GEOMETRIE IN POTSDAM

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Gauge Theory

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Contents

Preface

These are the lecture notes of an introductory course on gauge theory which I taught at Potsdam University in 2009. The aim was to develop the mathematical underpinnings of gauge theory such as bundle theory, characteristic classes etc. and to give applications both in physics (electrodynamics, Yang-Mills fields) as well as in mathematics (theory of 4-manifolds).

To keep the necessary prerequisites of the students at a minimum, there are introductory chapters on Lie groups and on algebraic topology. Basic differential geometric notions such as manifolds are assumed to be known.

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1 Lie groups and Lie algebras

1.1 Lie groups

Definition 1.1.1. A differentiable manifold G which is at the same time a group is called a ***Lie group*** iff the maps

$$\begin{aligned} G \times G \to G, &\qquad (g_1, g_2) \mapsto g_1 \cdot g_2 \\ G \to G, &\qquad g \mapsto g^{-1} \end{aligned}$$

are smooth.

Example 1.1.2
1. $G = \mathbb{R}^n$ with addition is a Lie group.

2. $G = \mathrm{GL}(n;\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \,|\, \det(A) \neq 0\} \subset \mathrm{Mat}(n \times n;\mathbb{R}) = \mathbb{R}^{n^2}$ is an open subset, since $\det : \mathrm{Mat}(n \times n;\mathbb{R}) \to \mathbb{R}$ is continuous. The multiplication map $(A, B) \mapsto A \cdot B$ is smooth, because the matrix coefficients of $A \cdot B$ are polynomials in the matrix coefficients of A and B. The inversion $A \mapsto A^{-1}$ is smooth, because the matrix coefficients of A^{-1} are rational functions of the matrix coefficients of A.

3. $\mathrm{GL}(n;\mathbb{C}) \subset \mathrm{Mat}(n \times n;\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{(2n)^2}$.

Theorem 1.1.3
Let G be a Lie group, let $H \subset G$ be a subgroup (algebraically) and closed as a subset. Then $H \subset G$ is a submanifold and a Lie group in its own right.

Example 1.1.4
1. $H = \mathrm{O}(n) := \{A \in \mathrm{GL}(n;\mathbb{R}) \,|\, A^t \cdot A = \mathbb{1}_n\}$ is called the ***orthogonal group***.

$\mathrm{O}(n)$ is a subgroup: For $A, B \in \mathrm{O}(n)$, we have:

$$(AB)^t \cdot (AB) = B^t A^t AB = B^t \mathbb{1}_n B = B^t B = \mathbb{1}_n\,,$$

hence $AB \in \mathrm{O}(n)$. Similarly, for $A \in \mathrm{O}(n)$, we have $A^{-1} = A^t$, hence $\mathbb{1}_n = A \cdot A^{-1} = (A^{-1})^t \cdot A^{-1}$. Thus $A^{-1} \in \mathrm{O}(n)$.

$\mathrm{O}(n) \subset \mathrm{GL}(n;\mathbb{R})$ is a closed subset, because the map $A \mapsto A^t A$ is continuous, i.e. $A^t \cdot A = \mathbb{1}_n$ is a closed condition.

2. $H = \mathrm{SL}(n;\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \,|\, \det(A) = 1\}$ is called the ***special linear group***.
3. $H = \mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n;\mathbb{R})$ is called the ***special orthogonal group***.
4. $H = \mathrm{U}(n) := \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) \,|\, A^* \cdot A = 1\}$ is called the ***unitary group***. Here $A^* := (\bar{A})^t$.
5. $H = \mathrm{SL}(n;\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) \,|\, \det(A) = 1\}$ is called the ***special linear group***.
6. $H = \mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n;\mathbb{C})$ is called the ***special unitary group***.

Example 1.1.5. Let G, G' be Lie groups. Then $G \times G'$ is a Lie group with the group structure given as follows:

$$\begin{aligned} (g_1, g_1') \cdot (g_2, g_2') &:= (g_1 \cdot g_2, g_1' \cdot g_2') \\ (g, g')^{-1} &:= \left(g^{-1}, (g')^{-1}\right) . \end{aligned}$$

Remark 1.1.6. Hilbert's 5^{th} problem, formulated at the International Congress of Mathematicians in Paris 1900: In the definiton of Lie group, can one replace "smooth" by "continuous"?
The answer (found around 1950's): Yes, replacing "smooth" by "continuous" does not change anything.

Definition 1.1.7. Let G, H be Lie groups. A smooth group homomorphism $\varphi : G \to H$ is called a ***homomorphism of Lie groups***.
A Lie group homomorphism $\varphi : G \to H$ is called an ***isomorphism of Lie groups*** if it is invertible and the inverse is again a Lie group homomorphism. In this case, G and H are called ***isomorphic*** as Lie groups.

Example 1.1.8. For any $A \in G = \mathrm{SO}(2)$, $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ the condition

$$1 = A^t \cdot A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

yields the equations

$$\begin{aligned} a^2 + b^2 &= 1 && (1.1) \\ c^2 + d^2 &= 1 && (1.2) \\ ac + bd &= 0\,. && (1.3) \end{aligned}$$

Further we have the condition

$$1 \stackrel{!}{=} \det(A) = ad - bc\,. \tag{1.4}$$

Multiplying (1.4) by c and d respectively, yields

$$\begin{aligned} c &= acd - bc^2 \stackrel{(1.3)}{=} -b\left(c^2 + d^2\right) \stackrel{(1.2)}{=} -b \\ d &= ad^2 - bcd \stackrel{(1.3)}{=} a\left(c^2 + d^2\right) \stackrel{(1.2)}{=} a\,. \end{aligned}$$

Hence any $A \in \mathrm{SO}(2)$ is of the form $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2 + b^2 = 1$. Thus there is a $\varphi \in \mathbb{R}$ such that $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$. We thus have

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \middle| \varphi \in \mathbb{R} \right\}.$$

For $H = \mathrm{U}(1)$, we find:

$$\begin{aligned} \mathrm{U}(1) &= \{(z) \in \mathrm{GL}(1;\mathbb{C}) \mid \bar{z} \cdot z = 1\} \\ &= \{(z) \mid |z| = 1\} \\ &= \left\{(e^{i\varphi}) \mid \varphi \in \mathbb{R}\right\}. \end{aligned}$$

Now the map

$$\mathrm{SO}(2) \to \mathrm{U}(1)\,, \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \mapsto \left(e^{i\varphi}\right)\,,$$

is an isomorphism of Lie groups: to see that it is a group homomorphism, use the addition theorems for sin and cos, to see that it is invertible, use Eulers formula. Hence $\mathrm{U}(1) \cong \mathrm{SO}(2)$. Both are diffeomorphic to the unit circle S^1.

