P^m$. Denote by $V = \ker d\pi \subset TS^{2m+1}$ the vertical vector bundle. Then we obtain an isomorphism

$$TS^{2m+1} = V \oplus \pi^* E \subset S^{2m+1} \times \mathbb{C}^{m+1}.$$

Let $p \in \mathbb{C}P^m$ und $x \in \pi^{-1}(p) \subset S^{2m+1}$. Then

$$d_x\pi\big|_{E_p} : (\pi^*E)_x = E_p \to T_p\mathbb{C}P^m$$

is a complex vector space isomorphism. Hence, the map

$$\begin{aligned} \mathrm{Hom}(\gamma_m, E) &\to T\mathbb{C}P^m, \\ \lambda &\mapsto d_x\pi(\lambda(x)) \quad \text{for some } x \in \pi^{-1}(p) \end{aligned}$$

is an isomorphism of complex vector bundles. The linearity of λ and the invariance of π under the action of $\mathrm{U}(1)$ on S^{2m+1} yield the well-definedness of this map. ✓

We thus obtain

$$\begin{aligned} T\mathbb{C}P^m \oplus \mathcal{E}^1 &\cong \mathrm{Hom}(\gamma_m, E) \oplus \mathrm{Hom}(\gamma_m, \gamma_m) \\ &\cong \mathrm{Hom}(\gamma_m, E \oplus \gamma_m) \\ &= \mathrm{Hom}\big(\gamma_m, \mathcal{E}^{m+1}\big) \\ &\cong \mathrm{Hom}\big(\gamma_m, \mathcal{E}^1\big) \oplus \ldots \oplus \mathrm{Hom}\big(\gamma_m, \mathcal{E}^1\big) \\ &\cong \gamma_m^* \oplus \ldots \oplus \gamma_m^* \end{aligned}$$

and

$$\begin{aligned} c\big(T\mathbb{C}P^m\big) &= c\big(T\mathbb{C}P^m \oplus \mathcal{E}^1\big) = c\big(\gamma_m^*\big) \ldots c\big(\gamma_m^*\big) \\ &= (1-a)^{m+1}. \end{aligned}$$

This implies

$$\begin{aligned} c\big(T\mathbb{C}P^m \otimes_{\mathbb{R}} \mathbb{C}\big) &= c\Big(T\mathbb{C}P^m \oplus \overline{T\mathbb{C}P^m}\Big) \\ &= (1-a)^{m+1}(1+a)^{m+1} \\ &= \big(1-a^2\big)^{m+1}. \end{aligned}$$

Hence we obtain for the Pontryagin class of $\mathbb{C}P^m$:

$$p\big(T\mathbb{C}P^m\big) = \big(1+a^2\big)^{m+1}.$$

Thus,

$$\begin{aligned} m = 1: &\quad p(T\mathbb{C}P^1) = (1+a^2)^2 = 1 \\ m = 2: &\quad p(T\mathbb{C}P^2) = (1+a^2)^3 = 1 + 3\,a^2 \\ m = 3: &\quad p(T\mathbb{C}P^3) = (1+a^2)^4 = 1 + 4\,a^2 \\ &\quad \vdots \end{aligned}$$

Now we are building multiplicative classes from Pontryagin classes. Here $R^j = H^{4j}_{\mathrm{dR}}(M)$.

Definition 4.3.5. For a given formal power series

$$f(x) = 1 + f_1 \cdot x + f_2 \cdot x^2 + \cdots \in \mathbb{R}[\![x]\!],$$

the endomorphism

$$\Lambda_{\log f} : H^{4*}_{\mathrm{dR}}(M) \to H^{4*}_{\mathrm{dR}}(M).$$

is defined by equation (4.14). The cohomology class

$$F_p(V) := \exp\big(\Lambda_{\log f}\big[\log\big(p(V)\big)\big]\big) \in H^{4*}_{\mathrm{dR}}(M)$$

is called the **multiplicative characteristic class** of the real vector bundle V associated with the formal power series f.

Example 4.3.6. The multiplicative class for the formal power series

$$\widehat{a}(x) = \frac{\frac{\sqrt{x}}{2}}{\sinh\left(\frac{\sqrt{x}}{2}\right)} = 1 - \frac{x}{24} + \frac{7x^2}{5760} + \dots$$

is called $\widehat{\boldsymbol{A}}$**-class**. We obtain

$$\log\left(\widehat{a}(x)\right) = -\frac{x}{24} + \frac{x^2}{2880} + \dots$$

and thus

$$\Lambda_{\log(\widehat{a}(x))} = \begin{pmatrix} 0 & & & \\ & -\frac{1}{24} & & \\ & & -\frac{1}{1440} & \\ & & & \ddots \end{pmatrix}.$$

It follows that

$$\begin{aligned} \widehat{A}(V) &= \exp\left(\Lambda_{\log(\widehat{a}(x))}\left[\log\left(p(V)\right)\right]\right) \\ &= \exp\left(\Lambda_{\log(\widehat{a}(x))}\left[p_1(V) + p_2(V) - \frac{p_1(V)^2}{2} + \cdots\right]\right) \\ &= \exp\left(-\frac{p_1(V)}{24} + \frac{p_1(V)^2 - 2p_2(V)}{2880} + \dots\right) \\ &= 1 - \frac{p_1(V)}{24} + \frac{p_1(V)^2 - 2p_2(V)}{2880} + \frac{p_1(V)^2}{1152} + \cdots \\ &= 1 - \frac{p_1(V)}{24} + \frac{7p_1(V)^2 - 4p_2(V)}{5760} + \cdots . \end{aligned}$$

Hence $\widehat{A}_1(V) = -\frac{p_1(V)}{24}$ and $\widehat{A}_2(V) = \frac{7p_1(V)^2 - 4p_2(V)}{5760}$.

The $\widehat{A}$-class occurs in the index theorem for the classical Dirac operator:

Theorem 4.3.7 (Atiyah-Singer index theorem). *Let $E \to M$ be a Hermitian vector bundle over a compact Riemannian spin manifold of even dimension. Let D^E be the classical Dirac operator twisted with E and let $D^+ \in \mathscr{Diff}_1(\Sigma^+ M \otimes_{\mathbb{C}} E, \Sigma^- M \otimes_{\mathbb{C}} E)$ be its positive part. Then we have:*

$$\operatorname{ind}(D^+) = \int_M \widehat{A}(TM) \cdot \operatorname{ch}(E).$$

Example 4.3.8. The multiplicative class of a real vector bundle associated with the formal power series

$$\ell(x) = \frac{\sqrt{x}}{\tanh \sqrt{x}}$$

is called **Hirzebruch *L*-class**.

The Hirzebruch L-class shows up in another index theorem, namely the one for the signature operator $D = d + d^*$ as defined in Example 1.3.19:

Theorem 4.3.9 (Hirzebruch signature theorem). *Let M be a compact oriented $4k$-dimensional manifold. Then the index of the signature operator is given by:*

$$\operatorname{ind}(d + d^*) \overset{(3.15)}{=} b^+(M) - b^-(M) = \int_M L(TM).$$

5. Index theorems for Dirac-type operators

5.1. Proof of the Atiyah-Singer index theorem

In this section we prove the Atiyah-Singer index theorem 4.3.7 for twisted Dirac operators. We follow the proof given in Chapter 11 of the first edition of Roe's book [10]. Let E be a Hermitian vector bundle with a metric connection ∇^E over a compact Riemannian spin manifold M of dimension n and let

$$D^E \in \mathscr{Diff}_1(\Sigma M \otimes_{\mathbb{C}} E, \Sigma M \otimes_{\mathbb{C}} E)$$

be the classical Dirac operator twisted with (E, ∇^E). Let R^E be the cuvature of the connection ∇^E on E. We define the endomorphism field $\mathscr{R}^E \in C^\infty(M, \mathrm{End}(\Sigma M \otimes E))$ by

$$\mathscr{R}^E(\phi \otimes f) := \frac{1}{2} \sum_{i,j=1}^{n} b_i \cdot b_j \cdot \phi \otimes R^E(b_i, b_j) f$$

where $b_1, \dots b_n$ is a local orthonormal tangent frame. Then the following generalization of Lichnerowicz' Theorem 2.5.11 holds.

Theorem 5.1.1. *Let $E \to M$ be a Hermitian vector bundle with a metric connection ∇^E over a (not necessarily compact) Riemannian spin manifold M. Then the twisted Dirac operator D^E satisfies*

$$(D^E)^2 = (\nabla^{\Sigma M \otimes E})^* \nabla^{\Sigma M \otimes E} + \frac{\mathrm{scal}}{4} + \mathscr{R}^E$$

Proof. Exercise. □

We abbreviate $S := \Sigma M \otimes_{\mathbb{C}} E$. If n is even then with respect to the splitting

$$S = (\Sigma^+ M \otimes_{\mathbb{C}} E) \oplus (\Sigma^- M \otimes_{\mathbb{C}} E) =: S^+ \oplus S^-$$

the twisted Dirac operator takes the form

$$D^E = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

where $D^+ \in \mathscr{D}\!\mathit{iff}_1(S^+, S^-)$ and $D^- = (D^+)^* \in \mathscr{D}\!\mathit{iff}_1(S^-, S^+)$ are Dirac-type operators. As in Section 3.4 we define

$$\Delta^+ := D^- D^+ \quad \text{and} \quad \Delta^- := D^+ D^-$$

and we denote by $\Phi_j^{\Delta^\pm} \in C^\infty(M \bowtie M, S^\pm \boxtimes (S^\pm)^*)$, $j \geq 0$, the coefficients of the formal heat kernel of Δ^+ and Δ^- respectively. Then by Theorem 3.4.4 the index of D^+ is given by

$$\operatorname{ind}(D^+) = (4\pi)^{-\frac{n}{2}} \int_M a_{\frac{n}{2}}(x)\, dvol(x) \tag{5.1}$$

where $a_{\frac{n}{2}} \in C^\infty(M)$ is given by

$$a_{\frac{n}{2}}(x) := \operatorname{tr}\big(\Phi_{\frac{n}{2}}^{\Delta^+}(x,x)\big) - \operatorname{tr}\big(\Phi_{\frac{n}{2}}^{\Delta^-}(x,x)\big).$$

Definition 5.1.2. Let V be a finite dimensional real or complex vector space with a decomposition $V = V^+ \oplus V^-$. We define $\varepsilon \in \operatorname{End}(V)$ by

$$\varepsilon := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \quad V^+ \oplus V^- \to V^+ \oplus V^-.$$

For any endomorphism $\varphi \in \operatorname{End}(V)$ the number $\operatorname{Str}(\varphi) := \operatorname{tr}(\varepsilon\varphi)$ is called the **supertrace** of φ with respect to the decomposition $V = V^+ \oplus V^-$.

Remark 5.1.3. Since D^E is formally self-adjoint we have

$$(D^E)^* D^E = (D^E)^2 = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} = \begin{pmatrix} \Delta^+ & 0 \\ 0 & \Delta^- \end{pmatrix}.$$

Therefore, if we denote by $\Phi_j \in C^\infty(M \bowtie M, S \boxtimes S^*)$, $j \geq 0$, the coefficients of the formal heat kernel of $(D^E)^* D^E$ we get for all $x \in M$

$$a_{\frac{n}{2}}(x) = \operatorname{Str}(\Phi_{\frac{n}{2}}(x,x))$$

with respect to the decomposition $S_x = S_x^+ \oplus S_x^-$ and thus by (5.1):

$$\operatorname{ind}(D^+) = (4\pi)^{-\frac{n}{2}} \int_M \operatorname{Str}(\Phi_{\frac{n}{2}}(x,x))\, dvol(x). \tag{5.2}$$

We fix $p \in M$. In the following let $x \colon U \to V$ be a Riemannian normal coordinate system of M centered at $p \in U \subset M$ mapping p to $0 \in V \subset \mathbb{R}^n$. At the point $p \in U$ we define $b_i|_p := \frac{\partial}{\partial x^i}|_p$, $i = 1, \ldots, n$. Let $(b_i)_{i=1}^n$ be the local orthonormal frame of $TM|_U$ obtained by parallel transport of the vectors $b_i|_p$ along the radial geodesics emanating from p. Then we have $\nabla b_i^{\mathrm{LC}}|_p = 0$ for all i, i.e., the b_i are synchronous at p. The local orthonormal frame $h := (b_i)_{i=1}^n$ defines a local section of the frame bundle $P^{\mathrm{SO}}(M)$. We can lift h to a local section H of the $\mathrm{Spin}(n)$-principal bundle $P^{\mathrm{Spin}}(M)$. Let $v_1, \ldots, v_{2^{n/2}}$ be a basis of Σ_n. We obtain a local trivialization of the spinor bundle ΣM over U by defining the local sections $[H, v_i]$, $i = 1, \ldots, 2^{n/2}$. Together with a local frame of E we obtain a local trivialization of S.

Remark 5.1.4. With this local trivialization of S over U we get an isomorphism $S_x \cong \Sigma_n \otimes_{\mathbb{C}} E_p$ for every $x \in U$ and thus for all j we can identify $\Phi_j(x,p) \in \mathrm{Hom}(S_p, S_x)$ with an endomorphism of $\Sigma_n \otimes_{\mathbb{C}} E_p$. Since n is even we have an isomorphism of complex algebras $\mathbb{C}\mathrm{l}_n \cong \mathrm{End}(\Sigma_n)$ by Proposition 2.3.11. Therefore $x \mapsto \Phi_j(x,p)$, $x \in U$, can be considered as a function on V with values in $\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathrm{End}(E_p)$. We abbreviate

$$\begin{aligned} S_n &:= \Sigma_n \otimes_{\mathbb{C}} E_p = (\Sigma_n^+ \otimes_{\mathbb{C}} E_p) \oplus (\Sigma_n^- \otimes_{\mathbb{C}} E_p) =: S_n^+ \oplus S_n^- \\ W_n &:= \mathrm{End}(S_n) \cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathrm{End}(E_p). \end{aligned}$$

We want to apply the twisted Dirac operator $D^E \otimes \mathrm{id}_{S_p^*}$ to the section

$$U \ni x \mapsto \Phi_j(x,p) \in \mathrm{Hom}(S_p, S_x) \cong S_x \otimes S_p^*.$$

To simplify the notation we write D^E instead of $D^E \otimes \mathrm{id}_{S_p^*}$. With the above identifications we can consider D^E over U as

$$D^E: \quad C^\infty(V, W_n) \to C^\infty(V, W_n).$$

The Riemannian normal coordinate system maps geodesics in U of length r starting at p to straight line segments in V of length r starting at 0. Thus if we identify $x \in U$ with its coordinate image in V then the Euclidean heat kernel of U at p is given by

$$q_t(\cdot, p) \in C^\infty(V, W_n), \quad q_t(x,p) = (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{|x|^2}{4t}\Big)$$

where $|\cdot|$ denotes the Euclidean distance in $\mathbb{R}^n$.

Definition 5.1.5. Let $\Delta\colon\ C^\infty(V, W_n) \to C^\infty(V, W_n)$ be a formally self-adjoint Laplace-type operator. A formal power series

$$\sigma(x,t) := (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{\infty} t^j u_j(x)$$

with $u_j \in C^\infty(V, W_n)$ for all j is called an **asymptotic solution** to the heat equation $\frac{\partial u}{\partial t} + \Delta u = 0$ at the point p if for all $N \in \mathbb{N}$ there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have as $t \to 0$

$$\Big(\frac{\partial}{\partial t} + \Delta\Big)\Big\{(4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{m} t^j u_j(x)\Big\} = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \cdot \mathrm{O}(t^N).$$

Remark 5.1.6. There is a unique asymptotic solution to the heat equation $\frac{\partial u}{\partial t} + \Delta u = 0$ at p such that $u_0(0) = \mathrm{id}_{S_n}$. This can be shown by determining the functions u_j recursively as in the proof of Proposition 3.2.7. If Φ_j denote the coefficients of the formal heat kernel of Δ then we have $u_j(x) = \Phi_j(x,p)$ for all $j \in \mathbb{N}_0$ and all $x \in V$.

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Then the elements $e_I := e_{i_1} \cdot \ldots \cdot e_{i_k}$, with multi-indices $I = (1 \le i_1 < \ldots < i_k \le n)$, $k \ge 0$, form a complex basis of $\mathbb{C}\mathrm{l}_n$.

Lemma 5.1.7. *Let n be even, $\alpha \in \mathrm{End}(E_p)$ and let $c = \sum_I c_I e_I \otimes \alpha \in \mathrm{End}(S_n)$ with $c_I \in \mathbb{C}$, where the sum is taken over all multi-indices I. Then with respect to the splitting $S_n = S_n^+ \oplus S_n^-$ we have*

$$\mathrm{Str}(c) = (-2i)^{n/2}\mathrm{tr}(\alpha)c_{12\ldots n}.$$

Proof. Let $\omega := e_1 \cdot \ldots \cdot e_n \in \mathbb{C}\mathrm{l}_n$ be the volume element. By the equation (2.9) we have

$$\Sigma_n^{\pm} = \{z \in \Sigma_n \,|\, i^{n/2}\omega \cdot z = \pm z\}.$$

Thus with

$$\varepsilon := i^{n/2}\omega \cdot \otimes \mathrm{id}_{E_p} \in \mathrm{End}(S_n)$$

we have $\mathrm{Str}(c) = \mathrm{tr}(\varepsilon c)$ for all $c \in \mathrm{End}(S_n)$. Let $e_I := e_{i_1} \cdot \ldots \cdot e_{i_k} \in \mathbb{C}\mathrm{l}_n \cong \mathrm{End}(\Sigma_n)$. We have $\mathrm{tr}(e_I \otimes \alpha) = \mathrm{tr}(e_I)\mathrm{tr}(\alpha)$.
Case 1: k is odd: Let $v \in \Sigma_n^+$. We have $\omega \cdot e_{i_j} = (-1)^{n-1} e_{i_j} \cdot \omega$ for all j. We get

$$i^{n/2}\omega \cdot e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot v = \underbrace{(-1)^{k(n-1)}}_{=-1} e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot i^{n/2}\omega \cdot v = -e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot v$$

and thus $e_I \cdot v \in \Sigma_n^-$. We have shown that $e_I(\Sigma_n^+) \subset \Sigma_n^-$ and similarly one shows that $e_I(\Sigma_n^-) \subset \Sigma_n^+$. It follows that $\mathrm{tr}(e_I) = 0$.
Case 2: k is even and $k \neq 0$: We have

$$e_{i_1} \cdot \ldots \cdot e_{i_k} = (-1)^{k-1} e_{i_2} \cdot \ldots \cdot e_{i_k} \cdot e_{i_1} = -e_{i_2} \cdot \ldots \cdot e_{i_k} \cdot e_{i_1}.$$

Using this together with the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in \mathrm{End}(\Sigma_n)$ we get

$$\mathrm{tr}(e_{i_1} \cdot \ldots \cdot e_{i_k}) = -\mathrm{tr}(e_{i_2} \cdot \ldots \cdot e_{i_k} \cdot e_{i_1}) = -\mathrm{tr}(e_{i_1} \cdot \ldots \cdot e_{i_k})$$

and thus $\mathrm{tr}(e_{i_1} \cdot \ldots \cdot e_{i_k}) = 0$.
Case 3: $I = \emptyset$: Then $e_I = \mathrm{id}_{\Sigma_n}$ and thus $\mathrm{tr}(e_I \otimes \alpha) = \dim(\Sigma_n)\mathrm{tr}(\alpha) = 2^{n/2}\mathrm{tr}(\alpha)$.
We have shown that for all $c = \sum_I c_I e_I \otimes \alpha$ we have $\mathrm{tr}(c) = 2^{n/2}\mathrm{tr}(\alpha)c_\emptyset$. Therefore

$$\begin{aligned}\mathrm{Str}(c) = \mathrm{tr}(\varepsilon c) = i^{n/2}2^{n/2}\mathrm{tr}(\alpha)(\omega c)_\emptyset &= (2i)^{n/2}\mathrm{tr}(\alpha)c_{12\ldots n}(\omega^2)_\emptyset \\ &= (-2i)^{n/2}\mathrm{tr}(\alpha)c_{12\ldots n}.\end{aligned}$$

∎

Definition 5.1.8. (i) An element $c = \sum_I c_I e_I \in \mathbb{C}l_n$ or $c = \sum_I c_I e_I \otimes \alpha \in W_n$ with $c_I \in \mathbb{C}$ is called of **degree** k if $c_I = 0$ for all multi-indices I whose cardinality is not equal to k.

(ii) For $\lambda > 0$ we define the **rescaling operator** $R_\lambda \colon C^\infty(V, W_n) \to C^\infty(\lambda^{-1}V, W_n)$ as follows: If $\varphi \in C^\infty(V, W_n)$ is of degree k everywhere, then we define

$$(R_\lambda \varphi)(x) := \lambda^{-k}\varphi(\lambda x), \quad x \in \lambda^{-1}V.$$

By Lemma 5.1.7 and the equation (5.2) we get a formula for $\operatorname{ind}(D^+)$ if for every $p \in M$ we can determine the n-degree part of the coefficient $\Phi_{\frac{n}{2}}(p,p)$ considered as an element of W_n. In order to achieve this we use a rescaling trick due to Ezra Getzler.
For $\lambda \geq 1$ we define

$$D_\lambda := \frac{1}{\lambda} R_\lambda^{-1} \circ D^E \circ R_\lambda : \quad C^\infty(V, W_n) \to C^\infty(V, W_n).$$

It is easy to see that D_λ^2 is a formally self-adjoint Laplace-type operator for every $\lambda \geq 1$. Getzler's idea is to consider the asymptotic solution to the heat equation for D_λ^2 and then consider the limit $\lambda \to \infty$.

Proposition 5.1.9. *Let σ be the asymptotic solution to the heat equation for $(D^E)^2$ at p with $u_0(0) = \mathrm{id}_{S_n}$. For $\lambda > 0$ define*

$$\sigma^\lambda(x,t) := \lambda^{-n}(R_\lambda^{-1}\sigma)\Big(x, \frac{t}{\lambda^2}\Big).$$

Then σ^λ is the asymptotic solution to the heat equation for D_λ^2 at p with $u_0^\lambda(0) = \mathrm{id}_{S_n}$.