1.2 Lie algebras

Definition 1.2.1. A vector space V together with a map $[\cdot,\cdot]: V \times V \to V$ is called a ***Lie algebra***, iff

(i) $[\cdot,\cdot]$ is bilinear.

(ii) $[\cdot,\cdot]$ is antisymmetric, i.e. $\forall\, v, w \in V$: $[v,w] = -[v,w]$.

(iii) $[\cdot,\cdot]$ satisfies the ***Jacobi identity***:

$$\forall\, u, v, w \in V : \; [[u,v],w] + [[v,w],u] + [[w,u],v] = 0\,.$$

The map $[\cdot,\cdot]$ is called the ***Lie bracket***.

Example 1.2.2

1. Every vector space together with the map $[\cdot,\cdot] \equiv 0$ is a Lie algebra. A Lie algebra with the trivial bracket $[\cdot,\cdot] \equiv 0$ is called ***abelian***.

2. The space $V = \mathrm{Mat}(n \times n; \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ together with the commutator $[A,B] := A \cdot B - B \cdot A$ is a Lie algebra. The Jacobi identity is given by a simple computation:

$$\begin{aligned} &[[A,B],C] + [[B,C],A] + [[C,A],B] \\ &\quad = \; (AB - BA)C - C(AB - BA) + (BC - CB)A - A(BC - CB) \\ &\quad\quad\; +(CA - AC)B - B(CA - AC) \\ &\quad = \; 0\,. \end{aligned}$$

   This computation shows that the Jacobi identity is a consequence of the associativity of matrix multiplication. In general, the Jacobi identity can be thought of as a replacement for associativity.

3. $V = \mathbb{R}^3$ together with the Lie bracket $[\cdot,\cdot] = (\cdot) \times (\cdot)$ defined as the vector product is a Lie algebra. Again, the verification of the Jacobi identity is a simple computation.

4. Let M be a differentiable manifold, let $V = \mathfrak{X}(M)$ be the space of smooth vector fields on M. Let $[\cdot,\cdot]$ be the usual Lie bracket of vector fields. Then $(V, [\cdot,\cdot])$ is an infinite dimensional Lie algebra.

Definition 1.2.3. Let $(V, [\cdot,\cdot])$ be a Lie algebra. A vector subspace $W \subset V$ together with the map $[\cdot,\cdot]|_{W\times W}$ is called a ***Lie subalgebra*** of V iff $\forall\, w, w' \in W$, $[w, w'] \in W$.

Obviously, a Lie subalgebra is a Lie algebra in its own right.
Now the goal is to associate in a natural way to each Lie group a Lie algebra. To this end, let G be a fixed Lie group. For a fixed $g \in G$, we have the following maps:

$$
\begin{aligned}
&L_g : G \to G, L_g(h) := g \cdot h\,, \quad (\textbf{\textit{left translation}} \text{ by } g)\\
&R_g : G \to G, R_g(h) := h \cdot g\,, \quad (\textbf{\textit{right translation}} \text{ by } g)\\
&\alpha_g : G \to G, \alpha_g(h) := \left(L_g \circ R_{g^{-1}}\right)(h) = g \cdot h \cdot g^{-1}\,, \quad (\textbf{\textit{conjugation}} \text{ by } g)\,.
\end{aligned}
$$

Note that conjugation is a Lie group isomorphism, whereas left and right translation are diffeomorphisms, but they are not group homomorphisms.

Remark 1.2.4. Let M be a differentiable manifold and $F : M \to M$ a diffeomorphism. For a smooth vector field X on M, $p \in M$, set

$$dF(X)(p) := d_{F^{-1}(p)}F\left(X\left(F^{-1}(p)\right)\right)\,.$$

Then $dF(X)$ is again a smooth vector field on M and the following diagram commutes:

$$
\begin{tikzcd}
TM \arrow[r, "dF"] & TM \\
M \arrow[u, "X"] \arrow[r, "F"] & M \arrow[u, "dF(X)"']
\end{tikzcd}
$$

Furthermore, $\forall\, X, Y \in \mathfrak{X}(M)$, we have

$$dF([X,Y]) = [dF(X), dF(Y)]\,. \tag{1.5}$$

Definition 1.2.5. Let $M = G$ be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called ***left-invariant*** iff $\forall\, g \in G$: $dL_g(X) = X$.

By (1.5), if $X, Y \in \mathfrak{X}(G)$ are left-invariant, then

$$dL_g\left([X,Y]\right) = [dL_g(X), dL_g(Y)] = [X,Y]\,,$$

so $[X,Y]$ is again left-invariant. Thus the vector space

$$\mathfrak{g} := \{X \in \mathfrak{X}(G) \mid X \text{ left-invariant}\}$$

of left-invariant smooth vector fields on G is a Lie subalgebra of $\mathfrak{X}(G)$.

Definition 1.2.6. $\mathfrak{g}$ is called the ***Lie algebra of*** G.

For $g \in G$ and $X \in \mathfrak{g}$

$$X(g) = dL_g(X)(g) = d_{(L_{g^{-1}}(g))}L_g\left(X\left(L_{g^{-1}}(g)\right)\right) = d_eL_g(X(e)),$$

where e is the neutral element in G. Conversely, given $X_0 \in T_eG$, then $X(g) := d_eL_g(X_0)$ yields a left-invariant vector field $X \in \mathfrak{g}$. We thus have a linear isomorphism $T_eG \to \mathfrak{g}$. In particular, $\dim \mathfrak{g}$ (as real vector space) equals $\dim G$ (as smooth manifold).

Example 1.2.7

1. For $G = \mathrm{GL}(n;\mathbb{R})$, $\mathfrak{g} = T_{\mathbb{1}_n}\mathrm{GL}(n;\mathbb{R}) = \mathrm{Mat}(n \times n;\mathbb{R})$. The Lie bracket $[\cdot,\cdot]$ is the commutator, as discussed in example 1.2.2 above.

2. For $G = \mathrm{O}(n)$,

$$\mathfrak{g} =: \mathfrak{o}(n) = T_{\mathbb{1}_n}\mathrm{O}(n) = \{\dot{c}(0) \mid c : (-\epsilon,\epsilon) \to \mathrm{O}(n)\,\text{smooth}, c(0) = \mathbb{1}_n\}\,.$$

We compute

$$\begin{aligned} c(s) \in \mathrm{O}(n) \quad &\Leftrightarrow \quad \mathbb{1}_n = c(s)^t \cdot c(s) \\ &\Longrightarrow \quad 0 = \frac{d}{ds}\Big|_{s=0}\left(c(s)^t \cdot c(s)\right) = \dot{c}(0)^t \cdot c(0) + c(0)^t \cdot \dot{c}(0) \\ &\qquad\quad = \dot{c}(0)^t \cdot \mathbb{1}_n + \mathbb{1}_n \cdot \dot{c}(0) = \dot{c}(0)^t + \dot{c}(0)\,. \end{aligned}$$

Hence $\mathfrak{o}(n) \subset \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \mid A^t + A = 0\}$. Further,

$$\dim \mathfrak{o}(n) = \dim \mathrm{O}(n) = \frac{n(n-1)}{2} = \dim\left\{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \mid A^t + A = 0\right\}$$

so that indeed

$$\mathfrak{o}(n) = \left\{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \mid A^t + A = 0\right\}\,.$$

3. Similarly, for $G = \mathrm{SL}(n;\mathbb{R})$, we have

$$\mathfrak{g} =: \mathfrak{sl}(n;\mathbb{R}) = T_{\mathbb{1}_n}\mathrm{SL}(n;\mathbb{R}) = \{\dot{c}(0) \mid c : (-\epsilon,\epsilon) \to \mathrm{SL}(n;\mathbb{R})\,\text{smooth}, c(0) = \mathbb{1}_n\}\,.$$

which yields

$$\begin{aligned} c(s) \in \mathrm{SL}(n;\mathbb{R}) \quad &\Leftrightarrow \quad 1 = \det c(s) \\ &\Longrightarrow \quad 0 = \frac{d}{ds}\Big|_{s=0} (\det c(s)) = \operatorname{tr}(\dot{c}(0)) \ . \end{aligned}$$

As before, a dimensional argument yields

$$\mathfrak{sl}(n;\mathbb{R}) = \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) \,|\, \operatorname{tr}(A) = 0\} \, .$$

4. For $G = \mathrm{SO}(n)$, we find

$$\mathfrak{g} =: \mathfrak{so}(n) = \mathfrak{o}(n) \cap \mathfrak{sl}(n;\mathbb{R}) = \mathfrak{o}(n) \, ,$$

since $\mathfrak{o}(n) \subset \mathfrak{sl}(n;\mathbb{R})$.

5. For $G = \mathrm{U}(n)$, we compute:

$$\begin{aligned} c(s) \in \mathrm{U}(n) \quad &\Leftrightarrow \quad \mathbb{1}_n = c(s)^* \cdot c(s) \\ &\Longrightarrow \quad 0 = \frac{d}{ds}\Big|_{s=0} (c(s)^* \cdot c(s)) = \dot{c}(0)^* \cdot c(0) + c(0) \cdot \dot{c}(0) \\ &\qquad\quad = \dot{c}(0)^* + \dot{c}(0) \, . \end{aligned}$$

Thus $\mathfrak{g} =: \mathfrak{u}(n) = \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) \,|\, A^* = -A\}$.

6. For $G = \mathrm{SL}(n;\mathbb{C})$, we find $\mathfrak{g} =: \mathfrak{sl}(n;\mathbb{C}) = \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) \,|\, \operatorname{tr}(A) = 0\}$.

7. For $G = \mathrm{SU}(n)$, we find $\mathfrak{g} =: \mathfrak{su}(n) = \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) \,|\, A^* = -A, \, \operatorname{tr}(A) = 0\}$.

1.3 Representations

Definition 1.3.1. A ***representation*** of a Lie group G is a Lie group homomorphism $\varrho : G \to \mathrm{Aut}(V)$ for some finite dimensional $\mathbb{K}$-vector space V (and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). If $\mathbb{K} = \mathbb{R}$, then ϱ is called a ***real representation***, whereas if $\mathbb{K} = \mathbb{C}$, ϱ is called a ***complex representation***.

Remark 1.3.2. Upon the choice of a basis $V \cong \mathbb{K}^n$ and $\mathrm{Aut}(V) \cong \mathrm{GL}(n;\mathbb{K})$.

Definition 1.3.3. A representation ϱ is called ***faithful***, iff it is injective.

Example 1.3.4

1. The ***trivial representation*** defined by $\varrho(g) := \mathrm{id}_V$ for any $g \in G$, is faithful only for the trivial Lie group $G = \{e\}$.

2. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The ***adjoint representation***

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

is defined as follows: For any $g \in G$, we have $\alpha_g(e) = g \cdot e \cdot g^{-1} = e$. By differentiating α_g at the neutral element e, we get a linear map

$$\mathrm{Ad}_g := d_e\alpha_g : \mathfrak{g} \cong T_eG \to T_eG \cong \mathfrak{g}\,.$$

We need to show that Ad is a group homomorphism, i.e. that $\mathrm{Ad}_{g_1 \cdot g_2} = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$. To this end, take $X \in \mathfrak{g}$, and let $c : (-\epsilon, \epsilon) \to G$ be a smooth curve such that $c(0) = e$ and $\dot{c}(0) = X$. Then we compute:

$$\begin{aligned}
\mathrm{Ad}_{g_1 \cdot g_2}(X) &= d_e\alpha_{g_1 \cdot g_2}(X) \\
&= \frac{d}{ds}\Big|_{s=0} (\alpha_{g_1 \cdot g_2}(c(s))) \\
&= \frac{d}{ds}\Big|_{s=0} ((\alpha_{g_1} \circ \alpha_{g_2})(c(s))) \\
&= d_e\alpha_{g_1}(d_e\alpha_{g_2}(X)) \\
&= \mathrm{Ad}_{g_1}(\mathrm{Ad}_{g_2}(X)) \\
&= \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}(X)\,.
\end{aligned}$$

(Here we have used the obvious property $\alpha_{g_1 \cdot g_2} = \alpha_{g_1} \circ \alpha_{g_2}$.) Further, we have $\mathrm{Ad}_e = \mathrm{id}_{\mathfrak{g}}$ and $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$. Thus we have obtained a group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. By the definition of a Lie group we know that $\alpha_g(h)$ depends smoothly on g. This implies that $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is a smooth map.

Remark 1.3.5. If G is abelian, then $\alpha_g = \mathrm{id}_G$ for any $g \in G$, so $\mathrm{Ad}_g = d_e\alpha_g = \mathrm{id}_{\mathfrak{g}}$. Thus the adjoint representation is trivial in this case.

Example 1.3.6. $G = \mathrm{U}(1)$ is abelian, so $\mathrm{Ad} = \mathrm{id}_{\mathfrak{g}}$.

Remark 1.3.7. Let G be any of the matrix groups from example 1.1.4. Conjugation in G is the ordinary conjugation by a matrix from G. Namely, for $X \in \mathfrak{g}$ let $c : (-\epsilon, \epsilon) \to G$ be a smooth curve with $c(0) = e$ and $\dot{c}(0) = X$. Then the adjoint representation is given by:

$$\mathrm{Ad}_g(X) = \frac{d}{ds}\Big|_{s=0} \alpha_g(c(s)) = \frac{d}{ds}\Big|_{s=0} \left(g \cdot c(s) \cdot g^{-1}\right) = g \cdot X \cdot g^{-1} \,.$$

Example 1.3.8. We now compute the adjoint representation of $G = \mathrm{SU}(2)$. The Lie algebra $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) = \{A \in \mathrm{Mat}(n \times n; \mathbb{C}) \,|\, A^* = -A, \, \mathrm{tr}(A) = 0\} = \left\{ \begin{pmatrix} it & z \\ -\bar{z} & -it \end{pmatrix} \middle| \, z \in \mathbb{C}, t \in \mathbb{R} \right\} ,$$

so a natural basis of $\mathfrak{su}(2)$ is given by $-i$ times the so called ***Pauli matrices*** $\sigma_1, \sigma_2, \sigma_3$:

$$-i\sigma_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; -i\sigma_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \; -i\sigma_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

For $g = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \in \mathrm{SU}(2)$, we get

$$\begin{aligned}
\mathrm{Ad}_g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \cdot \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{2i\varphi} \\ -e^{-2i\varphi} & 0 \end{pmatrix} \\
&= \cos(2\varphi) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \sin(2\varphi) \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\end{aligned}$$

By similar computations, we get

$$\begin{aligned}
\mathrm{Ad}_g \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} &= \cos(2\varphi) \cdot \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \sin(2\varphi) \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
\mathrm{Ad}_g \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned}$$

With respect to the basis $-i\sigma_1, -i\sigma_2, -i\sigma_3$ of $\mathfrak{su}(2)$, the adjoint representation of $\mathrm{SU}(2)$ thus has the matrix

$$\mathrm{Ad}_g = \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) & 0 \\ \sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.3.9. The adjoint representation of $\mathrm{SU}(2)$ is not faithful, since $\mathrm{Ad}_{-\mathbb{1}_n} = \mathbb{1}_n$.

Example 1.3.10. For the classical matrix groups from example 1.1.4, we have the ***standard representations***:

1. For $G = \mathrm{GL}(n;\mathbb{K})$, take $\varrho_{\mathrm{st}} := \mathrm{id} : G \to \mathrm{GL}(n;\mathbb{K}) = \mathrm{Aut}\,(\mathbb{K}^n)$.
2. For $G = \mathrm{O}(n)$, $\mathrm{SL}(n;\mathbb{R})$, $\mathrm{SO}(n)$, take the natural inclusion
$$\varrho_{\mathrm{st}} : G \hookrightarrow \mathrm{GL}(n;\mathbb{R}) = \mathrm{Aut}\,(\mathbb{R}^n)\,.$$
3. Similarly, for $G = \mathrm{U}(n)$, $\mathrm{SL}(n;\mathbb{C})$, $\mathrm{SU}(n)$, take the natural inclusion
$$\varrho_{\mathrm{st}} : G \hookrightarrow \mathrm{GL}(n;\mathbb{C}) = \mathrm{Aut}\,(\mathbb{C}^n)\,.$$

Now we consider several techniques to manufacture new representations of a fixed Lie group G out of given ones:

Definition 1.3.11. Let $\varrho : G \to \mathrm{Aut}(V)$ and $\varrho_j : G \to \mathrm{Aut}(V_j)$, $j = 1, 2$, be representations of a fixed Lie group G.