Proof. Denote the coefficients of σ by u_j, $j \geq 0$. Let $N \in \mathbb{N}$ and let $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$ we have

$$\Big(\frac{\partial}{\partial t} + (D^E)^2\Big) \underbrace{\Big\{(4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^m t^j u_j(x)\Big\}}_{=:\sigma_m(x,t)} = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \cdot S_N(x,t)$$

where $S_N(x,t) = \mathrm{O}(t^N)$ as $t \to 0$. We compute

$$\begin{aligned}\Big(\frac{\partial}{\partial t} + D_\lambda^2\Big)\sigma_m^\lambda(x,t) &= \Big(\frac{\partial}{\partial t} + \lambda^{-2}R_\lambda^{-1}(D^E)^2 R_\lambda\Big)\Big(\lambda^{-n}R_\lambda^{-1}\sigma_m\Big)\Big(x, \frac{t}{\lambda^2}\Big) \\ &= \lambda^{-n}R_\lambda^{-1}\lambda^{-2}\frac{\partial \sigma_m}{\partial t}\Big(x, \frac{t}{\lambda^2}\Big) + \lambda^{-2}R_\lambda^{-1}\lambda^{-n}((D^E)^2\sigma_m)\Big(x, \frac{t}{\lambda^2}\Big)\end{aligned}$$

$$
\begin{aligned}
&= \lambda^{-n-2} R_\lambda^{-1} \Big(\frac{\partial \sigma_m}{\partial t} + (D^E)^2 \sigma_m \Big) \Big(x, \frac{t}{\lambda^2} \Big) \\
&= \lambda^{-n-2} R_\lambda^{-1} \Big((4\pi t \lambda^{-2})^{-n/2} \exp\Big(- \frac{|x|^2 \lambda^2}{4t} \Big) S_N \Big) \Big(x, \frac{t}{\lambda^2} \Big) \\
&= \lambda^{-2} (4\pi t)^{-n/2} \exp\Big(- \frac{|x|^2}{4t} \Big) R_\lambda^{-1} S_N \Big(x, \frac{t}{\lambda^2} \Big).
\end{aligned}
$$

With $\tilde{S}_N(x,t) := R_\lambda^{-1} S_N(x, \frac{t}{\lambda^2})$ we have $\tilde{S}_N(x,t) = \mathrm{O}(t^N)$ as $t \to 0$. Therefore σ^λ is an asymptotic solution for D_λ^2 at p. Moreover we have

$$
\begin{aligned}
(4\pi t)^{-n/2} \exp\Big(- \frac{|x|^2}{4t} \Big) u_0^\lambda(x) &= \lambda^{-n} R_\lambda^{-1} \Big((4\pi t)^{-n/2} \exp\Big(- \frac{|x|^2}{4t} \Big) u_0 \Big) \Big(x, \frac{t}{\lambda^2} \Big) \\
&= \lambda^{-n} (4\pi t \lambda^{-2})^{-n/2} \exp\Big(- \frac{|\lambda^{-1} x|^2}{4t\lambda^{-2}} \Big) R_\lambda^{-1} u_0(x) \\
&= (4\pi t)^{-n/2} \exp\Big(- \frac{|x|^2}{4t} \Big) R_\lambda^{-1} u_0(x)
\end{aligned}
$$

and since $u_0(0) = \mathrm{id}_{S_n}$ has degree 0, we get $R_\lambda^{-1} u_0(0) = \mathrm{id}_{S_n}$ and thus $u_0^\lambda(0) = \mathrm{id}_{S_n}$. $\square$

Remark 5.1.10. Recall that the map $\Phi\colon \Lambda^\bullet \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{Cl}_n$ defined on standard basis elements by

$$
e_{i_1} \wedge \ldots \wedge e_{i_k} \mapsto e_{i_1} \cdot \ldots \cdot e_{i_k}, \quad 1 \le i_1 < \ldots < i_k \le n
$$

together with linear extension is an isomorphism of vector spaces. We define exterior multiplication on Cl_n as follows: For $v, w \in \mathrm{Cl}_n$:

$$
v \wedge w := \Phi(\Phi^{-1}(v) \wedge \Phi^{-1}(w))
$$

where on the right hand side $\wedge$ denotes exterior multiplication in $\Lambda^\bullet \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}$. In particular, if $v \in \mathrm{Cl}_n$ is of degree k and $w \in \mathrm{Cl}_n$ is of degree ℓ, then $v \wedge w$ is of degree $k + \ell$. Note that 0 is of any degree.

Lemma 5.1.11. *Let $c \in C^\infty(V, \mathrm{Cl}_n)$ be of degree k everywhere. For $\lambda \ge 1$ we define*

$$
M_{c,\lambda} := \lambda^{-k} R_\lambda^{-1} \circ c \circ R_\lambda : \quad C^\infty(V, W_n) \to C^\infty(V, W_n),
$$

where the map c is given by Clifford multiplication with c. Then as $\lambda \to \infty$ we get for all $\varphi \in C^\infty(V, W_n)$ and all $x \in V$: $(M_{c,\lambda}\varphi)(x) \to c(0) \wedge \varphi(x)$.

Proof. Let φ be of degree ℓ everywhere. Then $c(x) \cdot \varphi(\lambda x) = c(x) \wedge \varphi(\lambda x) + y$ where $c(x) \wedge \varphi(\lambda x)$ is of degree $k + \ell$ and y is a sum of terms of degree less than $k + \ell$. It follows that

$$\begin{aligned}(M_{c,\lambda}\varphi)(x) &= \lambda^{-k-\ell}R_\lambda^{-1}\big(c(.)\cdot\varphi(\lambda.)\big)(x)\\ &= \lambda^{-k-\ell}\big(\lambda^{k+\ell}c(\lambda^{-1}x)\wedge\varphi(x)+\mathrm{O}(\lambda^{k+\ell-1})\big)\\ &= c(\lambda^{-1}x)\wedge\varphi(x)+\mathrm{O}(\lambda^{-1})\\ &\to c(0)\wedge\varphi(x)\end{aligned}$$

as $\lambda\to\infty$. □

We define $\Theta_{jk}, F \in \Lambda^{\mathrm{even}}\mathbb{R}^n \otimes \mathrm{End}(E_p)$ by

$$\Theta_{jk} := \sum_{\alpha,\beta=1}^n R_{jk\alpha\beta}(0)e_\alpha\wedge e_\beta\otimes \mathrm{id}_{E_p}, \quad 1\le j,k\le n$$

$$F := \frac{1}{2}\sum_{i,j=1}^n e_i\wedge e_j\otimes R^E_{ij}(0)$$

where $e_1,\dots,e_n$ is the standard basis of $\mathbb{R}^n$, $R_{jk\alpha\beta}(0)\in\mathbb{R}$ denote the components of the Riemann curvature tensor of M at p and $R^E_{ij}(0)\in\mathrm{End}(E_p)$ denote the components of the curvature of ∇^E at p. Using $\Lambda^{\mathrm{even}}\mathbb{R}^n\subset\mathbb{Cl}_n$ we may also regard $\Theta_{jk}, F\in W_n$.

Proposition 5.1.12. *As $\lambda\to\infty$ the coefficients of $D^2_\lambda\colon C^\infty(V,W_n)\to C^\infty(V,W_n)$ tend to the coefficients of the operator $L\colon C^\infty(V,W_n)\to C^\infty(V,W_n)$ given by*

$$L := -\sum_{j=1}^n\Big(\frac{\partial}{\partial x^j}-\frac{1}{8}\sum_{k=1}^n x^k\Theta_{jk}\Big)^2+F.$$

Moreover, F commutes with every element Θ_{jk} in the algebra W_n.

Proof. We use the Christoffel symbols $\Gamma^\beta_{j\alpha}\colon V\to\mathbb{R}$ with respect to the local orthonormal frame $(b_i)_{i=1}^n$ of $TM|_U$ defined by

$$\nabla^{\mathrm{LC}}_{b_j}b_\alpha=\sum_{\beta=1}^n\Gamma^\beta_{j\alpha}b_\beta.$$

For all β,j,α we have $\Gamma^\beta_{j\alpha}(0)=0$ and thus we have

$$\Gamma^\beta_{j\alpha}(x)=\sum_{k=1}^n A_{jk\alpha\beta}x^k+\mathrm{O}(|x|^2)$$

where $A_{jk\alpha\beta} \in \mathbb{R}$ are such that $\partial_{b_i}\Gamma^{\beta}_{j\alpha}(0) = A_{ji\alpha\beta}$. Since the b_j are obtained by parallel transport along radial geodesics we get

$$0 = \sum_{j=1}^{n} x^j \nabla^{\mathrm{LC}}_{b_j} b_\alpha = \sum_{j,\beta=1}^{n} x^j \Gamma^{\beta}_{j\alpha}(x) b_\beta = \sum_{j,k,\beta=1}^{n} A_{jk\alpha\beta} x^j x^k + \mathrm{O}(|x|^3).$$

Thus for all k, j, α, β we have $A_{jk\alpha\beta} = -A_{kj\alpha\beta}$ i.e., $\partial_{b_i}\Gamma^{\beta}_{j\alpha}(0) = -\partial_{b_j}\Gamma^{\beta}_{i\alpha}(0)$. Now we have

$$\nabla_{b_i}\nabla_{b_j} b_\alpha = (\partial_{b_i}\Gamma^{\beta}_{j\alpha}) b_\beta + \sum_{\gamma=1}^{n} \Gamma^{\beta}_{j\alpha}\Gamma^{\gamma}_{i\beta} b_\gamma$$

and therefore

$$\begin{aligned} R_{ij\alpha\beta}(0) &= \langle \nabla_{b_i}\nabla_{b_j} b_\alpha - \nabla_{b_j}\nabla_{b_i} b_\alpha, b_\beta\rangle(0) = \partial_{b_i}\Gamma^{\beta}_{j\alpha}(0) - \partial_{b_j}\Gamma^{\beta}_{i\alpha}(0) \\ &= -2\partial_{b_j}\Gamma^{\beta}_{i\alpha}(0) = -2A_{ij\alpha\beta}. \end{aligned}$$

It follows that

$$\Gamma^{\beta}_{j\alpha}(x) = -\frac{1}{2}\sum_{k=1}^{n} R_{jk\alpha\beta}(0) x^k + \mathrm{O}(|x|^2).$$

Moreover we can write every spinor field on U in the form $\psi = [H, \varphi]$ with a local section H of $P^{\mathrm{Spin}}(M)$ and $\varphi\colon V \to \Sigma_n$. For the spinor connection we have

$$\begin{aligned} \nabla^{\Sigma}_{b_j}\psi &= \Big[H, \partial_{e_j}\varphi + \frac{1}{4}\sum_{\alpha,\beta=1}^{n} \Gamma^{\beta}_{j\alpha} e_\alpha \cdot e_\beta \cdot \varphi\Big] \\ &= \Big[H, \partial_{e_j}\varphi - \frac{1}{8}\sum_{\alpha,\beta,k=1}^{n} R_{jk\alpha\beta}(0) x^k e_\alpha \cdot e_\beta \cdot \varphi + |x|^2 v(x)\cdot\varphi\Big] \end{aligned}$$

where v is of degree 2. Let φ be of degree ℓ. Since $e_\alpha \wedge e_\beta$ and v are of degree 2 it follows from Lemma 5.1.11 that

$$\begin{aligned} \frac{1}{\lambda} R_\lambda^{-1}\partial_j R_\lambda \varphi(x) &= \frac{1}{\lambda} R_\lambda^{-1}\partial_j(\lambda^{-\ell}\varphi(\lambda x)) \\ &= \lambda^{-\ell-1} R_\lambda^{-1}\lambda \partial_j\varphi(\lambda x) \\ &= \partial_j\varphi(x), \\ \frac{1}{\lambda} R_\lambda^{-1} x^k e_\alpha \cdot e_\beta \cdot R_\lambda\varphi(x) &= \frac{x^k}{\lambda^2} R_\lambda^{-1} e_\alpha \cdot e_\beta \cdot R_\lambda \varphi(x) \\ &= \frac{x^k}{\lambda^2} R_\lambda^{-1}(e_\alpha \wedge e_\beta \wedge R_\lambda\varphi(x) + \text{lower degree}) \\ &= x^k e_\alpha \wedge e_\beta \wedge \varphi(x) + \mathrm{O}(\lambda^{-1}), \\ \frac{1}{\lambda} R_\lambda^{-1}|x|^2 v(x)\cdot R_\lambda\varphi(x) &= \frac{1}{\lambda}\underbrace{\frac{|x|^2}{\lambda^2} R_\lambda^{-1} v(x)\cdot R_\lambda\varphi(x)}_{\to v(0)\wedge\varphi(0)} = \mathrm{O}(\lambda^{-1}) \end{aligned}$$

and thus

$$\frac{1}{\lambda}R_\lambda^{-1}\nabla^\Sigma_{b_j}R_\lambda\psi = \Big[H, \partial_{e_j}\varphi - \frac{1}{8}\sum_{\alpha,\beta,k=1}^{n} R_{jk\alpha\beta}(0)x^k e_\alpha\wedge e_\beta\wedge\varphi + \mathrm{O}(\lambda^{-1})\Big].$$

Let $\eta_1,\dots,\eta_s$ be a local frame of E and define $\Gamma^{E,k}_{ji}\in C^\infty(V)$ by

$$\nabla^E_{b_j}\eta_i = \sum_{i=1}^{s}\Gamma^{E,k}_{ji}\eta_k.$$

We write a local section e of E as $e=\sum_{i=1}^s f_i\eta_i$ with $f_i\in C^\infty(V)$ and we get

$$\frac{1}{\lambda}R_\lambda^{-1}\nabla^E_{b_j}R_\lambda e = \sum_{i=1}^{s}\frac{1}{\lambda}(R_\lambda^{-1}\partial_{b_j}R_\lambda f_i)\eta_i + \frac{1}{\lambda}\sum_{i,k=1}^{s} f_i\Gamma^{E,k}_{ji}\eta_k = \sum_{i=1}^{s}(\partial_{b_j}f_i)\eta_i + \mathrm{O}(\lambda^{-1}).$$

Altogether we obtain

$$\nabla^{\Sigma M\otimes E}_{b_j}(\psi\otimes e) = \partial_{b_j}(\psi\otimes e) - \frac{1}{8}\sum_{k=1}^{n}\Theta_{jk}x^k\psi\otimes e + \mathrm{O}(\lambda^{-1}).$$

By Theorem 5.1.1 we have

$$(D^E)^2 = -\sum_{j=1}^{n}\nabla^{\Sigma M\otimes E}_{b_j}\nabla^{\Sigma M\otimes E}_{b_j} + \sum_{i,j=1}^{n}\Gamma^j_{ii}\nabla^{\Sigma M\otimes E}_{b_j} + \frac{1}{4}\mathrm{scal} + \mathscr{R}^E.$$

Since scal is of degree 0 and $\mathscr{R}^E$ is of degree 2 we get by Lemma 5.1.11

$$\begin{aligned}
D_\lambda^2 &= \lambda^{-2}R_\lambda^{-1}(D^E)^2R_\lambda\\
&= \frac{1}{\lambda^2}R_\lambda^{-1}\Big(-\sum_{j=1}^{n}\nabla^{\Sigma M\otimes E}_{b_j}\nabla^{\Sigma M\otimes E}_{b_j} + \sum_{i,j=1}^{n}\Gamma^j_{ii}\nabla^{\Sigma M\otimes E}_{b_j}\Big)R_\lambda\\
&\quad + \frac{1}{4\lambda^2}R_\lambda^{-1}\mathrm{scal}R_\lambda + \frac{1}{\lambda^2}R_\lambda^{-1}\mathscr{R}^E R_\lambda\\
&= -\sum_{j=1}^{n}\Big(\partial_j - \sum_{k=1}^{n}\frac{\Theta_{jk}}{8}x^k\Big)\Big(\partial_j - \sum_{\ell=1}^{n}\frac{\Theta_{j\ell}}{8}x^\ell\Big)\\
&\quad + \sum_{i,j=1}^{n}\frac{R_\lambda^{-1}\Gamma^j_{ii}}{\lambda^2}\Big(\partial_j - \sum_{k=1}^{n}\frac{\Theta_{jk}}{8}x^k\Big) + \mathrm{O}(\lambda^{-1}) + \frac{\mathrm{scal}}{4\lambda^2} + \frac{1}{\lambda^2}R_\lambda^{-1}\mathscr{R}^E R_\lambda\\
&\to -\sum_{j=1}^{n}\Big(\partial_j - \frac{1}{8}\sum_{k=1}^{n}x^k\Theta_{jk}\Big)^2 + F
\end{aligned}$$

as $\lambda\to\infty$. Obviously, F commutes with every element Θ_{jk}, since the algebra $\Lambda^{\mathrm{even}}\mathbb{R}^n$ is commutative and since id_{E_p} commutes with all elements $R^E_{ij}(0)$. $\square$

Lemma 5.1.13. *For every $p \in M$ choose a local trivialization of S as above and let*

$$\sigma_p^L(x,t) = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{\infty} t^j u_{p,j}^L(x)$$

be the asymptotic solution to the heat equation for L at p with $u_{p,j}^L \in C^\infty(V, W_n)$ for all j and $u_{p,0}^L(0) = \mathrm{id}_{S_n}$. Then for the smooth function $g\colon M \to \mathbb{R}$, $g(p) := \mathrm{Str}(u_{p,\frac{n}{2}}^L(0))$ we have

$$\mathrm{ind}(D^+) = (4\pi)^{-\frac{n}{2}} \int_M g(p)\, dvol(p).$$

Proof. By Remark 5.1.6 the asymptotic solution to the heat equation for $(D^E)^2$ at p with $u_0(0) = \mathrm{id}_{S_n}$ is given by

$$\sigma(x,t) = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{\infty} t^j \Phi_j(x,p),$$

where Φ_j are the coefficients of the formal heat kernel for $(D^E)^2$. The above local trivialization of S gives us an identification of $\Phi_j(x,p) \in \mathrm{Hom}(S_p, S_x)$ with an element $\sum_I \Phi_{j,I}(x) e_I \otimes \alpha_j(x)$ with $\Phi_{j,I}(x) \in \mathbb{C}$, $\alpha_j(x) \in \mathrm{End}(E_p)$ and the sum is taken over all multi-indices I. Therefore we can write

$$\sigma(x,t) = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{\infty} \sum_I t^j \Phi_{j,I}(x) e_I \otimes \alpha_j(x).$$

By Proposition 5.1.9 the asymptotic solution to the heat equation for D_λ^2 at p with $u_0^\lambda(0) = \mathrm{id}_{S_n}$ is given by

$$\sigma^\lambda(x,t) = (4\pi t)^{-n/2} \exp\Big(-\frac{|x|^2}{4t}\Big) \sum_{j=0}^{\infty} \sum_I t^j \lambda^{-2j+|I|} \Phi_{j,I}\Big(\frac{x}{\lambda}\Big) e_I \otimes \alpha_j\Big(\frac{x}{\lambda}\Big),$$

where $|I|$ is the cardinality of I. By Proposition 5.1.12 as $\lambda \to \infty$ the coefficients of the asymptotic solution σ^λ tend to the coefficients $u_{p,j}^L$ of the asymptotic solution σ_p^L. For $j = \frac{n}{2}$ we have $\lambda^{-2j+|I|} \to 0$ as $\lambda \to \infty$ for all I with $|I| < n$ and thus

$$u_{p,\frac{n}{2}}^L(0) = \Phi_{\frac{n}{2},12\ldots n}(0) e_1 \cdot \ldots \cdot e_n \otimes \alpha_{\frac{n}{2}}(0)$$

and together with Lemma 5.1.7 we get

$$\mathrm{Str}(u_{p,\frac{n}{2}}^L(0)) = (-2i)^{n/2} \mathrm{tr}(\alpha_{\frac{n}{2}}(0)) \Phi_{\frac{n}{2},12\ldots n}(0) = \mathrm{Str}(\Phi_{\frac{n}{2}}(p,p)).$$

The assertion follows from equation (5.2). $\square$

It remains to determine the coefficient $u^L_{p,\frac{n}{2}}(0)$. Now, the operator L defined above has coefficients $\Theta_{jk}, F \in \Lambda^{\mathrm{even}}\mathbb{R}^n \otimes \mathrm{End}(E_p)$. In order to solve the heat equation for L we first consider an operator with scalar coefficients instead.

Proposition 5.1.14 (Mehler's formula). *Let n be even, let $A \in \mathrm{Mat}(n \times n; \mathbb{R})$ be an antisymmetric matrix and let $B \in \mathbb{R}$. Then the heat equation for the operator*

$$H: \quad C^\infty(V,\mathbb{C}) \to C^\infty(V,\mathbb{C}), \quad H := -\sum_{j=1}^n \Big(\frac{\partial}{\partial x^j} - \frac{1}{8}\sum_{k=1}^n x^k A_{jk}\Big)^2 + B$$

for t close to 0 has a solution

$$w_t^H(x) = (4\pi t)^{-\frac{n}{2}} \det\Big(\frac{tA/4}{\sinh(tA/4)}\Big)^{1/2} \exp\Big(-\frac{1}{4t}\Big\langle \frac{tA}{4}\coth\Big(\frac{tA}{4}\Big)x, x\Big\rangle\Big)\exp(-tB)$$

where the matrices $\frac{tA/4}{\sinh(tA/4)}$ and $\frac{tA}{4}\coth(\frac{tA}{4})$ are defined by converging power series.

Proof. Let $S = (S_{ij})_{i,j} \in \mathrm{O}(n)$ be an orthogonal matrix and define the new coordinates $y^i := \sum_{j=1}^n S_{ji}x^j$, $i = 1, \dots, n$. A short calculation shows that

$$H = -\sum_{k=1}^n \Big(\frac{\partial}{\partial y^k} - \frac{1}{8}\sum_{\ell=1}^n y^\ell (S^\top A S)_{k\ell}\Big)^2 + B.$$

Since the matrix A is antisymmetric, we can choose S in such a way that $S^\top A S = D$ is in block diagonal form with 2×2 blocks

$$\begin{pmatrix} 0 & \theta_k \\ -\theta_k & 0 \end{pmatrix}$$

on the diagonal. Writing $x_1, y_1, \dots, x_{n/2}, y_{n/2}$ for the new coordinates we get

$$H = \sum_{k=1}^{n/2} \underbrace{\Big(-\Big(\frac{\partial}{\partial x_k} - \frac{1}{8}\theta_k y_k\Big)^2 - \Big(\frac{\partial}{\partial y_k} + \frac{1}{8}\theta_k x_k\Big)^2\Big)}_{=:H_k} + B.$$

For every k let $(x_k, y_k) \mapsto w_t^k(x_k, y_k)$ be a solution to the heat equation $(\frac{\partial}{\partial t} + H_k)w_t^k = 0$. Then $w_t := w_t^1 \cdot \ldots \cdot w_t^{n/2}$ is a solution to the heat equation $(\frac{\partial}{\partial t} + \sum_k H_k)w_t = 0$. Therefore it is sufficient to consider the 2-dimensional case $H_k = H_{k,0} + H_{k,1}$ where

$$H_{k,0} := -\Big(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial y_k^2}\Big) - \frac{1}{64}\theta_k^2(x_k^2 + y_k^2)$$
$$H_{k,1} := \frac{1}{4}\theta_k\Big(y_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial y_k}\Big).$$

One checks that

$$w_t^k(x,y) := \frac{i\theta_k}{16\pi \sinh(\frac{it\theta_k}{4})} \exp\Big(- \frac{i\theta_k(x_k^2+y_k^2)\coth(\frac{it\theta_k}{4})}{16}\Big)$$

solves the heat equation $(\frac{\partial}{\partial t} + H_k)w_t^k = 0$. Indeed we have

$$\begin{aligned}
\frac{\partial w_t^k}{\partial x_k}(x_k,y_k) &= -\frac{1}{8} i\theta_k x_k \coth\Big(\frac{it\theta_k}{4}\Big) w_t^k(x_k,y_k) \\
\frac{\partial^2 w_t^k}{\partial x_k^2}(x_k,y_k) &= \Big(- \frac{1}{64}\theta_k^2 x_k^2 \coth^2\Big(\frac{it\theta_k}{4}\Big) - \frac{1}{8} i\theta_k \coth\Big(\frac{it\theta_k}{4}\Big)\Big) w_t^k(x_k,y_k) \\
H_{k,0} w_t^k(x_k,y_k) &= \Big(\frac{1}{4} i\theta_k \coth\Big(\frac{it\theta_k}{4}\Big) + \frac{\theta_k^2(x_k^2+y_k^2)}{64\sinh^2(\frac{it\theta_k}{4})}\Big) w_t^k(x_k,y_k) = -\frac{\partial w_t^k}{\partial t}(x_k,y_k) \\
H_{k,1} w_t^k(x_k,y_k) &= 0.
\end{aligned}$$

Since $H = \sum_k H_k + B$ it follows that the function $w_t^H(x) := e^{-tB} w_t(x)$ solves the heat equation $(\frac{\partial}{\partial t} + H)w_t^H = 0$. □

Proposition 5.1.15. *Let σ_p^L be the asymptotic solution to the heat equation for L at p with $u_{p,0}(0) = \mathrm{id}_{S_n}$. Then at $x = 0$ we have*

$$\sigma_p^L(0,t) = (4\pi t)^{-\frac{n}{2}} \det\Big(\frac{t\Theta/4}{\sinh(t\Theta/4)}\Big)^{1/2} \exp(-tF)$$

where Θ is the matrix with entries Θ_{jk} and where $\Theta_{jk}, F \in W_n$ are defined as above.