1. The ***direct sum representation*** is defined as:
$$\begin{aligned} \varrho_1 \oplus \varrho_2 : G &\to \mathrm{Aut}\,(V_1 \oplus V_2) \\ (\varrho_1 \oplus \varrho_2)\,(g)\,(v_1 \oplus v_2) &:= \varrho_1(g)\,(v_1) \oplus \varrho_2(g)\,(v_2)\ . \end{aligned}$$
Thus with respect to a basis of $V_1 \oplus V_2$ induced from bases of V_1 and V_2 respectively, $\varrho_1 \oplus \varrho_2$ has block diagonal form:
$$(\varrho_1 \oplus \varrho_2)\,(g) = \begin{pmatrix} \varrho_1(g) & 0 \\ 0 & \varrho_2(g) \end{pmatrix}\,.$$
2. Similarly, the ***tensor product representation*** $\varrho_1 \otimes \varrho_2 : G \to \mathrm{Aut}\,(V_1 \otimes V_2)$ is defined on the homogeneous elements $v_1 \otimes v_2$ by
$$(\varrho_1 \otimes \varrho_2)\,(g)\,(v_1 \otimes v_2) := \varrho_1(g)\,(v_1) \otimes \varrho_2(g)\,(v_2)$$
and expanded linearly to all of $V_1 \otimes V_2$.
3. The ***antisymmetric tensor product representation*** (or ***wedge product representation***) is defined by:
$$\begin{aligned} \Lambda^k \varrho : G &\to \mathrm{Aut}\left(\Lambda^k V\right) \\ \left(\Lambda^k \varrho\right)(g)\,(v_1 \wedge \ldots \wedge v_k) &:= \varrho(g)v_1 \wedge \ldots \wedge \varrho(g)v_k\,. \end{aligned}$$

4. The ***symmetric tensor product representation*** is defined by:

$$\begin{aligned} \odot^k \varrho : G &\to \operatorname{Aut}\left(\odot^k V\right) \\ \left(\odot^k \varrho\right)(g)\,(v_1 \odot \ldots \odot v_k) &:= \varrho(g)v_1 \odot \ldots \odot \varrho(g)v_k\,. \end{aligned}$$

5. Associated to any $\mathbb{K}$-vector space V is the dual vector space V^* of all linear maps from V to the field $\mathbb{K}$. So we expect associated to any representation $\varrho : G \to \operatorname{Aut}(V)$ a dual representation $\varrho^* : G \to \operatorname{Aut}(V^*)$. Let's see how to define ϱ^*: Since for $g \in G$, the representation $\varrho(g)$ is a linear automorphism of V, we might take the dual automorphism $\varrho(g)^* : V^* \to V^*$, defined by $\varrho(g)^*(\lambda) := \lambda \circ \varrho(g)$, as a candidate for the dual representation. Now let's check whether the map $g \mapsto \varrho(g)^*$ is a group homomorphism $G \to \operatorname{Aut}(V^*)$:

$$\begin{aligned} g_1 \cdot g_2 \mapsto \varrho\left(g_1 \cdot g_2\right)^* &= \left(\varrho\left(g_1\right) \cdot \varrho\left(g_2\right)\right)^* \\ &= \varrho\left(g_2\right)^* \cdot \varrho\left(g_1\right)^* \\ &\neq \varrho\left(g_1\right)^* \cdot \varrho\left(g_2\right)^* \text{ in general}\,. \end{aligned}$$

To fix the problem, we define the ***dual representation*** as:

$$\begin{aligned} \varrho^* : G &\to \operatorname{Aut}\left(V^*\right) \\ \varrho^*(g) &:= \varrho\left(g^{-1}\right)^*\,. \end{aligned}$$

Now we compute

$$\begin{aligned} \varrho^*\left(g_1 \cdot g_2\right) &= \left(\varrho\left(\left(g_1 \cdot g_2\right)^{-1}\right)\right)^* \\ &= \left(\varrho\left(g_2^{-1} \cdot g_1^{-1}\right)\right)^* \\ &= \left(\varrho\left(g_2^{-1}\right) \cdot \varrho\left(g_1^{-1}\right)\right)^* \\ &= \varrho\left(g_1^{-1}\right)^* \cdot \varrho\left(g_2^{-1}\right)^* \\ &= \varrho^*\left(g_1\right) \cdot \varrho^*\left(g_2\right) \end{aligned}$$

so that $\varrho^* : G \to \operatorname{Aut}(V^*)$ is indeed a group homomorphism.

6. Suppose that the representation $\varrho : G \to \operatorname{Aut}(V)$ is real. We can manufacture a complex vector space out of V by setting $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. The ***complexification*** of ϱ is the complex representation

$$\varrho_{\mathbb{C}} : G \to \operatorname{Aut}\left(V_{\mathbb{C}}\right),\ \varrho_{\mathbb{C}} := \varrho \otimes \mathrm{id}_{\mathbb{C}}\,.$$

In terms of matrices this means that the representation ϱ is given by real matrices. If we now consider them as complex matrices, then we have the complexification.

Definition 1.3.12. Let G be a Lie group, let $\varrho : G \to \mathrm{Aut}(V)$, $\tilde{\varrho} : G \to \mathrm{Aut}(\tilde{V})$ be representations. Then ϱ and $\tilde{\varrho}$ are called ***equivalent*** , iff there exists an isomorphism $T : V \to \tilde{V}$ such that for every $g \in G$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varrho(g)"'] & \tilde{V} \arrow[d, "\tilde{\varrho}(g)"] \\ V \arrow[r, "T"] & \tilde{V} \end{tikzcd}$$

Example 1.3.13. Let $\varrho : G \to \mathrm{Aut}(V)$ be a given representation on a $\mathbb{K}$-vector space V. The choice of a basis on V yields an isomorphism $F_1 : V \to \mathbb{K}^n$ and a representation $\varrho_1 : G \to \mathrm{GL}(n; \mathbb{K}) = \mathrm{Aut}(\mathbb{K}^n)$. F_1 is an equivalence of representations from ϱ to ϱ_1. Another basis of V leads to another isomorphism $F_2 : V \to \mathbb{K}^n$ and another (equivalent) representation $\varrho_2 : G \to \mathrm{GL}(n; \mathbb{K}) = \mathrm{Aut}(\mathbb{K}^n)$. The automorphism $T := F_2 \circ F_1^{-1} : \mathbb{K}^n \to \mathbb{K}^n$ is an equivalence of the representations ϱ_1 and ϱ_2.