Remark 5.1.16. The matrix $(\frac{t\Theta/4}{\sinh(t\Theta/4)})^{1/2}$ is defined by the power series for the function $f(x) = (\frac{x}{\sinh(x)})^{1/2}$, i.e.,

$$\Big(\frac{t\Theta/4}{\sinh(t\Theta/4)}\Big)^{1/2} = \mathbb{1} - \frac{1}{48}t^2\Theta^2 + \frac{17}{7680}t^4\Theta^4 + \mathrm{O}(t^6\Theta^6)$$

and this series is a finite sum, since Θ has entries in $\Lambda^{\mathrm{even}}\mathbb{R}^n$ and is therefore nilpotent. The determinant of this matrix is then a polynomial in the entries Θ_{jk} and thus an element of W_n. For the same reason, $\exp(-tF)$ is a polynomial in F and an element of W_n.

Proof. The solution w_t^H to the heat equation for H from Proposition 5.1.14 satisfies

$$w_t^H(0) = (4\pi t)^{-\frac{n}{2}} \det\Big(\frac{tA/4}{\sinh(tA/4)}\Big)^{1/2} \exp(-tB).$$

Moreover, the formula for w_t^H shows that as $t \to 0$ we have

$$w_t^H(x) = (4\pi t)^{-\frac{n}{2}} \exp\Big(-\frac{|x|^2}{4t}\Big)(1 + \mathrm{O}(t)).$$

We define w_t^L by w_t^H, where we replace the scalars A_{jk} and B by the elements Θ_{jk} and F of W_n. Since F commutes with every element Θ_{jk} the map w_t^L solves the heat equation for L. In particular, w_t^L is an asymptotic solution to the heat equation for L at p whose coefficient of order 0 is equal to id_{S_n}. Since the asymptotic solution with this property is unique we conclude that $\sigma_p^L(0,t) = w_t^L(0)$. □

Lemma 5.1.17. *Let M be a Riemannian manifold of even dimension. For a formal power series*

$$f(x) = 1 + f_1 \cdot x + f_2 \cdot x^2 + \ldots \in \mathbb{R}[\![x]\!]$$

let $F_p(TM)$ denote the multiplicative characteristic class of TM associated with f as defined in Definition 4.3.5. Moreover, define the formal power series

$$\tilde{f}(x) := \sqrt{f(x^2)} \in \mathbb{R}[\![x]\!].$$

If Ω is the matrix of curvature 2-forms of some connection on $TM \otimes_{\mathbb{R}} \mathbb{C}$, then we have $F_p(TM) = \det(\tilde{f}(\frac{\Omega}{2\pi i}))$.

Proof. The matrix Ω is similar to a block diagonal matrix, and thus we may assume

$$\Omega = \begin{pmatrix} 0 & -\theta_1 & & & \\ \theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_m \\ & & & \theta_m & 0 \end{pmatrix}$$

with $\theta_k \in \Lambda^2\mathbb{R}^n$ for all k and $\dim(M) = 2m$. It follows that

$$\det\Big(\mathbb{1}_n + \frac{1}{2\pi i}\Omega\Big) = \prod_{k=1}^m \Big(1 + \frac{\theta_k^2}{(2\pi i)^2}\Big) = 1 + \sum_{j=1}^m \sigma_j\Big(\frac{\theta_1^2}{(2\pi i)^2}, \ldots, \frac{\theta_m^2}{(2\pi i)^2}\Big),$$

where σ_j denotes the j-th elementary-symmetric polynomial. Thus, for the Chern classes of $TM \otimes_{\mathbb{R}} \mathbb{C}$ we get for $1 \leq j \leq m$: $c_{2j-1}(TM \otimes_{\mathbb{R}} \mathbb{C}) = 0$ and

$$c_{2j}(TM \otimes_{\mathbb{R}} \mathbb{C}) = \sigma_j\Big(\frac{\theta_1^2}{(2\pi i)^2}, \ldots, \frac{\theta_m^2}{(2\pi i)^2}\Big).$$

Thus, for the Pontryagin classes of TM we get for $1 \leq j \leq m$:

$$p_j(TM) = (-1)^j c_{2j}(TM \otimes_{\mathbb{R}} \mathbb{C}) = \sigma_j\Big(\frac{(-\theta_1^2)}{(2\pi i)^2}, \ldots, \frac{(-\theta_m^2)}{(2\pi i)^2}\Big)$$

and thus the total Pontryagin class is equal to

$$p(TM) = \prod_{k=1}^{m} \Big(1 + \frac{(-\theta_k^2)}{(2\pi i)^2}\Big)$$

For $k = 1, \dots, m$ we write $x_k := \frac{-\theta_k^2}{(2\pi i)^2}$ and we get

$$\log(p(TM)) = \sum_{k=1}^{m} \log(1 + x_k) = \sum_{k=1}^{m}\sum_{j=1}^{\infty} h_j x_k^j \quad \text{with } h_j := \frac{(-1)^{j+1}}{j}$$

Writing $\log f = \sum_{j=1}^{\infty} (\log f)_j x^j$ we get

$$\begin{aligned}\Lambda_{\log f}[\log(p(TM))] &= \sum_{k=1}^{m}\sum_{j=1}^{\infty}(-1)^{j+1} j (\log f)_j h_j x_k^j = \sum_{k=1}^{m}\sum_{j=1}^{\infty}(\log f)_j x_k^j \\ &= \sum_{k=1}^{m} \log f(x_k) = \log \prod_{k=1}^{m} f(x_k)\end{aligned}$$

and thus $F_p(TM) = \prod_{k=1}^{m} f(x_k)$. On the other hand we have

$$\Omega^2 = \begin{pmatrix} -\theta_1^2 & & & & \\ & -\theta_1^2 & & & \\ & & \ddots & & \\ & & & -\theta_m^2 & \\ & & & & -\theta_m^2 \end{pmatrix}$$

and thus $\det f(\frac{\Omega^2}{(2\pi i)^2}) = \prod_{k=1}^{m} f(x_k)^2$. The assertion follows. □

Proof of the Atiyah-Singer index theorem 4.3.7. The matrix Ω of 2-forms given by the Riemann curvature of M is defined by the equation

$$R(b_\alpha, b_\beta) b_i = \sum_{j=1}^{n} \Omega_i^j(b_\alpha, b_\beta) b_j$$

where $b_1, \dots, b_n$ is a local orthonormal frame of TM. It follows that at the point p we have $\Omega_i^j(b_\alpha, b_\beta) = R_{\alpha\beta ij}(0)$ and thus

$$\Omega_i^j = \sum_{1 \le \alpha < \beta \le n} R_{\alpha\beta ij}(0) e_\alpha \wedge e_\beta = \frac{1}{2}\Theta_{ij}.$$

By definition the $\widehat{A}$-class of TM is the multiplicative class associated with

$$f(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}.$$

By Lemma 5.1.17 we get

$$\widehat{A}(TM) = \det \tilde{f}\Big(\frac{\Omega}{2\pi i}\Big) = (2\pi i)^{-n/2} \det\Big(\frac{\Omega/2}{\sinh(\Omega/2)}\Big)^{1/2}.$$

By Proposition 5.1.15 we have for every $p \in M$

$$\begin{aligned} u^L_{p,\frac{n}{2}}(0) &= \text{coefficient of } t^{n/2} \text{ in } \det\Big(\frac{t\Theta/4}{\sinh(t\Theta/4)}\Big)^{1/2}\exp(-tF) \\ &= \text{n-form part of } \det\Big(\frac{\Theta/4}{\sinh(\Theta/4)}\Big)^{1/2}\exp(-F) \\ &= \text{n-form part of } \det\Big(\frac{\Omega/2}{\sinh(\Omega/2)}\Big)^{1/2}\exp(-F) \\ &= \text{n-form part of } (2\pi i)^{n/2}\widehat{A}(TM)\cdot \mathrm{ch}(E). \end{aligned}$$

By Lemma 5.1.7 we get

$$\mathrm{Str}(u^L_{p,\frac{n}{2}}(0)) = \text{n-form part of } (4\pi)^{n/2}\widehat{A}(TM)\cdot \mathrm{ch}(E)$$

The assertion now follows from Lemma 5.1.13. □

5.2. Proof of the Hirzebruch signature theorem

Let M be a compact manifold of even dimension $n = 2m$. For $k \in \{0, \dots, n\}$ define

$$\tau = i^{k(k-1)+m} * : \quad \Lambda^k T^*M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^{n-k} T^*M \otimes_{\mathbb{R}} \mathbb{C}.$$

Consider the signature operator $d + d^* \in \mathscr{D}\!\mathit{iff}_1(E^+, E^-)$ introduced in Example 1.3.19, where $E^\pm$ denote the bundles of eigenvectors of τ for the eigenvalues ± 1.

Remark 5.2.1. Assume in addition that M is a spin manifold and denote by ΣM the spinor bundle over M.

a) Using the isomorphisms $\Lambda^\bullet T^*M \cong \mathrm{Cl}(TM)$ and $\mathrm{End}(\Sigma_n) \cong \mathbb{C}\mathrm{l}_n$ we obtain an isomorphism of complex vector bundles

$$\Phi : \quad \Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cl}(TM) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{End}(\Sigma M) \cong \Sigma M \otimes \Sigma M^*.$$

For every $x \in M$ the map

$$T_x^*M \times \Lambda^\bullet T_x^*M \otimes_{\mathbb{R}} \mathbb{C} \to \Lambda^\bullet T_x^*M \otimes_{\mathbb{R}} \mathbb{C}, \quad (\alpha, \varphi) \mapsto \alpha \cdot \varphi := \alpha \wedge \varphi - \alpha \lrcorner \varphi$$

satisfies the Clifford relation, i.e., for all $\alpha, \beta \in T_x^*M$, $\varphi \in \Lambda^\bullet T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$:

$$\alpha \cdot \beta \cdot \varphi + \beta \cdot \alpha \cdot \varphi = -2\langle \alpha, \beta\rangle \varphi.$$

On $\Lambda^\bullet T^*M \otimes_{\mathbb{R}} \mathbb{C}$ we define a Clifford multiplication using this map and on $\Sigma M \otimes \Sigma M^*$ we use the usual Clifford multiplication on the first factor. Then Φ is an isomorphism of Clifford modules, i.e., for all $x \in M$, $\alpha \in T_x^*M$, $\varphi \in \Lambda^\bullet T_x^*M \otimes_{\mathbb{R}} \mathbb{C}$ we have

$$\Phi(\alpha \cdot \varphi) = \alpha^\sharp \cdot \Phi(\varphi).$$

b) *Claim:* We have $\Phi(E^{\pm}) = \Sigma^{\pm}M \otimes \Sigma M^*$. In order to prove this claim we first fix $x \in M$, let $b_1, \ldots, b_n$ be an orthonormal basis of T_xM and let $b_1^*, \ldots b_n^*$ be the dual basis. We then compute for all $1 \leq j_1 < \ldots < j_k \leq n$,

$$b_{j_k}^* \cdot (b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*) = 0 - b_{j_k}^* \lrcorner (b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*) = (-1)^k b_{j_1}^* \wedge \ldots \wedge b_{j_{k-1}}^*$$

and we get inductively

$$b_{j_1}^* \cdot \ldots \cdot b_{j_k}^* \cdot (b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*) = (-1)^{k(k+1)/2}.$$

Writing $b_J^* := b_{j_1}^* \wedge \ldots \wedge b_{j_k}^*$ and denoting by $J^c =: \{r_1 < \ldots < r_{n-k}\}$ the multi-index complementary to J we get

$$\begin{aligned} b_1^* \cdot \ldots \cdot b_n^* \cdot b_J^* &= \mathrm{sign}(J^c, J) b_{r_1}^* \cdot \ldots \cdot b_{r_{n-k}}^* \cdot b_{j_1}^* \cdot \ldots b_{j_k}^* \cdot e_J^* \\ &= (-1)^{k(n-k)} \mathrm{sign}(J, J^c) b_{r_1}^* \cdot \ldots \cdot b_{r_{n-k}}^* \cdot (-1)^{k(k+1)/2} \\ &= (-1)^{-k^2 + k(k+1)/2} \mathrm{sign}(J, J^c) b_{r_1}^* \wedge \ldots \wedge b_{r_{n-k}}^* \\ &= (-1)^{k(k-1)/2} \mathrm{sign}(J, J^c) b_{J^c}^* \\ &= i^{-n/2} \tau(b_J^*) \end{aligned}$$

and thus

$$\Phi(\tau(b_J^*)) = i^{n/2} b_1 \cdot \ldots \cdot b_n \cdot \Phi(b_J^*).$$

The claim now follows, since by the equation (2.9) we have

$$\Sigma^{\pm}M = \{\varphi \in \Sigma M \mid i^{n/2} b_1 \cdot \ldots \cdot b_n \cdot \varphi = \pm\varphi\}.$$

c) We now write

$$S := \Sigma M \otimes \Sigma M^* = (\Sigma^+ M \otimes \Sigma M^*) \oplus (\Sigma^- M \otimes \Sigma M^*) =: S^+ \oplus S^-.$$

On ΣM^* we choose the connection $\nabla^{\Sigma M^*}$ induced by the spinor connection. We denote by $D^{\Sigma M^*} \in \mathscr{Diff}_1(S, S)$ the operator obtained from the classical Dirac operator on ΣM by twisting with $\nabla^{\Sigma M^*}$. From the splitting $S = S^+ \oplus S^-$ we obtain the Dirac-type operator $D^+ \in \mathscr{Diff}_1(S^+, S^-)$. We claim that the following diagram commutes.

$$\begin{tikzcd} C^\infty(M, E^+) \arrow[r, "\Phi"] \arrow[d, "d+d^*"'] & C^\infty(M, S^+) \arrow[d, "D^+"] \\ C^\infty(M, E^-) \arrow[r, "\Phi"] & C^\infty(M, S^-) \end{tikzcd}$$

Namely, for all $\xi \in T_x^*M$, $\varphi \in E_x^+$, $\psi \in S_x^+$, $x \in M$, we have

$$\begin{aligned} \sigma_1(d + d^*, \xi)\varphi &= \xi \wedge \varphi - \xi \lrcorner \varphi = \xi \cdot \varphi, \\ \sigma_1(D^+, \xi)\psi &= \xi \cdot \psi. \end{aligned}$$

Therefore, $D^+ \circ \Phi - \Phi \circ (d + d^*)$ is an operator of order 0. Now, the zero order terms of both D^+ and $d + d^*$ at $x \in M$ are linear in the Christoffel symbols at x. By choosing a local coordinate frame such that the Christoffel symbols at x vanish we see that the two operators coincide.

From the Atiyah-Singer index theorem with $E = \Sigma M^*$ we conclude that

$$\mathrm{ind}(d + d^*) = \mathrm{ind}(D^{\Sigma M^*}) = \int_M \widehat{A}(TM) \cdot \mathrm{ch}(\Sigma M^*).$$

Before we can prove the Hirzebruch signature theorem we need the following explicit formula.

Proposition 5.2.2. *Let M be an even dimensional spin manifold. Then we have*

$$\mathrm{ch}(\Sigma M^*) = \mathrm{ch}(\Sigma M) = F_p(TM) \quad \textit{with } f(z) := 2\cosh\Big(\frac{\sqrt{z}}{2}\Big).$$

Proof. Let $(b_i)_{i=1}^n$ be a local orthonormal frame of TM on $U \subset M$. This frame induces a local section h of the orthonormal frame bundle $P^{\mathrm{SO}}(M)$ on U which can be lifted to a local section H of $P^{\mathrm{Spin}}(M)$ on U. Let $z(j_1, \dots, j_k)$, $1 \le j_1 < \dots < j_k \le m := \frac{n}{2}$, be the basis of Σ_n defined in Section 2.3. We abbreviate the basis vectors by v_k, $k = 1, \dots, 2^{n/2} =: N$. Then $\varphi_k := [H, v_k]$, $k = 1, \dots, N$, is a local orthonormal frame of ΣM. We denote by φ_k^*, $k = 1, \dots, N$, the induced orthonormal frame of ΣM^*. Moreover, we denote by R, R^Σ and $\tilde{R}^\Sigma$ the curvatures of the Levi-Civita connection on TM, of the spinor connection on ΣM and of the induced connection on ΣM^* respectively. Finally, we denote by Ω, Ω^Σ and $\tilde{\Omega}^\Sigma$ the corresponding curvature 2-forms. By Lemma 2.4.13 we have for $k \in \{1, \dots, N\}$ and all $X, Y \in T_pM$, $p \in U$:

$$\begin{aligned}\sum_{j=1}^N (\Omega^\Sigma)_k^j(X,Y)\varphi_j &= R^\Sigma(X,Y)\varphi_k \\ &= -\frac{1}{4}\sum_{\ell=1}^n R(X,Y)b_\ell \cdot b_\ell \cdot \varphi_k = -\frac{1}{4}\sum_{\ell,r=1}^n \Omega_\ell^r(X,Y) b_r \cdot b_\ell \cdot \varphi_k.\end{aligned}$$

Taking the scalar product with φ_j yields for all X, Y

$$(\Omega^\Sigma)_k^j(X,Y) = -\frac{1}{4}\sum_{\ell,r=1}^n \Omega_\ell^r(X,Y)\langle b_r \cdot b_\ell \cdot \varphi_k, \varphi_j\rangle.$$

We may assume that Ω is a block diagonal matrix with blocks

$$\begin{pmatrix} 0 & -\theta_k \\ \theta_k & 0 \end{pmatrix}, \quad k = 1, \dots, \frac{n}{2} =: m$$

on the diagonal. It follows that

$$(\Omega^\Sigma)_k^j = \frac{1}{2}\sum_{\ell=1}^m \theta_\ell \langle b_{2\ell-1} \cdot b_{2\ell} \cdot \varphi_k, \varphi_j\rangle.$$

For every basis vector $z(J) := z(j_1, \ldots, j_k)$ of Σ_n we write

$$M_J := \{l \in \{1, \ldots, m\} \,|\, e_{2l},\, e_{2l-1} \text{ are not contained in } z_{j_1} \cdot \ldots \cdot z_{j_k}\}.$$

By the computation in Remark 2.3.7 it follows that Ω^Σ is a diagonal matrix and that the entry on the diagonal corresponding to $z(J)$ is given by

$$(\Omega^\Sigma)^J_J = \sum_{\ell \notin M_J} \frac{i\theta_\ell}{2} - \sum_{\ell \in M_J} \frac{i\theta_\ell}{2} = \sum_{\ell=1}^{m} \frac{i\theta_\ell}{2} - \sum_{\ell \in M_J} i\theta_\ell =: \lambda_J$$

where J runs through all multi-indices $(1 \leq j_1 < \ldots < j_k \leq m)$. Writing $x_J := \frac{\lambda_J}{2\pi i}$ for all J we get by equation (4.10) that $c(\Sigma M) = \prod_J (1 + x_J)$. It follows that

$$\log c(\Sigma M) = \sum_J \log(1 + x_J) = \sum_J \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x_J^k$$

and thus for any formal power series $g(x) = \sum_{k=0}^{\infty} g_k x^k$:

$$\Lambda_g \log c(\Sigma M) = \sum_J \sum_{k=1}^{\infty} (-1)^{k+1} k g_k \frac{(-1)^{k+1}}{k} x_J^k = \sum_J \sum_{k=1}^{\infty} g_k x_J^k = \sum_J (g(x_J) - g_0)$$

and therefore $g_c(\Sigma M) = \sum_J g(x_J)$. With $g(x) = e^x$ we therefore get

$$\begin{aligned}
g_c(\Sigma M) &= \sum_J \exp\Big(\sum_{\ell=1}^{m} \frac{\theta_\ell}{4\pi} - \sum_{\ell \in M_J} \frac{\theta_\ell}{2\pi} \Big) \\
&= \prod_{k=1}^{m} \exp\Big(\frac{\theta_k}{4\pi}\Big) \sum_J \prod_{\ell \in M_J} \exp\Big(-\frac{\theta_\ell}{2\pi}\Big) \\
&= \prod_{k=1}^{m} \exp\Big(\frac{\theta_k}{4\pi}\Big) \sum_{\ell=0}^{N} \sigma_\ell \Big(\exp\Big(-\frac{\theta_1}{2\pi}\Big), \ldots, \exp\Big(-\frac{\theta_m}{2\pi}\Big)\Big) \\
&= \prod_{k=1}^{m} \exp\Big(\frac{\theta_k}{4\pi}\Big) \prod_{a=1}^{m} \Big(1 + \exp\Big(-\frac{\theta_a}{2\pi}\Big)\Big) \\
&= \prod_{k=1}^{m} \Big(\exp\Big(\frac{\theta_k}{4\pi}\Big) + \exp\Big(-\frac{\theta_k}{4\pi}\Big)\Big) \\
&= \prod_{k=1}^{m} 2\cosh\Big(\frac{\theta_k}{4\pi}\Big).
\end{aligned}$$

For any complex number $a \in \mathbb{C}$ we have

$$\begin{aligned}
\cosh \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} &= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}^{2k} = \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{pmatrix} (-a^2)^k & \\ 0 & (-a^2)^k \end{pmatrix} \\
&= \begin{pmatrix} \cosh(ia) & 0 \\ 0 & \cosh(ia) \end{pmatrix}.
\end{aligned}$$

It follows that $\cosh(\frac{\Omega}{4\pi i})$ is a block diagonal matrix with blocks

$$\begin{pmatrix} \cosh(\frac{\theta_k}{4\pi}) & 0 \\ 0 & \cosh(\frac{\theta_k}{4\pi}) \end{pmatrix}$$

on the diagonal. With $f(x) = 2\cosh(\frac{\sqrt{x}}{2})$ we get by Lemma 5.1.17:

$$F_p(TM) = \det\Big(2\cosh\Big(\frac{\Omega}{4\pi i}\Big)\Big)^{1/2} = \prod_{k=1}^{m} 2\cosh\Big(\frac{\theta_k}{4\pi}\Big) = g_c(\Sigma M).$$

This shows the assertion for ΣM. From the definition of the induced connection on ΣM^* one easily sees that

$$(\tilde{R}^{\Sigma}(X,Y)\varphi_j^*)(\varphi_i) = -\varphi_j^*(R^{\Sigma}(X,Y)\varphi_i)$$

for all X, Y and all i, j. It follows that

$$(\tilde{\Omega}^{\Sigma})_j^k(X,Y) = \frac{1}{4}\sum_{\ell,r=1}^{n} \Omega_\ell^r(X,Y)\langle b_r \cdot b_\ell \cdot \varphi_k, \varphi_j\rangle,$$

i.e., $\tilde{\Omega}^{\Sigma} = -(\Omega^{\Sigma})^{\top} = -\Omega^{\Sigma}$. Now, the set of all diagonal entries λ_J of Ω^{Σ} is symmetric around 0. Thus the set of all x_J is symmetric around 0 and we get $g_c(\Sigma M) = g_c(\Sigma M^*)$. $\square$

Proof of the Hirzebruch signature theorem. First we assume that M is a spin manifold. We must compute the n-form part of $\widehat{A}(TM) \cdot \mathrm{ch}(\Sigma M^*) = F_p(TM)$ where

$$f(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)} \cdot 2\cosh(\sqrt{x}/2) = \frac{\sqrt{x}}{\tanh(\sqrt{x}/2)}.$$

By Lemma 5.1.17 we have $F_p(TM) = \det \tilde{f}(\frac{\Omega}{2\pi i})$ where $\tilde{f}(x) = (\frac{x}{\tanh(x/2)})^{1/2}$. Then with

$$\ell(x) := \frac{\sqrt{x}}{\tanh\sqrt{x}} \quad \text{and} \quad \tilde{\ell}(x) := \Big(\frac{x}{\tanh(x)}\Big)^{1/2}$$

we have $\tilde{f}(x) = \sqrt{2}\tilde{\ell}(\frac{x}{2})$. The multiplicative classes for f and ℓ do not coincide. However, if we denote by $(\cdot)_k$ the k-form part we get

$$\Big(\det \tilde{f}\Big(\frac{\Omega}{2\pi i}\Big)\Big)_k = 2^{\frac{n}{2}}\Big(\det \tilde{\ell}\Big(\frac{1}{2}\cdot\frac{\Omega}{2\pi i}\Big)\Big)_k = 2^{\frac{n-k}{2}}\Big(\det \tilde{\ell}\Big(\frac{\Omega}{2\pi i}\Big)\Big)_k.$$

Therefore, from Lemma 5.1.17 we conclude that the n-form parts of $F_p(TM)$ and $L(TM)$ and thus their integrals coincide if M is spinnable.
Assume now that M is not necessarily spinnable. Fix a point $p \in M$ and choose an open neighborhood U of p in M such that U is spinnable. We know from Equation (5.2) that $\mathrm{ind}(d + d^*) = (4\pi)^{-\frac{n}{2}} \int_M \mathrm{Str}\Phi_{n/2}(x,x)\,dx$. The computation for the case of a spin manifold shows that the integrand on U is pointwise given by the n-form part of $L(TM)$. But since $\mathrm{Str}(\Phi_{n/2})$ and the n-form part of $L(TM)$ are local quantities, we conclude that they coincide on all of M. $\square$

6. Semi-Riemannian Spin Geometry

6.1. The Spin Group

Let $r, s \in \mathbb{N}_0$, let $n := r + s$ and let

$$\varepsilon_1 = \ldots = \varepsilon_r = -1, \quad \varepsilon_{r+1} = \ldots = \varepsilon_n = 1.$$

Let $\mathbb{R}^n$ be equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle_{r,s}$ defined by

$$\langle x, y \rangle_{r,s} := \sum_{i=1}^{n} \varepsilon_i x_i y_i.$$

The pair (r, s) is called the **signature** of the symmetric bilinear form $\langle \cdot, \cdot \rangle_{r,s}$.