Example 1.3.14. We now construct several (complex) representations of $G = \mathrm{U}(1)$ out of the standard representation $\varrho_{\mathrm{st}} : \mathrm{U}(1) \to \mathrm{GL}(1; \mathbb{C}) = \mathbb{C} - \{0\}$. For any integer $k \in \mathbb{Z}$, we set:

$$\varrho_k : \mathrm{U}(1) \to \mathrm{GL}(1; \mathbb{C}),\ z \mapsto z^k .$$

ϱ_k is a representation, since

$$\varrho\left(z \cdot z'\right) = \left(z \cdot z'\right)^k = z^k \cdot \left(z'\right)^k = \varrho_k(z) \cdot \varrho_k\left(z'\right) .$$

For $k = 0$, we obtain the trivial representation: $\varrho_0(z) = z^0 = 1$. For $k = 1$, we obtain the standard representation $\varrho_1 = \varrho_{\mathrm{st}}$.
Note that we have a natural isomorphism $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ given by $u \otimes w \mapsto u \cdot w$. Under this isomorphism, the tensor product representation $\varrho_k \otimes \varrho_l$ is equivalent to the representation ϱ_{k+l}, since

$$\left(\varrho_k \otimes \varrho_l\right)(z)(u \otimes w) = \left(\varrho_k(z)u\right) \otimes \left(\varrho_l(z)w\right) = \left(z^k v\right) \otimes \left(z^l w\right) = z^{k+l} \cdot u \otimes w .$$

For $\lambda \in \mathbb{C}^*$, we find:

$$\varrho_k^*(z)(\lambda) := \left(\varrho_k(z^{-1})\right)^* (\lambda) := \lambda \circ \varrho_k\left(z^{-1}\right) = \lambda \circ z^{-k} = z^{-k} \cdot \lambda .$$

Thus $\varrho_k^* \cong \varrho_{-k}$.
It turns out that every complex representation of $\mathrm{U}(1)$ is equivalent to the direct sum of 1–dimensional representations ϱ_k. Thus we now know that whole complex representation theory of $\mathrm{U}(1)$.

Example 1.3.15. We study the (complex) representations of $G = \mathrm{SU}(2)$. We already know two of them, namely $\varrho_0 : \mathrm{SU}(2) \to \mathrm{GL}(2;\mathbb{C})$ – the trivial representation – and $\varrho_1 := \varrho_{\mathrm{st}} : \mathrm{SU}(2) \to \mathrm{GL}(2;\mathbb{C})$ – the standard representation. For $k \geq 2$, we set:

$$\varrho_k := \odot^k \varrho_1 .$$

Since a basis of $\odot^k\mathbb{C}^2$ is constructed from a basis e_1, e_2 of $\mathbb{C}^2$ by $e_1 \odot e_1 \odot \ldots \odot e_1$, $e_2 \odot e_1 \odot \ldots \odot e_1, \ldots, e_2 \odot e_2 \odot \ldots \odot e_2$, we find $\dim_{\mathbb{C}}(\odot^k\mathbb{C}^2) = k+1$. Since the (real) representation $\mathrm{Ad}_{\mathrm{SU}(2)}$ is 3–dimensional, ϱ_2 is the only of those representations, that could be equivalent to (the complexifation of) $\mathrm{Ad}_{\mathrm{SU}(2)}$.
To check this, let us compute ϱ_2 on the basis $e_1 \odot e_1$, $e_2 \odot e_2$, $e_2 \odot e_1$.
For $g = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$, we obtain on the first vector:

$$\varrho_2(g)(e_1 \odot e_1) = \varrho_1(g)e_1 \odot \varrho_1(g)e_1 = e^{i\varphi}e_1 \odot e^{i\varphi}e_1 = e^{2i\varphi}e_1 \odot e_1 .$$

Similarly, for the other two basis vectors, we obtain:

$$\begin{aligned} \varrho_2(g)(e_2 \odot e_2) &= e^{-2i\varphi}e_2 \odot e_2 , \\ \varrho_2(g)(e_2 \odot e_1) &= e^{-i\varphi}e_2 \odot e^{i\varphi}e_1 = e_2 \odot e_1 . \end{aligned}$$

Hence, in the basis $e_1 \odot e_1$, $e_2 \odot e_2$, $e_2 \odot e_1$, the element $\varrho_2(g)$ has the matrix

$$\varrho_2 \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} = \begin{pmatrix} e^{2i\varphi} & 0 & 0 \\ 0 & e^{-2i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In order to see that this is equivalent to the complexification of the adjoint representation we put

$$T := \begin{pmatrix} -i & 1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One computes

$$T^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{i}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$T \cdot \begin{pmatrix} \cos(2\varphi) & -\sin(2\varphi) & 0 \\ \sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot T^{-1} = \begin{pmatrix} e^{2i\varphi} & 0 & 0 \\ 0 & e^{-2i\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

It can be checked that the relation

$$T \cdot \mathrm{Ad}_g \cdot T^{-1} = \varrho_2(g)$$

holds for all $g \in \mathrm{SU}(2)$, not just for g of the form $g = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$.
Therefore T provides an equivalence of ϱ_2 and the complexification of Ad.
It turns out that every complex representation of $\mathrm{SU}(2)$ is equivalent to a direct sum of the representations ϱ_k.

Definition 1.3.16. A ***representation*** of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathrm{End}(V)$, where V is a finite dimensional $\mathbb{K}$–vector space. If $\mathbb{K} = \mathbb{R}$, then λ is called a ***real representation***, whereas if $\mathbb{K} = \mathbb{C}$, then λ is called a ***complex representation***.
Given representations $\lambda : \mathfrak{g} \to \mathrm{End}(V)$, $\tilde{\lambda} : \tilde{\mathfrak{g}} \to \mathrm{End}(\tilde{V})$, a linear isomorphism $T : V \to \tilde{V}$ is called an ***equivalence*** of λ and $\tilde{\lambda}$, iff for every $X \in \mathfrak{g}$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\lambda(X)"'] & \tilde{V} \arrow[d, "\tilde{\lambda}(X)"] \\ V \arrow[r, "T"] & \tilde{V} \end{tikzcd}$$

In this case, the representations λ and $\tilde{\lambda}$ are called ***equivalent***.