Definition 6.1.1. We define the **semi-orthogonal group $\mathbf{O}(r, s)$** by

$$\mathrm{O}(r, s) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \,|\, \langle Ax, Ay \rangle_{r,s} = \langle x, y \rangle_{r,s} \text{ for all } x, y \in \mathbb{R}^n\}.$$

Remark 6.1.2. We have $A \in \mathrm{O}(r, s)$ if and only if $A^\top J A = J$ where

$$J = \begin{pmatrix} -I_r & 0 \\ 0 & I_s \end{pmatrix}$$

and I_r, I_s denote the identity matrices in dimension r and s respectively. In particular, if $A \in \mathrm{O}(r, s)$ then we have $\det(A) \in \{\pm 1\}$. Thus $\mathrm{O}(r, s)$ is a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Definition 6.1.3. The subgroup

$$\mathrm{SO}(r, s) := \{A \in \mathrm{O}(r, s) \,|\, \det(A) = 1\}$$

of $\mathrm{O}(r, s)$ is called the **special semi-orthogonal group**.

Remark 6.1.4. If $r = 0$ or $s = 0$ then $\mathrm{SO}(r, s)$ is connected. If $r > 0$ and $s > 0$ then $\mathrm{SO}(r, s)$ has two connected components (see e. g. [9], Lemma 9.6).

Notation 6.1.5. In the following, we denote the Clifford algebra of $\mathbb{R}^n$ with the inner product $\langle\cdot,\cdot\rangle_{r,s}$ by $\mathbf{Cl}_{\boldsymbol{r,s}} := \mathrm{Cl}(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{r,s})$.

Remark 6.1.6. Upon identifying $\mathbb{R}^n$ with $\imath(\mathbb{R}^n) \subset \mathrm{Cl}_{r,s}$, for every $v \in \mathbb{R}^n$ with $\langle v,v\rangle_{r,s} \neq 0$ we have $v^2 = -\langle v,v\rangle_{r,s} \cdot 1$ and thus

$$-\frac{v}{\langle v,v\rangle_{r,s}} \cdot v = v \cdot \left(-\frac{v}{\langle v,v\rangle_{r,s}}\right) = 1.$$

Thus, $\{v \in \mathbb{R}^n \mid \langle v,v\rangle_{r,s} \neq 0\}$ is contained in the subgroup of (multiplicatively) invertible elements of $\mathrm{Cl}_{r,s}$.

Definition 6.1.7. We define the **Pin group Pin$(\boldsymbol{r,s})$** by

$$\mathrm{Pin}(r,s) := \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_{r,s} \;\middle|\; v_j \in \mathbb{R}^n,\ \langle v_j, v_j\rangle_{r,s} \in \{\pm 1\},\ m \in \mathbb{N}_0 \right\}.$$

Remark 6.1.8. The subset $\mathrm{Pin}(r,s) \subset \mathrm{Cl}_{r,s}$ is a group with respect to the multiplication in $\mathrm{Cl}_{r,s}$. The inverse element to $v_1 \cdot \ldots \cdot v_m$ is given by

$$(v_1 \cdot \ldots \cdot v_m)^{-1} = \left(-\frac{v_m}{\langle v_m, v_m\rangle_{r,s}}\right) \cdot \ldots \cdot \left(-\frac{v_1}{\langle v_1, v_1\rangle_{r,s}}\right) \in \mathrm{Pin}(r,s).$$

Definition 6.1.9. We define the **Spin group Spin$(\boldsymbol{r,s})$** by

$$\begin{aligned} \mathrm{Spin}(r,s) &:= \mathrm{Pin}(r,s) \cap \mathrm{Cl}^0_{r,s} \\ &= \left\{ v_1 \cdot \ldots \cdot v_m \in \mathrm{Cl}_{r,s} \;\middle|\; v_j \in \mathbb{R}^n,\ \langle v_j, v_j\rangle_{r,s} \in \{\pm 1\},\ m \in 2\mathbb{N}_0 \right\}. \end{aligned}$$

Remark 6.1.10. By the argument from Remark 6.1.8, $\mathrm{Spin}(r,s)$ is a subgroup of $\mathrm{Pin}(r,s)$.

For a fixed $v \in \mathbb{R}^n$ with $\langle v,v\rangle_{r,s} \in \{\pm 1\}$ and any $x \in \mathbb{R}^n$, we have:

$$\begin{aligned} v \cdot x \cdot v^{-1} &= -\frac{1}{\langle v,v\rangle_{r,s}} v \cdot x \cdot v \\ &= -\frac{1}{\langle v,v\rangle_{r,s}} \left(-x \cdot v - 2\langle x,v\rangle_{r,s}\, 1\right) \cdot v \\ &= -\frac{1}{\langle v,v\rangle_{r,s}} \left(-x \cdot v \cdot v - 2\langle x,v\rangle_{r,s} v\right) \end{aligned}$$

$$= -\Big(x - 2\frac{\langle x, v\rangle_{r,s}}{\langle v, v\rangle_{r,s}} v\Big)$$

The map $x \mapsto \big(x - 2\frac{\langle x,v\rangle_{r,s}}{\langle v,v\rangle_{r,s}} v\big)$ is the reflection about the hyperplane $v^\perp$ perpendicular to v. In particular, $(x \mapsto v \cdot x \cdot v^{-1}) \in \mathrm{O}(r, s)$. For any $a := v_1 \cdot \ldots \cdot v_m \in \mathrm{Spin}(r, s)$, the map

$$x \mapsto a \cdot x \cdot a^{-1} = v_1 \cdot \ldots \cdot v_m \cdot x \cdot v_m^{-1} \cdot \ldots \cdot v_1^{-1}$$

consists of an even number of hyperplane reflections and is thus contained in $\mathrm{SO}(r, s)$. We have thus defined a group homomorphism $\varrho : \mathrm{Spin}(r, s) \to \mathrm{SO}(r, s)$ by

$$\varrho(a)\, x := a \cdot x \cdot a^{-1}. \tag{6.1}$$

Example 6.1.11. Let $n = 2$ and $r = s = 1$. We have

$$\begin{aligned} \mathrm{SO}(1,1) &= \Big\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \Big| \, t \in \mathbb{R} \Big\} \cup \Big\{ \begin{pmatrix} -\cosh(t) & -\sinh(t) \\ -\sinh(t) & -\cosh(t) \end{pmatrix} \Big| \, t \in \mathbb{R} \Big\} \\ &\cong \mathbb{R} \times \mathbb{Z}_2 \end{aligned}$$

and $\langle e_1, e_1\rangle_{r,s} = -1$, $\langle e_2, e_2\rangle_{r,s} = 1$. Every element $a \in \mathrm{Spin}(1, 1)$ can be written as $a = v_1 \cdot \ldots \cdot v_m$ where $m \in 2\mathbb{N}_0$ and there exist $t_1, \ldots, t_m \in \mathbb{R}$ such that for all j

$$v_j = \cosh(t_j)e_1 + \sinh(t_j)e_2, \quad \text{or} \quad v_j = \sinh(t_j)e_1 + \cosh(t_j)e_2.$$

We compute

$$\begin{aligned} (\cosh(\vartheta)e_1 + \sinh(\vartheta)e_2) \cdot (\cosh(\varphi)e_1 + \sinh(\varphi)e_2) &= \cosh(\varphi - \vartheta) + \sinh(\varphi - \vartheta)e_1 \cdot e_2, \\ (\sinh(\vartheta)e_1 + \cosh(\vartheta)e_2) \cdot (\sinh(\varphi)e_1 + \cosh(\varphi)e_2) &= -\cosh(\varphi - \vartheta) - \sinh(\varphi - \vartheta)e_1 \cdot e_2, \\ (\cosh(\vartheta)e_1 + \sinh(\vartheta)e_2) \cdot (\sinh(\varphi)e_1 + \cosh(\varphi)e_2) &= \sinh(\varphi - \vartheta) + \cosh(\varphi - \vartheta)e_1 \cdot e_2, \\ (\sinh(\vartheta)e_1 + \cosh(\vartheta)e_2) \cdot (\cosh(\varphi)e_1 + \sinh(\varphi)e_2) &= -\sinh(\varphi - \vartheta) - \cosh(\varphi - \vartheta)e_1 \cdot e_2. \end{aligned}$$

and

$$\begin{aligned} (\cosh(\alpha) + \sinh(\alpha)e_1 \cdot e_2) \cdot (\cosh(\beta) + \sinh(\beta)e_1 \cdot e_2) &= \cosh(\alpha + \beta) + \sinh(\alpha + \beta)e_1 \cdot e_2, \\ (\cosh(\alpha) + \sinh(\alpha)e_1 \cdot e_2) \cdot (\sinh(\beta) + \cosh(\beta)e_1 \cdot e_2) &= \sinh(\alpha + \beta) + \cosh(\alpha + \beta)e_1 \cdot e_2, \\ (\sinh(\alpha) + \cosh(\alpha)e_1 \cdot e_2) \cdot (\sinh(\beta) + \cosh(\beta)e_1 \cdot e_2) &= \cosh(\alpha + \beta) + \sinh(\alpha + \beta)e_1 \cdot e_2. \end{aligned}$$

We conclude that

$$\begin{aligned} \mathrm{Spin}(1,1) &= \{\pm(\cosh(t) + \sinh(t)e_1 \cdot e_2) \,|\, t \in \mathbb{R}\} \cup \{\pm(\sinh(t) + \cosh(t)e_1 \cdot e_2) \,|\, t \in \mathbb{R}\} \\ &\cong \mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_2. \end{aligned}$$

For the group homomorphism ϱ: $\mathrm{Spin}(1,1) \to \mathrm{SO}(1,1)$ we get for $j = 1, 2$:

$$\begin{aligned}
&\varrho(\cosh(t) + \sinh(t)e_1 \cdot e_2)(e_j) \\
&\quad= (\cosh(t) + \sinh(t)e_1 \cdot e_2) \cdot e_j \cdot (\cosh(t) - \sinh(t)e_1 \cdot e_2) \\
&\quad= \cosh^2(t)e_j - \sinh^2(t)e_1 \cdot e_2 \cdot e_j \cdot e_1 \cdot e_2 + \sinh(t)\cosh(t)(e_1 \cdot e_2 \cdot e_j - e_j \cdot e_1 \cdot e_2) \\
&\quad= (\cosh^2(t) + \sinh^2(t))e_j + 2\sinh(t)\cosh(t)e_1 \cdot e_2 \cdot e_j \\
&\quad= \begin{cases} \cosh(2t)e_1 - \sinh(2t)e_2 & \text{if } j = 1, \\ \cosh(2t)e_2 - \sinh(2t)e_1 & \text{if } j = 2. \end{cases}
\end{aligned}$$

Thus we have

$$\varrho(\pm(\cosh(t) + \sinh(t)e_1 \cdot e_2)) = \begin{pmatrix} \cosh(-2t) & \sinh(-2t) \\ \sinh(-2t) & \cosh(-2t) \end{pmatrix}.$$

Similarly, one computes

$$\varrho(\pm(\sinh(t) + \cosh(t)e_1 \cdot e_2)) = -\begin{pmatrix} \cosh(-2t) & \sinh(-2t) \\ \sinh(-2t) & \cosh(-2t) \end{pmatrix}.$$

Proposition 6.1.12. *For any $r, s \in \mathbb{N}_0$, the sequence*

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(r,s) \xrightarrow{\varrho} \mathrm{SO}(r,s) \to 1$$

is exact.

Proof. For $r = 0$ the assertion follows from Proposition 2.2.10. Thus assume $r \geq 1$.

a) The map $\varrho : \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is surjective:

We prove by induction on n that every element of $\mathrm{O}(r,s)$ is a composition of reflections at hyperplanes $v^\perp$ where $v \in \mathbb{R}^n$ and $\langle v, v\rangle_{r,s} \in \{\pm 1\}$. Obviously, the claim is true for $n = 1$. Now, let $A \in \mathrm{O}(r,s)$ with $n = r + s \geq 2$ and assume that the claim holds for $n - 1$. We write $x := Ae_1$. Then we have $\langle x, x\rangle_{r,s} = \langle e_1, e_1\rangle_{r,s} = -1$ and thus

$$\langle x - e_1, x - e_1\rangle_{r,s} = -2 - 2\langle x, e_1\rangle_{r,s}, \quad \langle x + e_1, x + e_1\rangle_{r,s} = -2 + 2\langle x, e_1\rangle_{r,s}$$

and therefore not both of these numbers are zero. Denote the hyperplane reflection at $v^\perp$ by $R(v)$ and define

$$A_1 := \begin{cases} R\left(\frac{x - e_1}{|\langle x - e_1, x - e_1\rangle_{r,s}|^{1/2}}\right) & \text{if } \langle x - e_1, x - e_1\rangle_{r,s} \neq 0 \\ R(e_1) \circ R\left(\frac{x + e_1}{|\langle x + e_1, x + e_1\rangle_{r,s}|^{1/2}}\right) & \text{otherwise.} \end{cases}$$

Then A_1 is a composition of hyperplane reflections and $A_1 x = e_1$. Thus we get

$$A_1 \circ A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$$

with $B \in \mathrm{O}(r-1,s)$. By the induction hypothesis B is a composition of hyperplane reflections and therefore A is as well.

Thus, any given $A \in \mathrm{SO}(r,s)$ is the product of an even number of hyperplane reflections. Let the i-th hyperplane be the orthogonal complement to $v_i \in \mathbb{R}^n$ with $\langle v_i, v_i\rangle_{r,s} \in \{\pm 1\}$. Then we have $v_1 \cdot \ldots \cdot v_{2k} \in \mathrm{Spin}(r,s)$ and $\varrho(v_1 \cdot \ldots \cdot v_{2k}) = A$.

b) It remains to show that $\ker(\varrho) = \mathbb{Z}_2 = \{1,-1\}$. Obviously, we have $\{1,-1\} \subset \ker(\varrho)$. Conversely, let $a \in \ker(\varrho)$. Then for all $x \in \mathbb{R}^n$, we have:

$$x = \varrho(a)(x) = a \cdot x \cdot a^{-1}.$$

Equivalently, we have $x \cdot a = a \cdot x$ for all $x \in \mathbb{R}^n$ and in particular, $x \cdot a = a \cdot x$ for all $x \in \mathrm{Cl}_{r,s}$. Hence, a is contained in the center $\mathcal{Z}(\mathrm{Cl}_{r,s})$ of $\mathrm{Cl}_{r,s}$. Moreover, we have $a \in \mathrm{Spin}(r,s) \subset \mathrm{Cl}^0_{r,s}$. Now for any $r,s \in \mathbb{N}_0$ we have

$$\mathcal{Z}(\mathrm{Cl}_{r,s}) \cap \mathrm{Cl}^0_{r,s} = \mathbb{R}\cdot 1, \qquad \text{(exercise !)},$$

hence $a = \alpha\, 1$ for some $\alpha \in \mathbb{R}$. We finish the proof using the following lemma. □

Lemma 6.1.13. *We have* $\mathrm{Spin}(r,s) \cap \mathbb{R}\cdot 1 = \{-1,1\}$.

Proof. Let $\mathrm{Cl}^{\mathrm{opp}}_{r,s}$ be the additive group $\mathrm{Cl}_{r,s}$ equipped with the opposite multiplication $a \star b := b \cdot a$. Then $\mathrm{Cl}^{\mathrm{opp}}_{r,s}$ is a unital algebra. The map $j\colon \mathbb{R}^n \to \mathrm{Cl}^{\mathrm{opp}}_{r,s}$ defined by $j(v) := v$ satisfies the Clifford relation and thus by the universal property of $\mathrm{Cl}_{r,s}$ there is a unique algebra homomorphism $T\colon \mathrm{Cl}_{r,s} \to \mathrm{Cl}^{\mathrm{opp}}_{r,s}$ such that $T \circ \imath = j$.
Let $a = \alpha \cdot 1 \in \mathrm{Spin}(r,s)$. Then we can write $a = v_1 \cdot \ldots \cdot v_{2k}$ with $v_j \in \mathbb{R}^n$, $\langle v_j, v_j\rangle_{r,s} \in \{\pm 1\}$ and $k \in \mathbb{N}_0$. We get $T(a) = \alpha \cdot 1$ and $T(v_1 \cdot \ldots \cdot v_{2k}) = v_{2k} \cdot \ldots \cdot v_1$, since T is an algebra homomorphism. It follows that

$$\alpha^2 \cdot 1 = T(a) \star a = a \cdot v_{2k} \cdot \ldots \cdot v_1 = v_1 \cdot \ldots \cdot \underbrace{v_{2k} \cdot v_{2k}}_{=\pm 1} \cdot \ldots \cdot v_1 = \pm 1$$

and thus $\alpha \in \{-1,1\}$. □

Remark 6.1.14. We have seen that every $v \in \mathbb{R}^n$ with $\langle v, v\rangle_{r,s} \neq 0$ is a multiplicatively invertible element in $\mathrm{Cl}_{r,s}$. Let $\Gamma_{r,s} \subset \mathrm{Cl}_{r,s}$ be the group generated by all such elements v with multiplication given by Clifford multiplication. It is easy to see that multiplication and inversion are smooth maps and thus $\Gamma_{r,s}$ is a Lie group. Moreover, the map

$$N: \quad \Gamma_{r,s} \cap \mathrm{Cl}^0_{r,s} \to \mathbb{R}, \quad v_1 \cdot \ldots \cdot v_{2k} \mapsto \langle v_1, v_1\rangle_{r,s} \ldots \langle v_{2k}, v_{2k}\rangle_{r,s}$$

is smooth and we have $\mathrm{Spin}(r,s) = N^{-1}(\{-1,1\})$. In particular, $\mathrm{Spin}(r,s)$ is a closed subgroup of $\Gamma_{r,s}$ and thus a Lie group. By Proposition 6.1.12 the map $\varrho\colon \mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is a 2-fold covering.

Proposition 6.1.15. *Let $r > 0$ and $s > 0$ and let $r \geq 2$ or $s \geq 2$. Then the Spin group* $\mathrm{Spin}(r,s)$ *has two connected components.*

Proof. a) From the exact sequence in Proposition 6.1.12 we get the long exact homotopy sequence (base point = 1):

$$\to \underbrace{\pi_1(\mathbb{Z}_2)}_{=\{1\}} \to \pi_1(\mathrm{Spin}(r,s)) \to \pi_1(\mathrm{SO}(r,s)) \to \underbrace{\pi_0(\mathbb{Z}_2)}_{=\mathbb{Z}_2} \to \pi_0(\mathrm{Spin}(r,s)) \to \underbrace{\pi_0(\mathrm{SO}(r,s))}_{=\{-1,1\}}.$$

Claim: The map $\pi_0(\mathbb{Z}_2) \xrightarrow{\psi} \pi_0(\mathrm{Spin}(r,s))$ is trivial, that is, the image of ψ is $\{1\}$.

In fact, 1 and -1 can be connected by a continuous path in $\mathrm{Spin}(r,s)$: Since $r \geq 2$ or $s \geq 2$, we have two orthonormal vectors $e_i, e_j \in \mathbb{R}^n$ with $\langle e_i, e_i\rangle_{r,s} = \langle e_j, e_j\rangle_{r,s} =: \varepsilon_i \in \{\pm 1\}$ and we can define the smooth curve $c : \mathbb{R} \to \mathrm{Spin}(r,s)$,

$$t \mapsto \varepsilon_i\,(\cos(t)\,e_i + \sin(t)\,e_j) \cdot e_i,$$

satisfying $c(0) = -1$ and $c(\pi) = 1$.

b) By exactness at $\pi_0(\mathrm{Spin}(r,s))$ and the claim, the map

$$\pi_0(\mathrm{Spin}(r,s)) \to \pi_0(\mathrm{SO}(r,s)) = \{-1,1\}$$

is injective. On the other hand, $\mathrm{Spin}(r,s)$ has at least two connected components, since ϱ: $\mathrm{Spin}(r,s) \to \mathrm{SO}(r,s)$ is continuous and surjective and since $\mathrm{SO}(r,s)$ has two connected components. Hence, $\pi_0(\mathrm{Spin}(r,s)) = \{-1,1\}$. $\square$

Definition 6.1.16. We denote by $\boldsymbol{\mathrm{Spin}_0(r,s)} \subset \mathrm{Spin}(r,s)$ the connected component of the neutral element in $\mathrm{Spin}(r,s)$.

Proposition 6.1.17.

1. *Let $r > 0$ and $s > 0$ and let $r \geq 2$ or $s \geq 2$. Then the sequence*
$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}_0(r,s) \xrightarrow{\varrho_0} \mathrm{SO}_0(r,s) \to 1$$
is exact, where $\varrho_0 = \varrho|_{\mathrm{Spin}_0(r,s)}$.
2. *For $r = 1$ we have $\pi_1(\mathrm{Spin}_0(1,2)) = \mathbb{Z}$ and $\pi_1(\mathrm{Spin}_0(1,s)) = \{0\}$ if $s \geq 3$.*

Proof. a) The proof of Proposition 6.1.15 shows that $-1 \in \mathrm{Spin}_0(r,s)$. Thus we have $\ker(\varrho_0) = \{-1,1\}$. Let $A \in \mathrm{SO}_0(r,s)$. Then there is a continuous path c: $[0,1] \to$

$\mathrm{SO}_0(r,s)$ such that $c(0) = A$ and $c(1) = 1$. Let $a \in \mathrm{Spin}(r,s)$ such that $\varrho(a) = A$. From the lifting property of covering spaces it follows that there is a continuous path in $\mathrm{Spin}(r,s)$ from a to 1 or -1. Thus we have $a \in \mathrm{Spin}_0(r,s)$ and ϱ_0 is surjective.

b) Assume now that $r = 1$. The group $\mathrm{SO}_0(1,s)$ acts transitively on the hyperbolic space
$$H^s = \{x \in \mathbb{R}^{s+1} \mid \langle x, x\rangle_{1,s} = -1,\, x_1 > 0\}$$
and the isotropy group of the point $(1, 0, \ldots, 0) \in H^s$ is given by
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \,\middle|\, B \in \mathrm{SO}(s) \right\} \cong \mathrm{SO}(s).$$
Since all homotopy groups of H^s are trivial, the long exact homotopy sequence for the fiber bundle $\mathrm{SO}(s) \to \mathrm{SO}_0(1,s) \to H^s$ then yields isomorphisms $\pi_1(\mathrm{SO}_0(1,s)) \cong \pi_1(\mathrm{SO}(s))$ for all $s \geq 1$. We have computed the groups $\pi_1(\mathrm{SO}(s))$ in the proof of Proposition 2.2.13.

c) Consider the long exact homotopy sequence in part a) of the proof of Proposition 6.1.15. By exactness at $\pi_0(\mathbb{Z}_2)$ and the claim in part a) of this proof, the map χ: $\pi_1(\mathrm{SO}(r,s)) \to \pi_0(\mathbb{Z}_2)$ is surjective.
Let $s = 2$. We have $\pi_1(\mathrm{SO}_0(1,2)) \cong \pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$ and thus χ is the projection $\mathbb{Z} \to \mathbb{Z}_2$. Since $\pi_1(\mathrm{Spin}(1,2)) \to \pi_1(\mathrm{SO}(1,2))$ is injective we have $\pi_1(\mathrm{Spin}_0(1,2)) \cong \mathbb{Z}$.
Let $s \geq 3$. We have $\pi_1(\mathrm{SO}_0(1,s)) \cong \pi_1(\mathrm{SO}(s)) \cong \mathbb{Z}_2$. It follows that χ is also injective. Therefore the image of the map $\pi_1(\mathrm{Spin}(1,s)) \to \pi_1(\mathrm{SO}(1,s))$ is equal to $\{1\}$. By exactness of the sequence at $\pi_1(\mathrm{Spin}(1,s))$ this map is injective. Altogether we get that $\pi_1(\mathrm{Spin}_0(1,s)) = \{1\}$. $\square$

Remark 6.1.18. Several authors define the Spin group for the signature (r,s) as the group which we denote by $\mathrm{Spin}_0(r,s)$.

The Lie algebra of $\mathrm{SO}(r,s)$ is given by
$$\mathfrak{so}(r,s) = \left\{ A \in \mathrm{Mat}(n \times n; \mathbb{R}) \mid A^\top J + JA = 0 \right\}$$
and $\dim \mathrm{SO}(r,s) = \dim \mathfrak{so}(r,s) = \frac{1}{2}n(n-1)$.

For the Lie algebra of the Spin group, we have $\dim \mathfrak{spin}(r,s) = \dim \mathrm{Spin}(r,s) = \dim \mathrm{SO}(r,s) = \frac{1}{2}n(n-1)$. We want to identify the Lie algebra $\mathfrak{spin}(r,s)$ of $\mathrm{Spin}(r,s)$ as a vector subspace of $\mathrm{Cl}_{r,s}$:

For $i \neq j$ consider the smooth curve $c : \mathbb{R} \to \mathrm{Spin}(r,s)$, defined by
$$t \mapsto \begin{cases} (\varepsilon_i \cos(t) e_i + \sin(t) e_j) \cdot (-e_i) & \text{if } \varepsilon_i = \varepsilon_j, \\ (\varepsilon_i \cosh(t) e_i + \sinh(t) e_j) \cdot (-e_i) & \text{if } \varepsilon_i = -\varepsilon_j. \end{cases}$$

Then $c(0) = \varepsilon_i e_i \cdot (-e_i) = 1$ and $\dot{c}(0) = e_j \cdot (-e_i) = e_i \cdot e_j$. We thus have $e_i \cdot e_j \in T_1\mathrm{Spin}(r,s) \cong \mathfrak{spin}(r,s)$ for all $i \neq j$.
The products $\{e_i \cdot e_j\}$, $1 \le i < j \le n$ are linearly independent and there are $\frac{1}{2}n(n-1)$ of them. Since $\dim(\mathfrak{spin}(r,s)) = \frac{1}{2}n(n-1)$, we conclude that $\{e_i \cdot e_j\}_{i<j}$ is a basis of $\mathfrak{spin}(r,s)$.

We compute the Lie algebra homomorphism $\varrho_* : \mathfrak{spin}(r,s) \to \mathfrak{so}(r,s)$: Using the curve c defined above we get

$$\begin{aligned}
\varrho_*(e_i \cdot e_j)(e_k) &= \frac{d}{dt}\big|_{t=0}\varrho(c(t))(e_k) = \frac{d}{dt}\big|_{t=0}c(t)\cdot e_k \cdot c(t)^{-1} \\
&= \dot{c}(0)\cdot e_k - e_k \cdot \dot{c}(0) \\
&= e_i \cdot e_j \cdot e_k - e_k \cdot e_i \cdot e_j \\
&= \begin{cases} 0 & \text{for } k \notin \{i,j\} \\ 2\varepsilon_i e_j & \text{for } k = i \\ -2\varepsilon_j e_i & \text{for } k = j. \end{cases}
\end{aligned}$$

We thus have for $i < j$

$$\varrho_*(e_i \cdot e_j) = \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & & \cdots & -2\varepsilon_j & \cdots \\ & \vdots & & \vdots & \\ \cdots & 2\varepsilon_i & \cdots & \cdots & \cdots \\ & \vdots & & \vdots & \end{pmatrix}.$$

6.2. Spinors

Let $\mathrm{Cl}_{r,s}$ be the Clifford algebra of $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{r,s})$ and let $\mathbb{C}\mathrm{l}_{r,s} := \mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification.

The even dimensional case

In the following, let $n = 2m$. Let $e_1, \ldots, e_{2m}$ be the standard basis of $\mathbb{R}^n$. For $j = 1, \ldots, m$ define $z_j, \bar{z}_j \in \mathbb{C}\mathrm{l}_{r,s}$ by

$$z_j := \begin{cases} \frac{1}{2}\,(e_{2j-1} - i\,e_{2j}) & \text{if } \varepsilon_{2j-1} = \varepsilon_{2j} \\ \frac{1}{2}\,(e_{2j-1} - e_{2j}) & \text{if } \varepsilon_{2j-1} = -\varepsilon_{2j} \end{cases} \qquad \bar{z}_j := \begin{cases} \frac{1}{2}\,(e_{2j-1} + i\,e_{2j}) & \text{if } \varepsilon_{2j-1} = \varepsilon_{2j} \\ \frac{1}{2}\,(e_{2j-1} + e_{2j}) & \text{if } \varepsilon_{2j-1} = -\varepsilon_{2j} \end{cases}$$

Then products of the form

$$z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_{i_1} \cdot \ldots \cdot \bar{z}_{i_l}, \qquad \begin{array}{l} k, l = 0, \ldots, m \\ 1 \le j_1 < \ldots < j_k \le m,\ 1 \le i_1 < \ldots < i_l \le m, \end{array}$$

form a vector space basis of $\mathbb{C}\mathrm{l}_{r,s}$. Put

$$z(j_1,\dots,j_k) := z_{j_1}\cdot\ldots\cdot z_{j_k}\cdot\bar z_1\cdot\ldots\cdot\bar z_m.$$

Then

$$\Sigma_{r,s} := \operatorname{span}\{z(j_1,\dots,j_k)\,|\,k=0,\dots,m,\ 1\le j_1<\ldots<j_k\le m\}\subseteq\mathbb{C}\mathrm{l}_{r,s}$$

is a complex vector subspace of $\mathbb{C}\mathrm{l}_{r,s}$ of dimension 2^m. We call $\boldsymbol{\Sigma_{r,s}}$ the **spinor space** in signature (r,s). Elements of $\Sigma_{r,s}$ are called **spinors**.

For later purposes we want to compute $e_{2l}\cdot z(j_1,\dots,j_k)$ and $e_{2l-1}\cdot z(j_1,\dots,j_k)$. We have to distinguish two cases: e_{2l} and e_{2l-1} can be contained in $z_{j_1}\cdot\ldots\cdot z_{j_k}$ or not.

1) Let e_{2l} and e_{2l-1} not be contained in $z_{j_1}\cdot\ldots\cdot z_{j_k}$.

$$\begin{aligned} e_{2l}\cdot z(j_1,\dots,j_k) &= e_{2l}\cdot z_{j_1}\cdot\ldots\cdot z_{j_k}\cdot\bar z_1\cdot\ldots\cdot\bar z_m\\ &= (-1)^{k+(l-1)} z_{j_1}\cdot\ldots\cdot z_{j_k}\cdot\bar z_1\cdot\ldots\cdot\bar z_{l-1}\cdot e_{2l}\cdot\bar z_l\cdot\bar z_{l+1}\cdot\ldots\cdot\bar z_m. \end{aligned} \tag{6.2}$$

Case 1a) Assume that $\varepsilon_{2l}=\varepsilon_{2l-1}$. Then we have

$$\begin{aligned} e_{2l}\cdot\bar z_l &= \frac12\, e_{2l}\cdot(e_{2l-1}+i\,e_{2l})\\ &= \frac12\,(e_{2l}\cdot e_{2l-1}-i\varepsilon_{2l})\\ &= \frac12\,(-e_{2l-1}\cdot e_{2l}+i\,e_{2l-1}\cdot e_{2l-1})\\ &= i\,e_{2l-1}\cdot\frac12(e_{2l-1}+i\,e_{2l})\\ &= i\,e_{2l-1}\cdot\bar z_l, \end{aligned}$$

and inserting this into equation (6.2) we get

$$\begin{aligned} e_{2l}\cdot z(j_1,\dots,j_k) &= (-1)^{k+l-1}\,i\,z_{j_1}\cdot\ldots\cdot z_{j_k}\cdot\bar z_1\cdot\ldots\cdot\bar z_{l-1}\cdot e_{2l-1}\cdot\bar z_l\cdot\bar z_{l+1}\cdot\ldots\cdot\bar z_m\\ &= (-1)^{k+l-1}\,i\,(-1)^{k+l-1}\,e_{2l-1}\cdot z_{j_1}\cdot\ldots\cdot z_{j_k}\cdot\bar z_1\cdot\ldots\cdot\bar z_m\\ &= i\,e_{2l-1}\cdot z(j_1,\dots,j_k). \end{aligned} \tag{6.3}$$

Let ν such that $j_\nu<l<j_{\nu+1}$. Then we have:

$$\begin{aligned} e_{2l}\cdot z(j_1,\dots,j_k) &= \frac12\,e_{2l}\cdot z(j_1,\dots,j_k)+\frac12\,e_{2l}\cdot z(j_1,\dots,j_k)\\ &\overset{(6.3)}{=} \frac12\,e_{2l}\cdot z(j_1,\dots,j_k)+\frac{i}{2}\,e_{2l-1}\cdot z(j_1,\dots,j_k)\\ &= i\,\underbrace{\frac12\,(e_{2l-1}-i\,e_{2l})}_{=z_l}\cdot z(j_1,\dots,j_k)\\ &= i\,(-1)^\nu\,z(j_1,\dots,j_\nu,l,j_{\nu+1},\dots,j_k). \end{aligned} \tag{6.4}$$

Moreover, it follows from equations (6.3) and (6.4) that

$$\begin{aligned} e_{2l-1} \cdot z(j_1, \dots, j_k) &\overset{(6.3)}{=} -i\, e_{2l} \cdot z(j_1, \dots, j_k) \\ &\overset{(6.4)}{=} (-1)^{\nu}\, z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k). \end{aligned} \tag{6.5}$$

Case 1b) Assume that $\varepsilon_{2l} = -\varepsilon_{2l-1}$. Then we have

$$\begin{aligned} e_{2l} \cdot \bar{z}_l &= \frac{1}{2}\, e_{2l} \cdot (e_{2l-1} + e_{2l}) \\ &= \frac{1}{2}\, (e_{2l} \cdot e_{2l-1} - \varepsilon_{2l}) \\ &= \frac{1}{2}\, (-e_{2l-1} \cdot e_{2l} - e_{2l-1} \cdot e_{2l-1}) \\ &= -e_{2l-1} \cdot \frac{1}{2} (e_{2l} + e_{2l-1}) \\ &= -e_{2l-1} \cdot \bar{z}_l, \end{aligned}$$

and inserting this into equation (6.2) we get

$$\begin{aligned} e_{2l} \cdot z(j_1, \dots, j_k) &= (-1)^{k+l-1}\, (-1)\, z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_{l-1} \cdot e_{2l-1} \cdot \bar{z}_l \cdot \bar{z}_{l+1} \cdot \ldots \cdot \bar{z}_m \\ &= (-1)^{k+l-1}\, (-1)\, (-1)^{k+l-1}\, e_{2l-1} \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\ &= -e_{2l-1} \cdot z(j_1, \dots, j_k). \end{aligned} \tag{6.6}$$

Let ν such that $j_\nu < l < j_{\nu+1}$. Then we have:

$$\begin{aligned} e_{2l} \cdot z(j_1, \dots, j_k) &= \frac{1}{2}\, e_{2l} \cdot z(j_1, \dots, j_k) + \frac{1}{2}\, e_{2l} \cdot z(j_1, \dots, j_k) \\ &\overset{(6.6)}{=} -\frac{1}{2}\, e_{2l-1} \cdot z(j_1, \dots, j_k) + \frac{1}{2}\, e_{2l} \cdot z(j_1, \dots, j_k) \\ &= -\underbrace{\frac{1}{2}\, (e_{2l-1} - e_{2l})}_{= z_l} \cdot z(j_1, \dots, j_k) \\ &= (-1)^{\nu+1}\, z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k). \end{aligned} \tag{6.7}$$

Moreover, it follows from equations (6.6) and (6.7) that

$$\begin{aligned} e_{2l-1} \cdot z(j_1, \dots, j_k) &\overset{(6.6)}{=} -e_{2l} \cdot z(j_1, \dots, j_k) \\ &\overset{(6.7)}{=} (-1)^{\nu}\, z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k). \end{aligned} \tag{6.8}$$

2) Now let e_{2l} and e_{2l-1} be contained in $z_{j_1} \cdot \ldots \cdot z_{j_k}$.
Case 2a) Assume that $\varepsilon_{2l} = \varepsilon_{2l-1}$. Multiplying equation (6.4) with e_{2l}, we obtain:

$$e_{2l} \cdot z(j_1, \dots, j_\nu, l, j_{\nu+1}, \dots, j_k) = i\, (-1)^{\nu}\, \varepsilon_{2l}\, z(j_1, \dots, j_\nu, j_{\nu+1}, \dots, j_k). \tag{6.9}$$

Multiplying equation (6.5) with e_{2l-1} we obtain:

$$e_{2l-1} \cdot z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k) = (-1)^{\nu+1}\, \varepsilon_{2l}\, z(j_1, \ldots, j_\nu, j_{\nu+1}, \ldots, j_k). \tag{6.10}$$

Case 2b) Assume that $\varepsilon_{2l} = -\varepsilon_{2l-1}$. Multiplying equation (6.7) with e_{2l}, we obtain:

$$e_{2l} \cdot z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k) = (-1)^{\nu}\, \varepsilon_{2l}\, z(j_1, \ldots, j_\nu, j_{\nu+1}, \ldots, j_k). \tag{6.11}$$

Multiplying equation (6.8) with e_{2l-1} we obtain:

$$e_{2l-1} \cdot z(j_1, \ldots, j_\nu, l, j_{\nu+1}, \ldots, j_k) = (-1)^{\nu}\, \varepsilon_{2l}\, z(j_1, \ldots, j_\nu, j_{\nu+1}, \ldots, j_k). \tag{6.12}$$

Hence the spinor space $\Sigma_{r,s} \subset \mathbb{C}\mathrm{l}_{r,s}$ is invariant under **Clifford multiplication** by vectors in $\mathbb{R}^n$. Since the Clifford algebra $\mathrm{Cl}_{r,s}$ is generated by $\mathbb{R}^n$, the same holds for Clifford multiplication by elements of $\mathbb{C}\mathrm{l}_{r,s}$, thus $\Sigma_{r,s} \subset \mathbb{C}\mathrm{l}_{r,s}$ is a left ideal. In particular, $\Sigma_{r,s}$ is invariant under multiplication by elements of $\mathrm{Spin}(r,s)$.
We define:

$$\Sigma_{r,s}^{+} := \mathrm{span}\{z(j_1, \ldots, j_k) \,|\, k = 0, \ldots, m \text{ even}\}$$
$$\Sigma_{r,s}^{-} := \mathrm{span}\{z(j_1, \ldots, j_k) \,|\, k = 0, \ldots, m \text{ odd}\}.$$

The spinor space $\Sigma_{r,s}$ has the decomposition $\Sigma_{r,s} = \Sigma_{r,s}^{+} \oplus \Sigma_{r,s}^{-}$. Elements in $\Sigma_{r,s}^{\pm}$ are called spinors of positive and negative **chirality** respectively.
The equations (6.4), (6.5) and (6.7)–(6.12) show that the Clifford multiplication by elements of $\mathbb{R}^n$ satisfies:

$$\mathbb{R}^n \cdot \Sigma_{r,s}^{+} \subset \Sigma_{r,s}^{-}, \quad \mathbb{R}^n \cdot \Sigma_{r,s}^{-} \subset \Sigma_{r,s}^{+}.$$

However, Clifford multiplication by elements of $\mathbb{C}\mathrm{l}_{r,s}^0$ satisfies:

$$\mathbb{C}\mathrm{l}_{r,s}^0 \cdot \Sigma_{r,s}^{+} \subset \Sigma_{r,s}^{+}, \quad \mathbb{C}\mathrm{l}_{r,s}^0 \cdot \Sigma_{r,s}^{-} \subset \Sigma_{r,s}^{-}.$$

Thus, the restriction to $\mathrm{Spin}(r,s) \subset \mathrm{Cl}_{r,s}^0 \subset \mathbb{C}\mathrm{l}_{r,s}^0$ yields representations of $\mathrm{Spin}(r,s)$ on $\Sigma_{r,s}^{+}$ and $\Sigma_{r,s}^{-}$ and thus on $\Sigma_{r,s}$.

Definition 6.2.1. The representation $\sigma_{r,s} : \mathrm{Spin}(r,s) \to \mathrm{GL}(\Sigma_{r,s})$ is called the **spinor representation**.
The representations $\sigma_{r,s}^{\pm} : \mathrm{Spin}(r,s) \to \mathrm{GL}(\Sigma_{r,s}^{\pm})$ are called the **positive** and **negative** spinor representation, respectively.

Remark 6.2.2. The element

$$\omega := e_1 \cdot \ldots \cdot e_n \in \mathrm{Cl}_{r,s} \subset \mathbb{C}\mathrm{l}_{r,s}$$

is called the **volume element**. The equations (6.4), (6.5) and (6.7)–(6.12) show that for $\varepsilon_{2l} = \varepsilon_{2l-1}$ we have

$$e_{2l-1}\cdot e_{2l}\cdot z(j_1,\dots,j_k) = \begin{cases} -i\,\varepsilon_{2l}\,z(j_1,\dots,j_k) & \text{if } e_{2l},\, e_{2l-1} \text{ are not contained in } z_{j_1}\cdot\ldots\cdot z_{j_k} \\ i\,\varepsilon_{2l}\,z(j_1,\dots,j_k) & \text{if } e_{2l},\, e_{2l-1} \text{ are contained in } z_{j_1}\cdot\ldots\cdot z_{j_k} \end{cases}$$

and for $\varepsilon_{2l} = -\varepsilon_{2l-1}$ we have

$$e_{2l-1}\cdot e_{2l}\cdot z(j_1,\dots,j_k) = \begin{cases} -\varepsilon_{2l}\,z(j_1,\dots,j_k) & \text{if } e_{2l},\, e_{2l-1} \text{ are not contained in } z_{j_1}\cdot\ldots\cdot z_{j_k} \\ \varepsilon_{2l}\,z(j_1,\dots,j_k) & \text{if } e_{2l},\, e_{2l-1} \text{ are contained in } z_{j_1}\cdot\ldots\cdot z_{j_k}. \end{cases}$$

Let $r = 2a$ be even. Then we have $\varepsilon_{2l} = \varepsilon_{2l-1}$ for all l and thus

$$\omega\cdot z(j_1,\dots,j_k) = (-i)^{m-k} i^k \underbrace{\varepsilon_2\varepsilon_4\ldots\varepsilon_{2m}}_{=(-1)^a} z(j_1,\dots,j_k) = (-1)^{m-k+a}\, i^m\, z(j_1,\dots,j_k)$$

and therefore

$$i^{m+r}\omega\cdot z(j_1,\dots,j_k) = i^m(-1)^a\omega\cdot z(j_1,\dots,j_k) = (-1)^k\, z(j_1,\dots,j_k).$$

Let $r = 2a-1$ be odd. If e_{2a}, e_{2a-1} are not contained in $z_{j_1}\cdot\ldots\cdot z_{j_k}$ then we have

$$\begin{aligned}\omega\cdot z(j_1,\dots,j_k) &= (-i)^{m-k-1}(-1)i^k \underbrace{\varepsilon_2\varepsilon_4\ldots\varepsilon_{2m}}_{=(-1)^{a-1}} z(j_1,\dots,j_k) \\ &= (-1)^{m-k+a-1}\, i^{m-1} z(j_1,\dots,j_k)\end{aligned}$$

If e_{2a}, e_{2a-1} are contained in $z_{j_1}\cdot\ldots\cdot z_{j_k}$ then we have

$$\begin{aligned}\omega\cdot z(j_1,\dots,j_k) &= (-i)^{m-k} i^{k-1} \underbrace{\varepsilon_2\varepsilon_4\ldots\varepsilon_{2m}}_{=(-1)^{a-1}} z(j_1,\dots,j_k) \\ &= (-1)^{m-k+a-1} i^{m-1} z(j_1,\dots,j_k).\end{aligned}$$

Thus for $r = 2a-1$ we get

$$\begin{aligned}i^{m+r}\omega\cdot z(j_1,\dots,j_k) &= i^{2m+r-1}(-1)^{m-k+a-1} z(j_1,\dots,j_k) \\ &= (-1)^k z(j_1,\dots,j_k).\end{aligned}$$

Therefore, for all r we have

$$\Sigma_{r,s}^{\pm} = \{z\in\Sigma_{r,s}\,|\, i^{m+r}\omega\cdot z = \pm z\}.$$

Example 6.2.3. Let $r = s = 1$, i.e., $n = 2$, $m = 1$. Then we have:

$$\Sigma_{1,1}^{+} = \mathbb{C}\cdot z() \qquad\text{and}\qquad \Sigma_{1,1}^{-} = \mathbb{C}\cdot z(1).$$

By the equations (6.8) and (6.12), we have

$$\left.\begin{aligned} e_1\cdot z(\,) &= z(1) \\ e_1\cdot z(1) &= z(\,), \end{aligned}\right\} \quad \text{thus } e_1 \text{ acts on } \Sigma_{1,1}\cong\mathbb{C}^2 \text{ as } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the equations (6.7) and (6.11), we have

$$\left.\begin{aligned} e_2 \cdot z(\,) &= -z(1) \\ e_2 \cdot z(1) &= z(\,), \end{aligned}\right\} \quad \text{thus } e_2 \text{ acts on } \Sigma_{1,1} \cong \mathbb{C}^2 \text{ as } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Furthermore, from Example 6.1.11 we know that

$$\mathrm{Spin}(1,1) = \{\pm(\cosh(t) + \sinh(t)e_1 \cdot e_2) \,|\, t \in \mathbb{R}\} \cup \{\pm(\sinh(t) + \cosh(t)e_1 \cdot e_2) \,|\, t \in \mathbb{R}\}.$$

The element $e_1 \cdot e_2$ acts on $\Sigma_{1,1} \cong \mathbb{C}^2$ as

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, the elements $\pm(\cosh(t) + \sinh(t)e_1 \cdot e_2)$ and $\pm(\sinh(t) + \cosh(t)e_1 \cdot e_2)$ act as

$$\pm\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} -e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}$$

respectively.

We equip $\Sigma_{r,s}$ with the Hermitian scalar product $\langle \cdot, \cdot \rangle$ for which the vectors $z(j_1, \ldots, j_k)$ form an orthonormal basis of $\Sigma_{r,s}$. By our convention $\langle \cdot, \cdot \rangle$ is complex linear in the first argument and complex antilinear in the second argument. Moreover, $\Sigma_{r,s} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}$ is an orthogonal decomposition.

Lemma 6.2.4. *Let $n = r + s$ be even. Then we have for any vector $X \in \mathbb{R}^n$ and any spinors $\varphi, \psi \in \Sigma_{r,s}$:*

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, R(X) \cdot \psi \rangle,$$

where $R\colon \mathbb{R}^n \to \mathbb{R}^n$, $R(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n \varepsilon_j x_j e_j$ is a reflection.