Remark 1.3.17. Up to equivalence, a Lie algebra representation is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathrm{Mat}(n \times n; \mathbb{K})$.

Example 1.3.18

1. As for Lie groups, we have the ***trivial representation***: for any $X \in \mathfrak{g}$, set $\lambda(X) := 0$.

2. The adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined as $\mathrm{ad}(X)(Y) := [X, Y]$. Since the Lie bracket is linear in the second variable, $\mathrm{ad}(X) : Y \mapsto [X, Y]$ is indeed an endomorphism on $\mathfrak{g}$, i.e., $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$. Since the Lie bracket is linear in the first variable, the map $X \mapsto \mathrm{ad}(X) = [X, \cdot] \in \mathrm{End}(\mathfrak{g})$ is indeed a linear map. It remains to check that it is also a Lie algebra homomorphism, i.e., that $\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)] \in \mathrm{End}(\mathfrak{g})$. We compute, using the Jacobi identity and the antisymmetry of the Lie bracket on $\mathfrak{g}$:

$$\begin{aligned} \mathrm{ad}([X, Y])(Z) &= [[X, Y], Z] \\ &= -[[Y, Z], X] - [[Z, X], Y] \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= \mathrm{ad}(X)(\mathrm{ad}(Y)(Z)) - \mathrm{ad}(Y)(\mathrm{ad}(X)(Z)) \\ &= \big(\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)\big)(Z) \\ &= [\mathrm{ad}(X), \mathrm{ad}(Y)](Z)\,. \end{aligned}$$

Remark 1.3.19. If $\varrho : G \to \mathrm{Aut}(V)$ is a Lie group representation, then

$$\varrho_* := d_e\varrho : \mathfrak{g} \cong T_eG \to T_{\mathrm{id}_V}\mathrm{Aut}(V) \cong \mathrm{End}(V)$$

is a Lie algebra representation. The proof will be given later, see Corollary 1.4.10.

1.4 The exponential map

Exercise 1.4.1
Show that the maximal integral curves of left-invariant vector fields on Lie groups are defined on all of $\mathbb{R}$.

Lemma 1.4.2
Let G *be a Lie group and* $\gamma : \mathbb{R} \to G$ *a smooth curve with* $\gamma(0) = e$. *Then* γ *is a group homomorphism, i.e.* $\forall\, s, t \in \mathbb{R}$, $\gamma(s + t) = \gamma(s) \cdot \gamma(t)$ *iff* γ *is an integral curve to a left-invariant vector field on* G.

Proof.

$\underline{\Rightarrow}$: Suppose that $\forall\, s, t \in \mathbb{R}$, we have $\gamma(s + t) = \gamma(s) \cdot \gamma(t)$. Then

$$\dot{\gamma}(t) = \frac{d}{ds}\Big|_{s=0} \gamma(s + t) = \frac{d}{ds}\Big|_{s=0} \big(\gamma(t) \cdot \gamma(s)\big) = dL_{\gamma(t)} \dot{\gamma}(0)\,.$$

Let X be the unique left-invariant vector field on G with $X(e) = \dot{\gamma}(0)$. Then

$$\dot{\gamma}(t) = dL_{\gamma(t)} X(e) = X(\gamma(t))\,.$$

Hence γ is an integral curve to X.

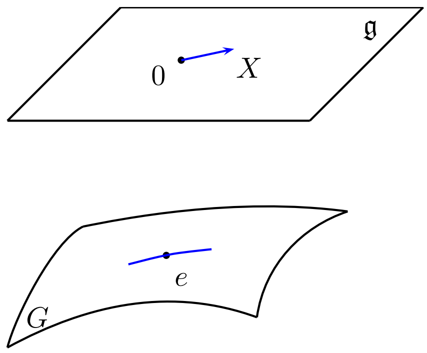
$\underline{\Leftarrow}$: This direction is slightly more involved.

$\square$

In the following let G be a Lie group and $\mathfrak{g}$ its Lie algebra. For any $X \in \mathfrak{g}$, let $\gamma_X : \mathbb{R} \to G$ denote the integral curve to X with $\gamma_X(0) = e$.

Definition 1.4.3. The map $\exp : \mathfrak{g} \to G$, $\exp(X) := \gamma_X(1)$, is called the ***exponential map*** of G.

By the general theory of ordinary differential equations the exponential map is a smooth map $\exp : \mathfrak{g} \to G$.



For a fixed $\alpha \in \mathbb{R}$, and $X \in \mathfrak{g}$, we set $\tilde{\gamma}(t) := \gamma_X(\alpha \cdot t)$. Then $\tilde{\gamma}$ is again a Lie group homomorphism $\tilde{\gamma} : \mathbb{R} \to G$ and thus an integral curve to a left-invariant vector field on G. Further, we have $\tilde{\gamma}(0) = \gamma_X(0) = e$, and $\dot{\tilde{\gamma}}(t) = \alpha \cdot \dot{\gamma}_X(t) = \alpha \cdot X(\tilde{\gamma}(t))$. Since $\tilde{\gamma}$ is uniquely determined as an integral curve to a left-invariant vector field on G, we find $\tilde{\gamma} = \gamma_{\alpha X}$. We thus have

$$\gamma_X(\alpha) = \tilde{\gamma}(1) = \gamma_{\alpha X}(1) = \exp(\alpha X)\,.$$

Renaming α by t, we found the relation:

$$\gamma_X(t) = \exp(tX)\,. \tag{1.6}$$

Hence the curve $t \mapsto \exp(tX)$ coincides with the integral curve γ_X to le left-invariant vector field $X \in \mathfrak{g}$. By Lemma 1.4.2, we have $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$, so that in particular $\exp(0) = e$ and $\exp(-X) = (\exp(X))^{-1}$.