Proof. It is sufficient to prove the statement for $X = e_j$, $\varphi = z(j_1, \ldots, j_k)$, $\psi = z(i_1, \ldots, i_l)$.
We only consider the case $e_j = e_{2l}$ and $\varphi = z(j_1, \ldots, j_\nu, l, j_{\nu+2}, \ldots, j_k)$. The remaining cases are treated analogously. By Equations (6.9), (6.11) we get

$$e_{2l} \cdot \varphi = \begin{cases} i(-1)^\nu \varepsilon_{2l}\, z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k) & \text{if } \varepsilon_{2l} = \varepsilon_{2l-1} \\ (-1)^\nu \varepsilon_{2l}\, z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k) & \text{if } \varepsilon_{2l} = -\varepsilon_{2l-1} \end{cases}$$

and thus

$$\langle e_{2l} \cdot \varphi, \psi \rangle = \begin{cases} i\,(-1)^\nu \varepsilon_{2l} & \text{if } \psi = z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k) \text{ and } \varepsilon_{2l} = \varepsilon_{2l-1} \\ (-1)^\nu \varepsilon_{2l} & \text{if } \psi = z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k) \text{ and } \varepsilon_{2l} = -\varepsilon_{2l-1} \\ 0 & \text{otherwise} \end{cases}$$

If $\psi = z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k)$ then by Equations (6.4), (6.7) we get

$$e_{2l} \cdot \psi = \begin{cases} i\,(-1)^\nu\, z(j_1, \ldots, j_\nu, l, j_{\nu+2}, \ldots, j_k) = i\,(-1)^\nu\, \varphi & \text{if } \varepsilon_{2l} = \varepsilon_{2l-1} \\ (-1)^{\nu+1}\, z(j_1, \ldots, j_\nu, l, j_{\nu+2}, \ldots, j_k) = (-1)^{\nu+1}\, \varphi & \text{if } \varepsilon_{2l} = -\varepsilon_{2l-1} \end{cases}$$

and thus

$$\langle \varphi, e_{2l} \cdot \psi \rangle = \begin{cases} i(-1)^{\nu+1} = -\varepsilon_{2l} \langle e_{2l} \cdot \varphi, \psi \rangle & \text{if } \varepsilon_{2l} = \varepsilon_{2l-1} \\ (-1)^{\nu+1} = -\varepsilon_{2l} \langle e_{2l} \cdot \varphi, \psi \rangle & \text{if } \varepsilon_{2l} = -\varepsilon_{2l-1}. \end{cases}$$

If $\psi \neq z(j_1, \ldots, j_\nu, j_{\nu+2}, \ldots, j_k)$ then $\langle e_{2l} \cdot \varphi, \psi \rangle = 0 = -\varepsilon_{2l} \langle \varphi, e_{2l} \cdot \psi \rangle$. □

Remark 6.2.5. If $X \in \mathrm{span}\{e_1, \ldots, e_r\}$ then $R(X) = -X$. In this case we get for any two spinors $\varphi, \psi \in \Sigma_{r,s}$:

$$\langle X \cdot \varphi, \psi \rangle = \langle \varphi, X \cdot \psi \rangle,$$

i.e., Clifford multiplication by X is symmetric. If in addition $\langle X, X \rangle_{r,s} = -1$, we have

$$\langle X \cdot \varphi, X \cdot \psi \rangle = \langle \varphi, X \cdot X \cdot \psi \rangle = -\langle X, X \rangle_{r,s} \langle \varphi, \psi \rangle = \langle \varphi, \psi \rangle,$$

i.e., Clifford multiplication by X is an isometry.
If $X \in \mathrm{span}\{e_{r+1}, \ldots, e_n\}$ then $R(X) = X$. In this case we get for any two spinors $\varphi, \psi \in \Sigma_{r,s}$:

$$\langle X \cdot \varphi, \psi \rangle = -\langle \varphi, X \cdot \psi \rangle$$

i.e., Clifford multiplication by X is skew-symmetric. If in addition $\langle X, X \rangle_{r,s} = 1$, we have

$$\langle X \cdot \varphi, X \cdot \psi \rangle = -\langle \varphi, X \cdot X \cdot \psi \rangle = \langle X, X \rangle_{r,s} \langle \varphi, \psi \rangle = \langle \varphi, \psi \rangle,$$

i.e., Clifford multiplication by X is an isometry.
However, if $r > 0$ and $s > 0$ then for $X \in \mathbb{R}^n$ with $\langle X, X \rangle_{r,s} \in \{\pm 1\}$, Clifford multiplication by X is in general neither symmetric nor skew-symmetric nor an isometry for $\langle \cdot, \cdot \rangle$. As an example one might take $X = \frac{1}{\sqrt{3}}(e_1 + 2e_2)$ where $\langle e_1, e_1 \rangle_{r,s} = -1$ and $\langle e_2, e_2 \rangle_{r,s} = 1$. In particular, if $r > 0$ and $s > 0$ the spinor representation is not a unitary representation for $\langle \cdot, \cdot \rangle$.

Proposition 6.2.6. *Let $n = r + s = 2m$ be even. Then the map*

$$\Phi : \quad \mathbb{C}\mathrm{l}_{r,s} \to \mathrm{End}(\Sigma_{r,s}), \quad \Phi(X)(z) := X \cdot z$$

is an isomorphism of complex algebras.

Proof. Obviously Φ is a homomorphism of complex algebras. We prove that Φ is surjective. Note first that for all $\ell \in \{1, \dots, m\}$ we have

$$z_\ell \cdot \bar{z}_\ell + \bar{z}_\ell \cdot z_\ell = -\varepsilon_{2\ell-1} \tag{6.13}$$
$$\bar{z}_\ell \cdot \bar{z}_\ell = 0 \tag{6.14}$$
$$z_\ell \cdot z_\ell = 0. \tag{6.15}$$

Let $i, \ell \in \{1, \dots, m\}$ and let $z(j_1, \dots, j_k) \in \Sigma_{r,s}$.
a) Assume $\ell \in \{j_1, \dots, j_k\}$. From the equations (6.13) and (6.14) we get

$$\begin{aligned}
\Phi(\bar{z}_\ell)(z(j_1, \dots, j_k)) &= \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_\ell \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_\ell \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot z_\ell \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= \pm(-\varepsilon_{2l-1} - z_\ell \cdot \bar{z}_\ell) \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_\ell} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m + 0 \\
&= \pm z(j_1, \dots, \widehat{\ell}, \dots, j_k),
\end{aligned}$$

where the signs $\pm$ may change in every line.
b) Assume $\ell \notin \{j_1, \dots, j_k\}$. Then by the equation (6.14) we get

$$\begin{aligned}
\Phi(\bar{z}_\ell)(z(j_1, \dots, j_k)) &= \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_\ell \cdot \ldots \cdot \bar{z}_m \\
&= \pm \bar{z}_\ell \cdot \bar{z}_\ell \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \widehat{\bar{z}_\ell} \cdot \ldots \cdot \bar{z}_m \\
&= 0.
\end{aligned}$$

c) Assume $i \in \{j_1, \dots, j_k\}$. By the equation (6.15) we get

$$\begin{aligned}
\Phi(z_i)(z(j_1, \dots, j_k)) &= z_i \cdot z_{j_1} \cdot \ldots \cdot z_i \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\
&= \pm z_i \cdot z_i \cdot z_{j_1} \cdot \ldots \cdot \widehat{z_i} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m \\
&= 0.
\end{aligned}$$

d) If $i \notin \{j_1, \dots, j_k\}$ then we get

$$\Phi(z_i)(z(j_1, \dots, j_k)) = z_i \cdot z_{j_1} \cdot \ldots \cdot z_{j_k} \cdot \bar{z}_1 \cdot \ldots \cdot \bar{z}_m = \pm z(j_1, \dots, i, \dots, j_k).$$

For any multi-index $I = \{i_1, \dots, i_s\}$ we write

$$z_I := z_{i_1} \cdot \ldots \cdot z_{i_s}, \quad \bar{z}_I := \bar{z}_{i_1} \cdot \ldots \cdot \bar{z}_{i_s}, \quad z(I) := z(i_1, \dots, i_s)$$

and we denote by I^c the complementary multi-index of I. Let now I and K be multi-indices. The calculations in a) - d) show that for all multi-indices J we have

$$z_1 \cdot \ldots \cdot z_m \cdot z(J) = \begin{cases} 0 & \text{if } J \neq \emptyset \\ \pm z_1 \cdot \ldots \cdot z_m & \text{if } J = \emptyset \end{cases}$$

and thus

$$z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I \cdot z(J) = \begin{cases} 0 & \text{if } J \neq I \\ \pm z_1 \cdot \ldots \cdot z_m & \text{if } J = I \end{cases}$$

and therefore

$$\bar{z}_{K^c} \cdot z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I \cdot z(J) = \begin{cases} 0 & \text{if } J \neq I \\ \pm z(K) & \text{if } J = I \end{cases}$$

Thus every endomorphism of $\Sigma_{r,s}$ can be obtained by composing endomorphisms of the form $\Phi(\bar{z}_{K^c} \cdot z_1 \cdot \ldots \cdot z_m \cdot \bar{z}_I)$. This shows that Φ is surjective. Since $\mathbb{C}\mathrm{l}_{r,s}$ and $\mathrm{End}(\Sigma_{r,s})$ have the same dimension we conclude that Φ is an isomorphism. □

The odd dimensional case

In the following, let $n = 2m - 1$. To construct the spinor space $\Sigma_{r,s}$, we make the following observation:

Lemma 6.2.7. *Let $n = r + s \in \mathbb{N}$. The linear map $j : \mathbb{R}^n \to \mathrm{Cl}^0_{r,s+1}$,*

$$X \mapsto j(X) := X \cdot e_{n+1},$$

induces an algebra isomorphism $\mathrm{Cl}_{r,s} \to \mathrm{Cl}^0_{r,s+1}$.

Remark 6.2.8. Lemma 6.2.7 also holds for $\mathbb{C}\mathrm{l}_{r,s}$ instead of $\mathrm{Cl}_{r,s}$.

Proof. The proof is analogous to the proof of Lemma 2.3.12. □

For $n = r + s$ odd we define the spinor space $\Sigma_{r,s}$ by:

$$\Sigma_{r,s} := \Sigma^+_{r,s+1}.$$

In particular, we have $\dim \Sigma_{r,s} = 2^{\lfloor \frac{n}{2} \rfloor}$ for both even and odd n. The Clifford algebra $\mathrm{Cl}_{r,s}$ acts on the spinor space $\Sigma_{r,s}$ via the map α: For $X \in \mathrm{Cl}_{r,s}$ and $\phi \in \Sigma_{r,s}$ put

$$X \bullet \phi := \alpha(X) \cdot \phi \in \Sigma^+_{r,s+1} = \Sigma_{r,s}.$$

The restriction of this action to $\mathrm{Spin}(r,s) \subset \mathrm{Cl}_{r,s} \subset \mathbb{C}\mathrm{l}_{r,s}$ defines the **spinor representation** $\sigma_{r,s} : \mathrm{Spin}(r,s) \to \mathrm{GL}(\Sigma_{r,s})$ in odd dimensions. We define a Hermitian scalar product $\langle \cdot, \cdot \rangle$ on $\Sigma_{r,s}$ by restricting the Hermitian scalar product of $\Sigma_{r,s+1}$ to $\Sigma_{r,s} = \Sigma^+_{r,s+1}$.

Lemma 6.2.9. *Let $n = r + s$ be odd. Then we have for any vector $X \in \mathbb{R}^n$ and any spinors $\varphi, \psi \in \Sigma_{r,s}$:*

$$\langle X \bullet \varphi, \psi \rangle = -\langle \varphi, R(X) \bullet \psi \rangle,$$

where R: $\mathbb{R}^n \to \mathbb{R}^n$, $R(\sum_{j=1}^n x_j e_j) = \sum_{j=1}^n \varepsilon_j x_j e_j$ is a reflection.

Proof. We compute using Lemma 6.2.4:

$$\begin{aligned}
\langle X \bullet \varphi, \psi\rangle &= \langle \alpha(X)\cdot\varphi, \psi\rangle \\
&= \langle X \cdot e_{n+1}\cdot \varphi, \psi\rangle \\
&= -\langle e_{n+1}\cdot\varphi, R(X)\cdot\psi\rangle \\
&= \langle \varphi, e_{n+1}\cdot R(X)\cdot\psi\rangle \\
&= -\langle \varphi, R(X)\cdot e_{n+1}\cdot\psi\rangle \\
&= -\langle \varphi, \alpha(R(X))\cdot\psi\rangle \\
&= -\langle \varphi, R(X)\bullet\psi\rangle.
\end{aligned}$$

□

Remark 6.2.10. As in Remark 6.2.5 one sees that if $r > 0$ and $s > 0$ the spinor representation in odd dimensions is not a unitary representation for $\langle\cdot,\cdot\rangle$.

From now let $n = 2m$ or $n = 2m-1$ be even or odd. We denote the Clifford multiplication in both cases by $\cdot$. Assume that $r > 0$ and $s > 0$. Our aim is to define an inner product on $\Sigma_{r,s}$ for which the restriction of the spinor representation to $\mathrm{Spin}_0(r, s)$ is unitary. We proceed as in Chapter 1.5 of the book [3] by Helga Baum. We define $\beta \in \mathbb{C}\mathrm{l}_{r,s}$ by

$$\beta := \begin{cases} e_1\cdot\ldots\cdot e_r & \text{if } r \equiv 0,1 \bmod 4 \\ ie_1\cdot\ldots\cdot e_r & \text{if } r\equiv 2,3 \bmod 4. \end{cases}$$

Lemma 6.2.11. *We have:*

1. $\beta\cdot\beta = 1$.
2. *For all* $\varphi, \psi \in \Sigma_{r,s}$ *we have* $\langle\beta\cdot\varphi, \psi\rangle = \langle\varphi, \beta\cdot\psi\rangle$.

Proof. 1. We compute

$$\begin{aligned}
e_1\cdot\ldots\cdot e_r\cdot e_1\cdot\ldots\cdot e_r &= (-1)^{\frac{r(r-1)}{2}} e_1^2\cdot\ldots\cdot e_r^2 \\
&= (-1)^{\frac{r(r-1)}{2}} = \begin{cases} 1 & r\equiv 0,1 \bmod 4 \\ -1 & r\equiv 2,3 \bmod 4 \end{cases}
\end{aligned}$$

The assertion follows from the definition of β.

2. We have $\varepsilon_j = -1$ for $j = 1, \ldots, r$ and thus by Lemmas 6.2.4, 6.2.9:

$$\langle e_1\cdot\ldots\cdot e_r\cdot\varphi,\psi\rangle = \langle\varphi, e_r\cdot\ldots\cdot e_1\cdot\psi\rangle = (-1)^{\frac{r(r-1)}{2}}\langle\varphi, e_1\cdot\ldots\cdot e_r\cdot\psi\rangle.$$

For $r \equiv 0,1 \bmod 4$ the assertion follows immediately, for $r\equiv 2,3 \bmod 4$ we use that multiplication by i is skew-symmetric. □

Definition 6.2.12. Let $r > 0$ and $s > 0$. For $\varphi, \psi \in \Sigma_{r,s}$ we define

$$(\varphi, \psi) := \langle \beta \cdot \varphi, \psi \rangle.$$

We call $(\cdot, \cdot)$ the **indefinite inner product** on $\Sigma_{r,s}$.

Lemma 6.2.13. *Let $n = 2m$ or $n = 2m - 1$ and let $r > 0$ and $s > 0$. The following holds:*

1. *$(\cdot, \cdot)$ is a non-degenerate sesquilinear form on $\Sigma_{r,s}$ of signature $(2^{\lfloor \frac{n}{2} \rfloor - 1}, 2^{\lfloor \frac{n}{2} \rfloor - 1})$.*
2. *For all $X \in \mathbb{R}^n$ and all $\varphi, \psi \in \Sigma_{r,s}$ we have $(X \cdot \varphi, \psi) = (-1)^{r-1}(\varphi, X \cdot \psi)$.*
3. *For all $a \in \mathrm{Spin}_0(r, s)$ and all $\varphi, \psi \in \Sigma_{r,s}$ we have $(a \cdot \varphi, a \cdot \psi) = (\varphi, \psi)$.*
4. *For $n = 2m$ and $r = 1$ the spaces $\Sigma^{\pm}_{1,2m-1}$ are isotropic with respect to $(\cdot, \cdot)$.*

Proof. 1. By Lemma 6.2.11 we have for all $\varphi, \psi \in \Sigma_{r,s}$:

$$(\varphi, \psi) = \langle \beta \cdot \varphi, \psi \rangle = \langle \varphi, \beta \cdot \psi \rangle = \overline{\langle \beta \cdot \psi, \varphi \rangle} = \overline{(\psi, \varphi)}$$

and thus $(\cdot, \cdot)$ is sesquilinear.
By Lemma 6.2.11 we have $\beta \cdot \beta = 1$ and thus $\Sigma_{r,s} = E(\beta, 1) \oplus E(\beta, -1)$ where $E(\beta, \pm 1)$ denote the eigenspaces of β for the eigenvalues ± 1.
If r is even then we have $\beta \cdot e_1 = -e_1 \cdot \beta$ and thus $e_1 \cdot$ is an isomorphism $E(\beta, 1) \to E(\beta, -1)$. If r is odd then we have $\beta \cdot e_{r+1} = -e_{r+1} \cdot \beta$ and thus $e_{r+1} \cdot$ is an isomorphism $E(\beta, 1) \to E(\beta, -1)$. In both cases we get $\dim E(\beta, 1) = \dim E(\beta, -1) = 2^{\lfloor \frac{n}{2} \rfloor - 1}$.

2. Let $X \in \mathbb{R}^n$, $X = X_1 + X_2$ where $X_1 \in \mathrm{span}\{e_1, \ldots, e_r\}$ and $X_2 \in \mathrm{span}\{e_{r+1}, \ldots, e_n\}$. Using the Lemmas 6.2.4 and 6.2.9 we get for all $\varphi, \psi \in \Sigma_{r,s}$:

$$\begin{aligned}
(X \cdot \varphi, \psi) &= \langle \beta \cdot X \cdot \varphi, \psi \rangle \\
&= \langle \beta \cdot X_1 \cdot \varphi, \psi \rangle + \langle \beta \cdot X_2 \cdot \varphi, \psi \rangle \\
&= (-1)^{r-1} \langle X_1 \cdot \beta \cdot \varphi, \psi \rangle + (-1)^r \langle X_2 \cdot \beta \cdot \varphi, \psi \rangle \\
&= (-1)^{r-1} \langle \beta \cdot \varphi, X_1 \cdot \psi \rangle + (-1)^{r-1} \langle \beta \cdot \varphi, X_2 \cdot \psi \rangle \\
&= (-1)^{r-1} (\varphi, X \cdot \psi).
\end{aligned}$$

3. Let $\varphi, \psi \in \Sigma_{r,s}$. For $x \in \mathbb{R}^n$, $\langle x, x \rangle_{r,s} = \pm 1$, we get by part 2

$$(x \cdot \varphi, x \cdot \psi) = (-1)^{r-1} (\varphi, x \cdot x \cdot \psi) = (-1)^r \langle x, x \rangle_{r,s} (\varphi, \psi).$$

and thus for $a = x_1 \cdot \ldots \cdot x_{2k} \in \mathrm{Spin}(r,s)$, $x_j \in \mathbb{R}^n$, $\langle x_j, x_j\rangle_{r,s} = \pm 1$, $j = 1, \ldots, 2k$, we get:

$$(a\cdot\varphi, a\cdot\psi) = \langle x_1, x_1\rangle_{r,s} \ldots \langle x_{2k}, x_{2k}\rangle_{r,s}(\varphi,\psi) = \pm(\varphi,\psi).$$

This equation holds for all $a \in \mathrm{Spin}(r,s)$. Now fix $a \in \mathrm{Spin}_0(r,s)$. Then there is a continuous path c: $[0,1] \to \mathrm{Spin}_0(r,s)$ with $c(0) = 1$ and $c(1) = a$. The function

$$t \mapsto (c(t)\cdot\varphi, c(t)\cdot\psi) = \pm(\varphi,\psi)$$

is continuous and thus constant. In particular $(a\cdot\varphi, a\cdot\psi) = (\varphi,\psi)$.

4. For $\varphi \in \Sigma^{\pm}_{1,2m-1}$ we have $i^{m+1}\omega\cdot\varphi = \pm\varphi$ and thus by 2. with $r = 1$:

$$\begin{aligned}(\varphi,\varphi) &= (i^{m+1}\omega\cdot\varphi, i^{m+1}\omega\cdot\varphi) = (e_1\cdot\ldots\cdot e_n\cdot\varphi, e_1\cdot\ldots\cdot e_n\cdot\varphi)\\ &= (\varphi, \underbrace{e_n\cdot\ldots\cdot e_1\cdot e_1\cdot\ldots\cdot e_n}_{=(-1)^{n-1}=-1}\cdot\varphi) = -(\varphi,\varphi).\end{aligned}$$ □

Remark 6.2.14. Let $r > 0$ and $s > 0$. Then the indefinite inner product $(\cdot,\cdot)$ on $\Sigma_{r,s}$ has the best invariance property in the following sense (see [3], p. 69):

1. There is no inner product on $\Sigma_{r,s}$ which is invariant under $\mathrm{Spin}(r,s)$.
2. There is no positive definite inner product on $\Sigma_{r,s}$ which is invariant under $\mathrm{Spin}_0(r,s)$.

Proof. 1. Assume that $(\cdot,\cdot)$ is such an inner product. Let c: $(-\varepsilon,\varepsilon) \to \mathrm{Spin}(r,s)$ be a smooth curve with $c(0) = 1$ and $\dot c(0) = e_i\cdot e_j$ where $\varepsilon_i = -\varepsilon_j$. Then for all t and all $\varphi,\psi \in \Sigma_{r,s}$ we have by assumption $(c(t)\cdot\varphi, c(t)\cdot\psi) = (\varphi,\psi)$ and thus

$$0 = \frac{d}{dt}\Big|_{t=0}(c(t)\cdot\varphi, c(t)\cdot\psi) = (e_i\cdot e_j\cdot\varphi,\psi) + (\varphi, e_i\cdot e_j\cdot\psi). \tag{6.16}$$

Let $\varphi,\psi \in \Sigma_{r,s}$ with $(\varphi,\psi) \neq 0$. Since $e_i\cdot e_j \in \mathrm{Spin}(r,s)$ we get by the assumption and by equation (6.16)

$$(\varphi,\psi) = (e_i\cdot e_j\cdot\varphi, e_i\cdot e_j\cdot\psi) = -(\varphi, e_i\cdot e_j\cdot e_i\cdot e_j\cdot\psi) = \varepsilon_i\varepsilon_j(\varphi,\psi) = -(\varphi,\psi)$$

which is a contradiction.

2. Assume that $(\cdot,\cdot)$ is such an inner product. Choose i,j with $\varepsilon_i = -\varepsilon_j$. Then we have for all $\varphi \in \Sigma_{r,s}$ by equation (6.16)

$$0 \leq (e_i\cdot e_j\cdot\varphi, e_i\cdot e_j\cdot\varphi) = -(\varphi, e_i\cdot e_j\cdot e_i\cdot e_j\cdot\varphi) = \varepsilon_i\varepsilon_j(\varphi,\varphi) = -(\varphi,\varphi)$$

which is a contradiction. □

6.3. Spin structures

Let M be an oriented semi-Riemannian manifold of signature (r,s) and of dimension $n = r+s$. For $x \in M$ put

$$P_x^{\mathrm{SO}}(M) := \big\{h : (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{r,s}) \to T_xM \,|\, h \text{ orientation preserving isometry}\big\}.$$

Each element $h \in P_x^{\mathrm{SO}}(M)$ induces an oriented semi-orthonormal basis $h(e_1),\dots,h(e_n)$ of T_xM. Conversely, for any oriented semi-orthonormal basis $b_1,\dots,b_n$ of T_xM, there is a unique $h \in P_x^{\mathrm{SO}}(M)$ such that $b_1 = h(e_1),\dots,b_n = h(e_n)$. The space

$$P^{\mathrm{SO}}(M) := \bigsqcup_{x\in M} P_x^{\mathrm{SO}}(M)$$

is a $\mathrm{SO}(r,s)$-principal bundle over M and is called the oriented semi-orthonormal frame bundle of M.

A **spin structure** on M is a $\mathrm{Spin}(r,s)$-principal bundle over M with properties analogous to Definition 2.4.3. An oriented semi-Riemannian manifold M is called **spinnable** if there exists a spin structure on M. A **semi-Riemannian spin manifold** is an oriented spinnable semi-Riemannian manifold with a fixed spin structure. A detailed discussion of existence and uniqueness of spin structures on oriented semi-Riemannian manifolds can be found in Chapter 2 of the book [3] by Helga Baum.

Definition 6.3.1. Let $r > 0$ and $s > 0$ and let

$$\sigma_{r,s}^0 := \sigma_{r,s}|_{\mathrm{Spin}_0(r,s)} : \quad \mathrm{Spin}_0(r,s) \to \mathrm{GL}(\Sigma_{r,s})$$

be the restriction of the spinor representation to $\mathrm{Spin}_0(r,s) \subset \mathrm{Spin}(r,s)$. Let M be a semi-Riemannian spin manifold of dimension n and signature (r,s) with a spin structure $P^{\mathrm{Spin}}(M)$. The **spinor bundle** of M for the spin structure $P^{\mathrm{Spin}}(M)$ is the associated vector bundle

$$\Sigma M := P^{\mathrm{Spin}}(M) \times_{\sigma_{r,s}^0} \Sigma_{r,s}.$$

Sections of ΣM are called **spinor fields** on M. If $r+s$ is even and $\sigma_{r,s}^{0,\pm} := \sigma_{r,s}^{\pm}|_{\mathrm{Spin}_0(r,s)}$ then the vector bundles

$$\Sigma^{\pm} M := P^{\mathrm{Spin}}(M) \times_{\sigma_{r,s}^{0,\pm}} \Sigma_{r,s}^{\pm}$$

are called the **positive** and the **negative spinor bundle** of M respectively.

The spinor bundle ΣM carries a sesquilinear indefinite bundle metric $(\cdot,\cdot)$ of signature $(2^{\lfloor \frac{n}{2}\rfloor -1}, 2^{\lfloor \frac{n}{2}\rfloor -1})$ defined by

$$(\llbracket H,\varphi\rrbracket, \llbracket H,\psi\rrbracket) := (\varphi,\psi), \quad \text{for } H \in P^{\mathrm{Spin}}(M),\ \varphi,\psi \in \Sigma_{r,s}.$$

This assignment is well-defined, since for any $a \in \mathrm{Spin}_0(r,s)$ we have by Lemma 6.2.13

$$([\![H \cdot a, \sigma^0_{r,s}(a^{-1})\varphi]\!], [\![H \cdot a, \sigma^0_{r,s}(a^{-1})\psi]\!]) = (\sigma^0_{r,s}(a^{-1})\varphi, \sigma^0_{r,s}(a^{-1})\psi) = (\varphi, \psi).$$

We define Clifford multiplication on ΣM as in equation (2.17). By Lemma 6.2.13 we have for all $X \in T_pM$ and for all $\varphi, \psi \in \Sigma_p M$, $p \in M$

$$(X \cdot \varphi, \psi) = (-1)^{r-1}(\varphi, X \cdot \psi). \tag{6.17}$$

The spinor connection

The Levi-Civita connection ∇ on TM induces a connection 1-form $\omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{SO}}(M), \mathfrak{so}(r,s))$. By pull-back with $\bar{\varrho}$, we obtain an $\mathfrak{so}(r,s)$-valued 1-form $\bar{\varrho}^*\omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{Spin}}(M), \mathfrak{so}(r,s))$. Applying the isomorphism $\varrho_*^{-1} : \mathfrak{so}(r,s) \to \mathfrak{spin}(r,s)$ yields the connection 1-form

$$\widetilde{\omega}^{\mathrm{LC}} := \varrho_*^{-1}\bar{\varrho}^*\omega^{\mathrm{LC}} \in \Omega^1(P^{\mathrm{Spin}}(M), \mathfrak{spin}(r,s))$$

and a corresponding **spinor connection** ∇^Σ on ΣM. The covariant derivative with respect to ∇^Σ of a local section $[\![H, \varphi]\!] \in C^\infty(U, \Sigma M)$ is given by:

$$\nabla^\Sigma_X [\![H, \varphi]\!] = [\![H, \partial_X \varphi + (\sigma^0_{r,s})_*(\widetilde{\omega}^{\mathrm{LC}}(dH(X))) \cdot \varphi]\!]. \tag{6.18}$$

Here $U \subset M$ is an open subset, $x \in U$, $X \in T_xM$, and $H : U \to P^{\mathrm{Spin}}(M)$ is a local smooth section, and $\varphi : U \to \Sigma_{r,s}$ a smooth function.

In order to write the spinor connection in terms of Christoffel symbols, we fix a local smooth section $H : U \to P^{\mathrm{Spin}}(M)$. Then $h := \bar{\varrho} \circ H : U \to P^{\mathrm{SO}}(M)$ is a smooth local oriented semi-orthonormal tangent frame and the vector fields

$$b_1 := h(e_1), \ldots, b_n := h(e_n)$$

form an oriented semi-orthonormal basis of T_xM at each $x \in U$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. The Christoffel symbols $\Gamma^k_{ij} : U \to \mathbb{R}$ of this orthonormal frame are defined by the equation

$$\nabla^{\mathrm{LC}}_{b_i} b_j = \sum_{k=1}^n \Gamma^k_{ij}\, b_k \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

Note that unlike the Christoffel symbols of a local coordinate system the Γ^k_{ij} are in general not symmetric in i, j. Instead we have $\Gamma^k_{ij} = -\Gamma^j_{ik}$ for all i, j, k. We compute the covariant derivative of $b_j = [\![h, e_j]\!]$ in terms of the connection 1-form ω^{LC}:

$$\begin{aligned}
\left[\!\!\left[h, \sum_{k=1}^n \Gamma^k_{ij}\, e_k\right]\!\!\right] &= \nabla^{\mathrm{LC}}_{b_i} b_j \\
&= \nabla^{\mathrm{LC}}_{b_i} [\![h, e_j]\!] \\
&= [\![h, \underbrace{\partial_{b_i} e_j}_{=0} + \lambda_*(\omega^{\mathrm{LC}}(dh(b_i)))e_j]\!] \\
&= \left[\!\!\left[h, \sum_{k=1}^n \omega^{\mathrm{LC}}_{kj}(dh(b_i)) e_k\right]\!\!\right].
\end{aligned}$$

Hence

$$\Gamma_{ij}^{k} = \omega_{kj}^{\mathrm{LC}}(dh(b_i)). \tag{6.19}$$

For the local section $H : U \to P^{\mathrm{Spin}}(M)$ with $\bar{\varrho} \circ H = h$, we then have:

$$\bar{\varrho}^{*}\omega^{\mathrm{LC}}\big(dH(b_i)\big) = \omega^{\mathrm{LC}}\big(d\bar{\varrho} \circ dH(b_i)\big) = \omega^{\mathrm{LC}}\big(d(\bar{\varrho} \circ H)(b_i)\big) = \omega^{\mathrm{LC}}\big(dh(b_i)\big).$$

Upon writing

$$\varrho_{*}^{-1}\big(\bar{\varrho}^{*}\omega^{\mathrm{LC}}(dH(b_i))\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, e_\mu \cdot e_\nu \ \in \mathfrak{spin}(r,s), \tag{6.20}$$

we obtain

$$\omega^{\mathrm{LC}}\big(dh(b_i)\big) = \bar{\varrho}^{*}\omega^{\mathrm{LC}}\big(dH(b_i)\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, \varrho_{*}(e_\mu \cdot e_\nu).$$

We apply this to $e_j \in \mathbb{R}^n$ and obtain

$$\begin{aligned}
\omega^{\mathrm{LC}}\big(dh(b_i)\big)(e_j) &= \sum_{\mu<\nu} \gamma_{\mu\nu i}\, \varrho_{*}(e_\mu \cdot e_\nu)(e_j) \\
&= \sum_{\mu<\nu} \gamma_{\mu\nu i} \begin{cases} 2\,\varepsilon_\mu\, e_\nu, & j=\mu \\ -2\,\varepsilon_\nu\, e_\mu, & j=\nu \\ 0 & \text{otherwise} \end{cases} \\
&= 2\sum_{\nu>j} \gamma_{j\nu i}\, \varepsilon_j\, e_\nu - 2\sum_{\mu<j} \gamma_{\mu j i}\, \varepsilon_j\, e_\mu .
\end{aligned}$$

Comparing the coefficients with equation (6.19) yields

$$\Gamma_{ij}^{k} = \begin{cases} 2\,\varepsilon_j\, \gamma_{jki} & k>j \\ -2\,\varepsilon_j\, \gamma_{kji} & k<j \\ 0 & k=j \end{cases} .$$

Thus, we can replace the coefficients in (6.20) by Christoffel symbols and obtain:

$$\widetilde{\omega}^{\mathrm{LC}}\big(dH(b_i)\big) = \sum_{\mu<\nu} \gamma_{\mu\nu i}\, e_\mu \cdot e_\nu = \frac{1}{2} \sum_{\mu<\nu} \varepsilon_\mu \Gamma_{i\mu}^{\nu}\, e_\mu \cdot e_\nu$$

Thus, the covariant derivative of a local section $[\![H,\varphi]\!] \in C^\infty(U,\Sigma M)$ can be written in terms of Christoffel symbols:

$$\begin{aligned}
\nabla_{b_i}^{\Sigma}[\![H,\varphi]\!] &= \big[\!\big[H, \partial_{b_i}\varphi + (\sigma_{r,s}^{0})_{*}\big(\widetilde{\omega}^{\mathrm{LC}}(dH(b_i))\big)\cdot\varphi\big]\!\big] \\
&= \Big[\!\Big[H, \partial_{b_i}\varphi + \frac{1}{2}\sum_{\substack{j,k=1 \\ j<k}}^{n} \varepsilon_j\, \Gamma_{ij}^{k}\, e_j \cdot e_k \cdot \varphi\Big]\!\Big].
\end{aligned} \tag{6.21}$$

and the spinor connection ∇^{Σ} on ΣM as in equation (6.18).

Remark 6.3.2. 1. The spinor connection is a metric connection on ΣM with respect to the indefinite bundle metric $(\cdot,\cdot)$. This follows from the general principle explained in Remark 2.4.11 using that $\sigma^0_{r,s}$ is a unitary representation of $\mathrm{Spin}_0(r,s)$ by Lemma 6.2.13.

2. On an even dimensional semi-Riemannian spin manifold M the spinor connection preserves chirality: for every vector field X on M and every spinor field $\phi \in C^\infty(M,\Sigma^\pm M)$ we have $\nabla^\Sigma_X\phi \in C^\infty(M,\Sigma^\pm M)$. This follows immediately from equation (6.21).

Now we prove a Leibniz rule for the Clifford multiplication:

Lemma 6.3.3. *Let M be a semi-Riemannian spin manifold with spinor bundle ΣM and spinor connection ∇^Σ. Then for all vector fields $X, Y \in C^\infty(M,TM)$ and all spinor fields $\phi \in C^\infty(M,\Sigma M)$ we have*

$$\nabla^\Sigma_X(Y\cdot\phi) = (\nabla^{\mathrm{LC}}_X Y)\cdot\phi + Y\cdot\nabla^\Sigma_X\phi. \tag{6.22}$$

Proof. Fix $x \in M$ and let U be a neighborhood of x. Let $H : U \to P^{\mathrm{Spin}}(M)$ be a local section and $h = \bar\varrho \circ H : U \to P^{\mathrm{SO}}(M)$ be the corresponding local section of $P^{\mathrm{SO}}(M)$. Then the vector fields $b_1 := h(e_1), \ldots, b_n := h(e_n)$ form an oriented semi-orthonormal local frame of TM.
Since the spinor connection is tensorial in the vector fields, it suffices to prove the assertion for $X = b_i$. We thus write $Y = [\![h, Y']\!]$ and $\phi = [\![H,\varphi]\!]$ on U, where $Y' : U \to \mathbb{R}^n$ and $\varphi : U \to \Sigma_{r,s}$. Now we compute:

$$\begin{aligned}
\nabla^\Sigma_{b_i}(Y\cdot\phi) &= \nabla^\Sigma_{b_i}[\![H, Y'\cdot\varphi]\!] \\
&\overset{(6.21)}{=} \Bigg[\!\!\Bigg[H, \partial_{b_i}(Y'\cdot\varphi) + \frac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^n \varepsilon_j\Gamma^k_{ij}\, e_j\cdot e_k\cdot Y'\cdot\varphi\Bigg]\!\!\Bigg] \\
&= \Bigg[\!\!\Bigg[H, (\partial_{b_i}Y')\cdot\varphi + Y'\cdot\partial_{b_i}\varphi - \frac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^n \varepsilon_j\Gamma^k_{ij}\, e_j\cdot Y'\cdot e_k\cdot\varphi \\
&\qquad - \sum_{\substack{j,k=1\\ j<k}}^n \varepsilon_j\Gamma^k_{ij}\, e_j\langle e_k, Y'\rangle\cdot\varphi\Bigg]\!\!\Bigg] \\
&= \Bigg[\!\!\Bigg[H, (\partial_{b_i}Y')\cdot\varphi + Y'\cdot\partial_{b_i}\varphi + \frac{1}{2}\sum_{\substack{j,k=1\\ j<k}}^n \varepsilon_j\Gamma^k_{ij}\, Y'\cdot e_j\cdot e_k\cdot\varphi
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{j,k=1\\ j<k}}^{n} \varepsilon_j \Gamma_{ij}^k \langle e_j, Y' \rangle e_k \cdot \varphi - \sum_{\substack{j,k=1\\ j<k}}^{n} \varepsilon_j \Gamma_{ij}^k \, e_j \langle e_k, Y' \rangle \cdot \varphi \Big]\!\Big] \\
= &\Big[\!\Big[H, Y' \cdot \Big(\partial_{b_i} \varphi + \frac{1}{2} \sum_{\substack{j,k=1\\ j<k}}^{n} \varepsilon_j \Gamma_{ij}^k \, e_j \cdot e_k \cdot \varphi \Big) \Big]\!\Big] \\
&+ \Big[\!\Big[H, \Big(\partial_{b_i} Y' + \sum_{j,k=1}^{n} \langle Y', e_j \rangle \varepsilon_j \Gamma_{ij}^k \, e_k \Big) \cdot \varphi \Big]\!\Big] \\
= &\, Y \cdot \nabla_{b_i}^{\Sigma} \phi + \nabla_{b_i}^{\mathrm{LC}} Y \cdot \phi.
\end{aligned}
$$
□

6.4. The classical Dirac operator on spinors

Let M be an n-dimensional semi-Riemannian spin manifold. We have the spinor connection

$$
\nabla^{\Sigma}: \quad C^{\infty}(M, \Sigma M) \to C^{\infty}(M, T^*M \otimes \Sigma M), \quad \psi \mapsto \sum_{i=1}^{n} v_i^* \otimes \nabla_{v_i}^{\Sigma} \psi,
$$

where $v_1, \dots, v_n$ is a local frame of TM and $v_1^*, \dots, v_n^*$ is the dual frame, i.e., v_i^* are local sections of T^*M such that $v_i^*(v_j) = \delta_{ij}$ for all i, j. We define

$$
c: \quad C^{\infty}(M, T^*M \otimes \Sigma M) \to C^{\infty}(M, \Sigma M), \quad \alpha \otimes \psi \mapsto \alpha^{\#} \cdot \psi,
$$

where $\alpha^{\#} \in C^{\infty}(M, TM)$ is given by $\alpha(X) = g(\alpha^{\#}, X)$ for all $X \in C^{\infty}(M, TM)$ and g is the semi-Riemannian metric on TM.

Definition 6.4.1. The Dirac operator is defined as the composition $D := c \circ \nabla^{\Sigma}$.

Remark 6.4.2. Let $b_1, \dots, b_n$ be a local g-semi-orthonormal frame of TM, i.e., $g(b_i, b_j) = \varepsilon_i \delta_{ij}$ for all i, j with $\varepsilon_i \in \{\pm 1\}$. Then we have $b_i^* = \varepsilon_i g(b_i, \cdot)$ and thus $(b_i^*)^{\#} = \varepsilon_i b_i$. Thus we have for all $\varphi \in C^{\infty}(M, \Sigma M)$:

$$
D\varphi = \sum_{i=1}^{n} \varepsilon_i b_i \cdot \nabla_{b_i}^{\Sigma} \varphi.
$$

Remark 6.4.3. If $\dim M = n$ is even then with respect to the splitting $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ the Dirac operator takes the form

$$
D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}
$$

where $D^+ \in \mathscr{Diff}_1(\Sigma^+ M, \Sigma^- M)$ and $D^- \in \mathscr{Diff}_1(\Sigma^- M, \Sigma^+ M)$. The Dirac operator D interchanges chirality, since Clifford multiplication by b_i does.

Proposition 6.4.4. *Assume $r > 0$ and $s > 0$. Then for all spinor fields $\varphi, \psi \in C^\infty(M, \Sigma M)$ with compact support we have*

$$\int_M (D\varphi, \psi)\, dvol_g = (-1)^r \int_M (\varphi, D\psi)\, dvol_g,$$

where $(\cdot,\cdot)$ denotes the indefinite inner product on ΣM.

Proof. Let $\varphi, \psi \in C^\infty(M, \Sigma M)$ and let $X \in C^\infty(M, TM \otimes_{\mathbb{R}} \mathbb{C})$ be the unique complex vector field such that

$$(Y \cdot \varphi, \psi) = g(X, Y) \quad \text{for all } Y \in C^\infty(M, TM).$$

Let $b_1, \ldots, b_n$ be a local semi-orthonormal frame of TM. Using that ∇^Σ is a metric connection with respect to $(\cdot,\cdot)$ and using the equations (6.17) and (6.22) we get

$$\begin{aligned}
\operatorname{div}(X) &= \sum_{i=1}^n \varepsilon_i g(\nabla^{\mathrm{LC}}_{b_i} X, b_i) = \sum_{i=1}^n \varepsilon_i \partial_{b_i} g(X, b_i) - \sum_{i=1}^n \varepsilon_i g(X, \nabla^{\mathrm{LC}}_{b_i} b_i)\\
&= \sum_{i=1}^n \varepsilon_i \partial_{b_i} (b_i \cdot \varphi, \psi) - \sum_{i=1}^n \varepsilon_i ((\nabla^{\mathrm{LC}}_{b_i} b_i) \cdot \varphi, \psi)\\
&= \sum_{i=1}^n \varepsilon_i (\nabla^\Sigma_{b_i}(b_i \cdot \varphi), \psi) + \sum_{i=1}^n \varepsilon_i (b_i \cdot \varphi, \nabla^\Sigma_{b_i} \psi) - \sum_{i=1}^n \varepsilon_i ((\nabla^{\mathrm{LC}}_{b_i} b_i) \cdot \varphi, \psi)\\
&= \sum_{i=1}^n \varepsilon_i ((\nabla^{\mathrm{LC}}_{b_i} b_i) \cdot \varphi + b_i \cdot \nabla^\Sigma_{b_i} \varphi, \psi)\\
&\qquad + (-1)^{r-1} \sum_{i=1}^n \varepsilon_i (\varphi, b_i \cdot \nabla^\Sigma_{b_i} \psi) - \sum_{i=1}^n \varepsilon_i ((\nabla^{\mathrm{LC}}_{b_i} b_i) \cdot \varphi, \psi)\\
&= (D\varphi, \psi) + (-1)^{r-1} (\varphi, D\psi).
\end{aligned}$$

We integrate over M, and by the divergence theorem we obtain

$$\int_M (D\varphi, \psi)\, dvol_g + (-1)^{r-1} \int_M (\varphi, D\psi)\, dvol_g = \int_M \operatorname{div}(X)\, dvol_g = 0. \qquad \square$$

6.5. Spacelike hypersurfaces of Lorentzian manifolds

Let $\mathbb{R}^{n+1}$ be equipped with the inner product $\langle \cdot, \cdot \rangle_{1,n}$. Denote the standard basis of $\mathbb{R}^{n+1}$ by $e_0, \ldots, e_n$ where $\langle e_0, e_0 \rangle_{1,n} = -1$ and $\langle e_i, e_i \rangle_{1,n} = 1$ for $1 \leq i \leq n$. It is easy to see that the complexified Clifford algebra $\mathbb{C}\mathrm{l}_n := \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the Clifford algebra of $(\mathbb{C}^n, \beta)$ with the symmetric bilinear form $\beta(x,y) = \sum_{j=1}^n x_j y_j$ on $\mathbb{C}^n$. The map

$$j: \quad \mathbb{C}^n \to \mathbb{C}\mathrm{l}^0_{1,n}, \quad j(X) := ie_0 \cdot X$$

satisfies $j(X)\cdot j(X) = -\beta(X,X)$ for all $X \in \mathbb{C}^n$ and thus induces an algebra homomorphism J: $\mathbb{C}\mathrm{l}_n \cong \mathbb{C}\mathrm{l}^0_{1,n}$. It is easy to see that J is an isomorphism of algebras.
Since $\mathrm{Spin}(n) \subset \mathbb{C}\mathrm{l}_n$ and $\mathrm{Spin}(1,n) \subset \mathbb{C}\mathrm{l}^0_{1,n}$, we obtain a map

$$\begin{array}{rcl} \mathrm{Spin}(n) & \hookrightarrow & \mathrm{Spin}(1,n) \\ a = v_1 \cdot v_2 \cdot \ldots \cdot v_{2m} & \mapsto & i\,e_0 \cdot v_1 \cdot \ldots \cdot i\,e_0 \cdot v_{2m} = v_1 \cdot \ldots \cdot v_{2m}. \end{array}$$

With this embedding we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Spin}(n) \arrow[r, hook] \arrow[d, "\varrho"'] & \mathrm{Spin}(1,n) \arrow[d, "\varrho"'] \\ \mathrm{SO}(n) \arrow[r, hook] & \mathrm{SO}(1,n) \end{tikzcd} .$$

Let M be a semi-Riemannian spin manifold of dimension $n+1$ with a metric g of signature $(1,n)$, i.e., M is a Lorentzian spin manifold. Let $N \subset M$ be an orientable hypersurface such that the restriction of g to TN is a Riemannian metric on TN, i.e., $N \subset M$ is a spacelike orientable hypersurface. We want to construct a spin structure on N and relate the spinor bundles ΣM and ΣN and the Dirac operators D^M and D^N.

Let ν be a normal vector field along N such that $g(\nu,\nu) = -1$ on N and equip N with the orientation such that a basis $b_1, \ldots, b_n$ of T_xN is positively oriented if and only if the basis $\nu, b_1, \ldots, b_n$ of T_xM is positively oriented. Using the canonical embedding

$$\begin{array}{rcl} \mathrm{SO}(n) & \hookrightarrow & \mathrm{SO}(1,n) \\ A & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \end{array}$$

the action of $\mathrm{SO}(n)$ on $(\nu(x), b_1, \ldots, b_n)$ preserves the normal $\nu(x)$.
Moreover, we have a canonical embedding of frame bundles

$$\begin{gathered} P^{\mathrm{SO}}(N) \hookrightarrow P^{\mathrm{SO}}(M), \\ \big(h : \mathbb{R}^n \to T_pN\big) \mapsto \big(h' : \mathbb{R}^{n+1} \to T_pM\big), \end{gathered}$$

where $h'(0, x_1, \ldots, x_n) = h(x_1, \ldots, x_n)$ and $h'(1,0,\ldots,0) := \nu(p)$. This embedding is compatible with the embedding $\mathrm{SO}(n) \hookrightarrow \mathrm{SO}(1,n)$ defined above. Thus, the diagram

$$\begin{tikzcd} P^{\mathrm{SO}}(N) \times \mathrm{SO}(n) \arrow[r] \arrow[d, hook] & P^{\mathrm{SO}}(N) \arrow[d, hook] \\ P^{\mathrm{SO}}(M) \times \mathrm{SO}(1,n) \arrow[r] & P^{\mathrm{SO}}(M) \end{tikzcd} .$$

commutes.

Now let $\bar{\varrho}$: $P^{\mathrm{Spin}}(M) \to P^{\mathrm{SO}}(M)$ be a spin structure on M. We set

$$P^{\mathrm{Spin}}(N) := \bar{\varrho}^{-1}\big(P^{\mathrm{SO}}(N)\big)$$

This defines a spin structure on N:

- The action of $\mathrm{Spin}(1,n)$ on $P^{\mathrm{Spin}}(M)$ restricts to an action of $\mathrm{Spin}(n)$ on $P^{\mathrm{Spin}}(N)$: For $H \in P^{\mathrm{Spin}}(N)$ and $a \in \mathrm{Spin}(n)$, we have $H \cdot a \in P^{\mathrm{Spin}}(M)$ and

$$\bar{\varrho}(H \cdot a) = \underbrace{\bar{\varrho}(H)}_{\in P^{\mathrm{SO}}(N)} \cdot \underbrace{\varrho(a)}_{\in \mathrm{SO}(n)} \in P^{\mathrm{SO}}(N).$$

  Thus $H \cdot a \in P^{\mathrm{Spin}}(N)$.

- Obviously, the action of $\mathrm{Spin}(n)$ on $P^{\mathrm{Spin}}(N)$ is compatible with the action of $\mathrm{SO}(n)$ on $P^{\mathrm{SO}}(N)$, hence $\bar{\varrho} : P^{\mathrm{Spin}}(N) \to P^{\mathrm{SO}}(N)$ is a spin structure on N.

In particular, orientable spacelike hypersurfaces of spinnable Lorentzian manifolds are again spinnable.

Spinor bundles

We study how the spinor bundles of N and M are related to one another.

Case 1: $\mathbf{n+1}$ is even

In this case, $\Sigma_n = \Sigma^+_{1,n}$. For any $x \in N$, we have[1]

$$\Sigma_x N = P^{\mathrm{Spin}}_x(N) \times_{\sigma_n} \Sigma_n = P^{\mathrm{Spin}}_x(N) \times_{\sigma^{0,+}_{1,n}|_{\mathrm{Spin}(n)}} \Sigma^+_{1,n} \cong P^{\mathrm{Spin}}_x(M) \times_{\sigma^{0,+}_{1,n}} \Sigma^+_{1,n}$$

Thus, $\Sigma N = \Sigma^+ M|_N$.
The Clifford multiplication of $\mathbb{R}^n$ on $\Sigma_n = \Sigma^+_{1,n}$ is given by

$$X \cdot \varphi = i\, e_0 \cdot X \cdot \varphi,$$

where the $\cdot$ on the left hand side is the Clifford multiplication in Cl_n, while the $\cdot$ on the right hand side is the Clifford multiplication in $\mathrm{Cl}_{1,n}$. Thus, the Clifford multiplication in ΣN is given by

$$X \cdot \varphi = i\, \nu \cdot X \cdot \varphi,$$

[1] Let $X \subset Y$ be sets and $H \subset G$ be groups. Let G act simply transitively from the right on Y such that the action restricts to a simply transitive right action of H on X. Then for any representation of G on Σ, the inclusions induce a bijection $X \times_H \Sigma \cong Y \times_G \Sigma$.

where $X \in T_xN$ and $\varphi \in \Sigma_x N$.

Case 2: $\mathbf{n+1}$ is odd

The inclusion of Clifford algebras

$$\mathbb{C}\mathrm{l}_n \xrightarrow{\cong} \mathbb{C}\mathrm{l}^0_{1,n} \hookrightarrow \mathbb{C}\mathrm{l}_{1,n} \xrightarrow{\cong} \mathbb{C}\mathrm{l}^0_{1,n+1} \hookrightarrow \mathbb{C}\mathrm{l}_{1,n+1}$$

together with the inclusions $\Sigma_n \subset \mathbb{C}\mathrm{l}_n$ and $\Sigma_{1,n} \cong \Sigma^+_{1,n+1} \subset \mathbb{C}\mathrm{l}_{1,n+1}$ induces an isomorphism $\Xi_n : \Sigma_n \to \Sigma_{1,n}$ such that the diagram

$$\begin{tikzcd} \Sigma_n \arrow[r, "\Xi_n"] \arrow[d, "X\cdot()"'] & \Sigma_{1,n} = \Sigma^+_{1,n+1} \arrow[d, "i\,e_0\cdot X\cdot() = i\,e_0\cdot X\cdot e_{n+1}\cdot()"] \\ \Sigma_n \arrow[r, "\Xi_n"] & \Sigma_{1,n} = \Sigma^+_{1,n+1} \end{tikzcd}$$

of Clifford multiplications with $X \in \mathbb{R}^n$ commutes.
As in case 1 we obtain the canonical isomorphism $\Sigma N \cong \Sigma M|_N$ such that again

$$X \cdot \varphi = i\,\nu \cdot X \cdot \varphi.$$

for $X \in T_xN$, $\varphi \in \Sigma_x N$.
In the following we treat both cases simultaneously using the notation

$$\Sigma^{(+)}M := \begin{cases} \Sigma^+ M & \text{if } n+1 \text{ is even,} \\ \Sigma M & \text{if } n+1 \text{ is odd.} \end{cases}$$

Spinor connections

The Levi-Civita connections on TM and TN are related by the **Gauß equation**

$$\underbrace{\nabla^M_X Y}_{\in T_xM} = \underbrace{\nabla^N_X Y}_{\in T_xN} + \underbrace{\mathrm{II}(X,Y)}_{\in (T_xN)^\perp}, \tag{6.23}$$

where $X \in T_xN$ and $Y \in C^\infty(N, TN)$. The second fundamental form is a symmetric bilinear map $\mathrm{II} : T_xN \times T_xN \to (T_xN)^\perp$, given by the orthogonal projection of $\nabla^M_X Y$ to $(T_xN)^\perp$. The Weingarten map is the corresponding symmetric endomorphism $B : T_xN \to T_xN$ such that for all $X, Y \in T_xN$

$$\mathrm{II}(X,Y) = g(B(X),Y)\,\nu =: \langle B(X), Y\rangle\,\nu\,.$$

The mean curvature field $\mathcal{H} \in C^\infty(N, TN^\perp)$ is defined by

$$\mathcal{H} = \frac{1}{n}\sum_{i=1}^n \langle B(b_i), b_i\rangle\,\nu = \frac{1}{n}\,\mathrm{tr}(B)\,\nu = H\,\nu,$$

where $b_1, \ldots, b_n$ is a local orthonormal tangent frame for N and $H : N \to \mathbb{R}$ is the mean curvature of the hypersurface $N \subset M$.

The spinor connections of M and N are related by the Weingarten map. Let $(b_1, \ldots, b_n)$ be a local oriented orthonormal tangent frame for N. Then $(b_0 = \nu, b_1, \ldots, b_n)$ is a local orthonormal tangent frame for M along N. The Christoffel symbols for the Levi-Civita connections ∇^M and ∇^N are defined by

$$\nabla^M_{b_i} b_j = \sum_{k=0}^{n} {}^M\Gamma^k_{ij}\, b_k \qquad \text{and} \qquad \nabla^N_{b_i} b_j = \sum_{k=1}^{n} {}^N\Gamma^k_{ij}\, b_k.$$

By the Gauß equation (6.23), we have for $i, j \in \{1, \ldots, n\}$:

$$\nabla^M_{b_i} b_j = \nabla^N_{b_i} b_j + \big\langle B(b_i), b_j \big\rangle\, \nu = \sum_{k=1}^{n} {}^N\Gamma^k_{ij}\, b_k + \big\langle B(b_i), b_j \big\rangle\, b_0.$$

Comparing coefficients yields

$$\begin{aligned} {}^M\Gamma^k_{ij} &= {}^N\Gamma^k_{ij} && \forall\, i, j, k = \{1, \ldots, n\}, \\ {}^M\Gamma^0_{ij} &= -\, {}^M\Gamma^j_{i,0} = \big\langle B(b_i), b_j \big\rangle && \forall\, i, j = \{1, \ldots, n\}. \end{aligned}$$

For the covariant derivative of a section of $\Sigma^{(+)}M$, we compute for $i \in \{1, \ldots, n\}$:

$$\begin{aligned} {}^M\nabla^\Sigma_{b_i} [\![H, \varphi]\!] &\overset{(6.21)}{=} \Big[\!\!\Big[H, \partial_{b_i}\varphi + \frac{1}{2} \sum_{\substack{j,k=0 \\ j<k}}^{n} \varepsilon_j\, {}^M\Gamma^k_{ij}\, e_j \cdot e_k \cdot \varphi \Big]\!\!\Big] \\ &= \Big[\!\!\Big[H, \partial_{b_i}\varphi + \frac{1}{2} \sum_{\substack{j,k=1 \\ j<k}}^{n} {}^N\Gamma^k_{ij}\, e_j \cdot e_k \cdot \varphi + \frac{1}{2} \sum_{k=1}^{n} \big\langle B(b_i), b_k \big\rangle\, e_0 \cdot e_k \cdot \varphi \Big]\!\!\Big] \\ &= {}^N\nabla^\Sigma_{b_i} [\![H, \varphi]\!] + \frac{1}{2} \sum_{j=1}^{n} \big\langle B(b_i), b_j \big\rangle\, b_0 \cdot b_j \cdot [\![H, \varphi]\!] \\ &= {}^N\nabla^\Sigma_{b_i} [\![H, \varphi]\!] + \frac{1}{2}\, \nu \cdot B(b_i) \cdot [\![H, \varphi]\!]. \end{aligned}$$

Hence for all $\phi \in C^\infty(M, \Sigma^{(+)}M)$ and for all $X \in TN$, we have along N:

$${}^M\nabla^\Sigma_X \phi = {}^N\nabla^\Sigma_X \phi + \frac{1}{2}\, \nu \cdot B(X) \cdot \phi.$$

Dirac operators

For a spinor field $\phi \in C^\infty(M, \Sigma^{(+)}M)$ we have along the hypersurface N:

$$\begin{aligned}
D^M\phi &= -\nu \cdot {}^M\nabla^\Sigma_\nu \phi + \sum_{j=1}^n b_j \cdot {}^M\nabla^\Sigma_{b_j}\phi \\
&= -\nu \cdot {}^M\nabla^\Sigma_\nu \phi + \sum_{j=1}^n b_j \cdot \Big({}^N\nabla^\Sigma_{b_j}\phi + \frac{1}{2}\,\nu \cdot B(b_j)\cdot\phi\Big) \\
&= -\nu \cdot {}^M\nabla^\Sigma_\nu \phi - i\nu \cdot \sum_{j=1}^n i\nu \cdot b_j \cdot {}^N\nabla^\Sigma_{b_j}\phi - \frac{1}{2}\sum_{j=1}^n \nu \cdot b_j \cdot B(b_j)\cdot\phi \\
&= -\nu \cdot \Big({}^M\nabla^\Sigma_\nu \phi + iD^N\phi + \frac{1}{2}\sum_{j=1}^n b_j \cdot B(b_j)\cdot\phi\Big).
\end{aligned}$$

Since B is a symmetric endomorphism, we may choose $b_1, \ldots, b_n$ as an eigenbasis at $x \in M$, thus $B(b_j) = \kappa_j \cdot b_j$ for $j = 1, \ldots, n$. Then we have

$$\begin{aligned}
D^M\phi &= -\nu \cdot \Big({}^M\nabla^\Sigma_\nu \phi + iD^N\phi - \frac{1}{2}\mathrm{tr}(B)\phi\Big) \\
&= -\nu \cdot \Big(iD^N\phi - \frac{n}{2}\,H\phi + {}^M\nabla^\Sigma_\nu\phi\Big)
\end{aligned}$$

Hence

$$-\nu \cdot D^M\phi = iD^N\phi - \frac{n}{2}\,H\phi + {}^M\nabla^\Sigma_\nu\phi.$$

A. Existence of Friedrichs mollifiers

Lemma A.0.1. *The family of operators $J_\varepsilon : C^\infty(T^n) \to C^\infty(T^n)$ defined in Example 1.4.11 is a Friedrichs mollifier on T^n for the trivial line bundle $E = T^n \times \mathbb{C}$.*

Proof. For ε small enough the support of j_ε is contained in $[-\pi, \pi]^n$ and we extend j_ε to a $2\pi\mathbb{Z}^n$-periodic smooth function $\mathbb{R}^n \to \mathbb{R}$ again denoted by j_ε. Then we have for all $u \in C^\infty(T^n)$:

$$(J_\varepsilon u)(x) = \int_{T^n} j_\varepsilon(x-y)\, u(y)\, dy = \int_{[0,2\pi]^n} j_\varepsilon(x-y)\, u(y)\, dy.$$

We show that J_ε satisfies the properties i)-iv) of Definition 1.4.10.

i) J_ε is a smoothing operator, since j_ε is a smooth function. Obviously, J_ε is self-adjoint.

ii) Let $u \in C^\infty(T^n)$. The function

$$([0,2\pi]^n)^3 \to \mathbb{R}, \quad (x,y,z) \mapsto j_\varepsilon(y-x) j_\varepsilon(y-z)|u(x)||u(z)|$$

is integrable and thus we may use Fubini's theorem to get

$$\begin{aligned}
\|J_\varepsilon u\|_{L^2}^2 &= \int_{[0,2\pi]^n} \int_{[0,2\pi]^n} j_\varepsilon(y-x)\, u(x)\, dx\, \overline{\int_{[0,2\pi]^n} j_\varepsilon(y-z)\, u(z)\, dz}\, dy \\
&\le \int_{[0,2\pi]^n} \int_{[0,2\pi]^n} \int_{[0,2\pi]^n} j_\varepsilon(y-x)\, j_\varepsilon(y-z) \underbrace{|u(x)|\,|u(z)|}_{\le \frac{1}{2}(|u(x)|^2 + |u(z)|^2)} dx\, dy\, dz \\
&\le \frac{1}{2} \int_{[0,2\pi]^n} \int_{[0,2\pi]^n} |u(x)|^2 j_\varepsilon(y-x) \underbrace{\int_{[0,2\pi]^n} j_\varepsilon(y-z)\, dz}_{=1}\, dx\, dy \\
&\quad + \frac{1}{2} \int_{[0,2\pi]^n} \int_{[0,2\pi]^n} |u(z)|^2 j_\varepsilon(y-z) \underbrace{\int_{[0,2\pi]^n} j_\varepsilon(y-x)\, dx}_{=1}\, dz\, dy \\
&= \int_{[0,2\pi]^n} |u(x)|^2 \underbrace{\int_{[0,2\pi]^n} j_\varepsilon(y-x)\, dy}_{=1}\, dx \\
&= \|u\|_{L^2}^2.
\end{aligned}$$

Therefore we obtain a bounded linear extension $J_\varepsilon\colon L^2(M,E) \to L^2(M,E)$ and we have $\|J_\varepsilon\|_{L^2,L^2} \le 1$ for all ε.

iii) Let $k \in \mathbb{N}$ and $P \in \mathscr{D}\!\mathit{iff}_k(E,E)$. We write

$$P = \sum_{|\alpha|\le k} A_\alpha(x)\partial^\alpha$$

with smooth functions $A_\alpha \in C^\infty(T^n)$ and multi-indices $\alpha \in \mathbb{N}_0^n$ and $\partial^\alpha = \partial_{y_1}^{\alpha_1}\dots\partial_{y_n}^{\alpha_n}$. Now, for $\alpha = 0$ and for every $u \in C^\infty(T^n)$ we get using property ii) of J_ε

$$\begin{aligned}
\|A_0 J_\varepsilon u\|_{L^2} &\le \sup_{x\in T^n} |A_0(x)| \|J_\varepsilon u\|_{L^2} \le C_1 \sup_{x\in T^n} |A_0(x)| \|u\|_{L^2},\\
\|J_\varepsilon A_0 u\|_{L^2} &\le C_1 \|A_0 u\|_{L^2} \le C_1 \sup_{x\in T^n} |A_0(x)| \|u\|_{L^2}
\end{aligned}$$

and thus

$$\|[A_0, J_\varepsilon]u\|_{L^2} \le C_2 \|u\|_{L^2} \le C_2 \|u\|_{H^{k-1}},$$

where $C_1, C_2 > 0$ are independent of ε. Thus it is sufficient to consider

$$P = A_\alpha(x)\partial^\alpha$$

with a smooth function $A_\alpha \in C^\infty(T^n)$ and a multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \ge 1$ and $\partial^\alpha = \partial_{y_1}^{\alpha_1}\dots\partial_{y_n}^{\alpha_n}$. We choose i with $\alpha_i \ge 1$ and we write

$$\bar\alpha := (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_n).$$

Then we have $\partial^\alpha = \partial^{\bar\alpha}\partial_{y_i}$. For all $u \in C^\infty(T^n)$ we get using integration by parts

$$\begin{aligned}
PJ_\varepsilon u(x) &= A_\alpha(x) \int_{T^n} (\partial_x^\alpha j_\varepsilon(x-y)) u(y)\,dy\\
&= A_\alpha(x)(-1)^{|\alpha|} \int_{T^n} (\partial_y^\alpha j_\varepsilon(x-y)) u(y)\,dy\\
&= A_\alpha(x)(-1) \int_{T^n} (\partial_{y_i} j_\varepsilon(x-y))(\partial^{\bar\alpha} u)(y)\,dy\\
&= A_\alpha(x)\varepsilon^{-n-1} \int_{T^n} (\partial_{y_i} j)\Big(\frac{x-y}{\varepsilon}\Big)(\partial^{\bar\alpha} u)(y)\,dy
\end{aligned}$$

and

$$\begin{aligned}
J_\varepsilon P u(x) &= \int_{T^n} j_\varepsilon(x-y) A_\alpha(y)(\partial^\alpha u)(y)\,dy\\
&= -\int_{T^n} (\partial_{y_i} j_\varepsilon(x-y)) A_\alpha(y)(\partial^{\bar\alpha} u)(y)\,dy\\
&\quad - \int_{T^n} j_\varepsilon(x-y)(\partial_{y_i} A_\alpha)(y)(\partial^{\bar\alpha} u)(y)\,dy\\
&= \varepsilon^{-n-1} \int_{T^n} (\partial_{y_i} j)\Big(\frac{x-y}{\varepsilon}\Big) A_\alpha(y)(\partial^{\bar\alpha} u)(y)\,dy\\
&\quad - \int_{T^n} j_\varepsilon(x-y)(\partial_{y_i} A_\alpha)(y)(\partial^{\bar\alpha} u)(y)\,dy
\end{aligned}$$

and thus

$$
\begin{aligned}
[P, J_\varepsilon]u(x) &= \int_{T^n} j_\varepsilon(x-y)(\partial_{y_i} A_\alpha)(y)(\partial^{\bar\alpha} u)(y)\,dy \\
&\quad + \varepsilon^{-n-1} \int_{T^n} (\partial_{y_i} j)\Big(\frac{x-y}{\varepsilon}\Big)(A_\alpha(x) - A_\alpha(y))(\partial^{\bar\alpha} u)(y)\,dy \\
&= J_\varepsilon((\partial_{y_i} A_\alpha)(\partial^{\bar\alpha} u))(x) \\
&\quad + \underbrace{\varepsilon^{-n-1} \int_{T^n} (\partial_{y_i} j)\Big(\frac{x-y}{\varepsilon}\Big)(A_\alpha(x) - A_\alpha(y))(\partial^{\bar\alpha} u)(y)\,dy}_{=:(M_\varepsilon u)(x)}
\end{aligned}
$$

By property ii) of J_ε and using that $|\bar\alpha| \leq k-1$ we get

$$
\begin{aligned}
\|J_\varepsilon((\partial_{y_i} A_\alpha)(\partial^{\bar\alpha} u))\|_{L^2} &\leq C\|(\partial_{y_i} A_\alpha)(\partial^{\bar\alpha} u)\|_{L^2} \\
&\leq C \sup_{x\in T^n} |\partial_{y_i} A_\alpha(x)|\, \|\partial^{\bar\alpha} u\|_{L^2} \\
&\leq C \sup_{x\in T^n} |\partial_{y_i} A_\alpha(x)|\, \|u\|_{H^{k-1}}.
\end{aligned}
$$

We now estimate $(M_\varepsilon u)(x)$. First we note that there exists $C_1 > 0$ such that $|\partial_{y_i} j| \leq C_1$ on T^n. Moreover, we have $(\partial_{y_i} j)(\frac{x-y}{\varepsilon}) = 0$ if $|x-y| > \varepsilon$. Since A_α is smooth there exists $C_2 > 0$ such that for all $x, y \in T^n$ we have

$$
|A_\alpha(x) - A_\alpha(y)| \leq C_2|x-y|.
$$

It follows that

$$
|(M_\varepsilon u)(x)| \leq \varepsilon^{-n-1} \int_{B_\varepsilon(x)} C_1\, C_2\, |x-y|\, |\partial^{\bar\alpha} u(y)|\,dy \leq \varepsilon^{-n} C_1\, C_2 \int_{B_\varepsilon(x)} |(\partial^{\bar\alpha} u)(y)|\,dy.
$$

Let $\zeta > 0$. Since $\partial^{\bar\alpha} u$ is uniformly continuous we can choose ε so small that for all $x, y \in T^n$ with $|x-y| < \varepsilon$ we have $|(\partial^{\bar\alpha} u)(x) - (\partial^{\bar\alpha} u)(y)| < \zeta$ and thus $|(\partial^{\bar\alpha} u)(y)| \leq |(\partial^{\bar\alpha} u)(x)| + \zeta$. Thus for all $x \in T^n$ we have

$$
|(M_\varepsilon u)(x)| \leq \varepsilon^{-n} C_1\, C_2 \operatorname{vol}(B_\varepsilon(x))(|(\partial^{\bar\alpha} u)(x)| + \zeta) \leq C_3(|(\partial^{\bar\alpha} u)(x)| + \zeta),
$$

where $C_3 > 0$ is independent of ε, since $\operatorname{vol}(B_\varepsilon(x)) \leq C'\varepsilon^n$ for some $C' > 0$. It follows that

$$
|(M_\varepsilon u)(x)|^2 \leq C_3^2(|(\partial^{\bar\alpha} u)(x)| + \zeta)^2 \leq 2C_3^2(|(\partial^{\bar\alpha} u)(x)|^2 + \zeta^2)
$$

and thus

$$
\|M_\varepsilon u\|_{L^2}^2 \leq 2C_3^2(\|\partial^{\bar\alpha} u\|_{L^2}^2 + \zeta^2 \operatorname{vol}(T^n)).
$$

Let $\zeta \to 0$ and obtain

$$
\|M_\varepsilon u\|_{L^2}^2 \leq 2C_3^2 \|\partial^{\bar\alpha} u\|_{L^2}^2 \leq 2C_3^2 \|u\|_{H^{k-1}}^2.
$$

iv) Let $u \in C^\infty(T^n)$. We prove that $J_\varepsilon u \to u$ in $C^0(T^n)$ as $\varepsilon \to 0$. Let $\alpha > 0$. Since u is uniformly continuous we can choose $\varepsilon > 0$ such that for all $x, y \in [0, 2\pi]^n$ with $|x - y| < \varepsilon$ we have $|u(x) - u(y)| < \alpha$. Using that $\int_{[0,2\pi]^n} j_\varepsilon(x - y)\, dy = 1$ we get for all $x \in [0, 2\pi]^n$

$$(J_\varepsilon u)(x) - u(x) = \int_{[0,2\pi]^n} j_\varepsilon(x-y)\,(u(y) - u(x))\, dy = \int_{B_\varepsilon(x)} j_\varepsilon(x-y)\,(u(y)-u(x))\, dy$$

and thus for all $x \in [0, 2\pi]^n$

$$|(J_\varepsilon u)(x) - u(x)| \le \int_{B_\varepsilon(x)} j_\varepsilon(x - y)\, \underbrace{|u(y) - u(x)|}_{<\alpha}\, dy < \alpha \int_{B_\varepsilon(x)} j_\varepsilon(x - y)\, dy = \alpha.$$

It follows that $J_\varepsilon u \to u$ in $C^0(T^n)$ and thus also in $L^2(T^n)$. □

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