Lemma 1.4.4
The differential at 0 *of the exponential map is the identity:*

$$d_0 \exp = \mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}\,.$$

Proof. Directly from the definition, we compute:

$$d_0 \exp(X) = \frac{d}{ds}\Big|_{s=0} \exp(sX) = X\,.$$

Corollary 1.4.5
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. There exist neighbourhoods $U \subset \mathfrak{g}$ of 0 in $\mathfrak{g}$ and $V \subset G$ of e in G such that $\exp|_U : U \to V$ is a diffeomorphism.

Proof. This follows from Lemma 1.4.4 and the inverse function theorem. □

Corollary 1.4.6
Let $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$, *be the* **inversion map** *of a Lie group G. Then*

$$d_e\mathrm{inv} = -\mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}.$$

Proof. Choose open sets U, V as in Corollary 1.4.5 such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "-\mathrm{id}_{\mathfrak{g}}"] \arrow[d, "\exp"', "\cong"] & U \arrow[d, "\cong"', "\exp"] \\ V \arrow[r, "\mathrm{inv}"] & V \end{tikzcd}$$

Differentiating the diagram at $0 \in \mathfrak{g}$ yields

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "-\mathrm{id}_{\mathfrak{g}}"] \arrow[d, "\mathrm{id}_{\mathfrak{g}}"'] & \tilde{\mathfrak{g}} \arrow[d, "\mathrm{id}_{\mathfrak{g}}"] \\ \mathfrak{g} \arrow[r, "d_0\mathrm{inv}"] & \mathfrak{g} \end{tikzcd}$$

which proves the claim. □

Corollary 1.4.7
For any Lie group homomorphism $\varphi : G \to H$, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_e\varphi"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

Proof. For any $X \in \mathfrak{g}$, the map $\mathbb{R} \to H$, $t \mapsto \varphi(\exp(tX))$ is a Lie group homomorphism and $\frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX)) = d_e\varphi(X)$. By Lemma 1.4.2, $t \mapsto \varphi(\exp(t))$ is an integral curve to a left-invariant vector field on H defined by $d_e\varphi(X) \in T_eH$. Evaluating at $t = 1$, we thus get

$$\exp\left(d_e\varphi(X)\right) = \varphi(\exp(X))\,.$$

Remark 1.4.8. For any Lie group G with Lie algebra $\mathfrak{h}$, the adjoint representations of G and $\mathfrak{g}$ are related as $\mathrm{Ad}_* = \mathrm{ad}$. This is easily checked for the matrix groups $G \subset \mathrm{Mat}(n \times n; \mathbb{R})$: Here, $\mathrm{Ad}_g(X) = g \cdot X \cdot g^{-1}$. We compute:

$$\begin{aligned} \mathrm{Ad}_*(X)(Y) &= \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y) \\ &= \frac{d}{dt}\Big|_{t=0} (\exp(tX) \cdot Y \cdot \exp(-tX)) \\ &= X \cdot Y \cdot \mathbb{1}_n + \mathbb{1}_n \cdot Y \cdot (-X) \\ &= [X, Y] \\ &= \mathrm{ad}(X)(Y)\,. \end{aligned}$$

Lemma 1.4.9
If $\varphi : G \to H$ is a Lie group homomorphism, then $\varphi_* := d_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

Proof. For $X, Y \in \mathfrak{g}$, we compute:[1]

$$\begin{aligned}
\varphi_*([X,Y]) &= \varphi_*(\mathrm{Ad}_*(X)(Y)) \\
&= \varphi_*\left(\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \varphi_*\left(\mathrm{Ad}_{\exp(tX)}(Y)\right) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi_*\left(\left.\frac{\partial}{\partial s}\right|_{s=0} \alpha_{\exp(tX)}(\exp(sY))\right) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \varphi\left(\alpha_{\exp(tX)}(\exp(sY))\right) \\
&= \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \alpha_{\varphi(\exp(tX))}\varphi(\exp(sY)) \\
&\overset{1.4.7}{=} \left.\frac{\partial}{\partial t}\right|_{t=0} \left.\frac{\partial}{\partial s}\right|_{s=0} \alpha_{\exp(t\varphi_*(X))} \exp\left(s\varphi_*(Y)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(t\varphi_* X)}\left(\varphi_* Y\right) \\
&= \mathrm{Ad}_*\left(\varphi_*(X)\right)\left(\varphi_*(Y)\right) \\
&= \mathrm{ad}\left(\varphi_*(X)\right)\left(\varphi_*(Y)\right) \\
&= \left[\varphi_*(X), \varphi_*(Y)\right] .
\end{aligned}$$

Hence φ_* is a Lie algebra homomorphism. □

Corollary 1.4.10
If $\varphi : G \to \mathrm{Aut}(V)$ is a Lie group representation, then $\varphi_* : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra representation.

Remark 1.4.11. If G is an abelian Lie group, then the inversion $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$, is a Lie group homomorphism, since

$$\mathrm{inv}(g \cdot h) = (g \cdot h)^{-1} = h^{-1} \cdot g^{-1} = g^{-1} \cdot h^{-1} = \mathrm{inv}(g) \cdot \mathrm{inv}(h) .$$

By Corollary 1.4.6 and Lemma 1.4.9, $-\mathrm{id}_{\mathfrak{g}} = \mathrm{inv}_* : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism, and we have:

$$-[X,Y] = \mathrm{inv}_*([X,Y]) = [\mathrm{inv}_*(X), \mathrm{inv}_*(Y)] = [-X,-Y] = [X,Y] .$$

[1] Note, that since φ is a Lie group homomorphism, we have

$$\varphi\left(\alpha_g\left(g'\right)\right) = \varphi\left(g \cdot g' \cdot g^{-1}\right) = \varphi(g) \cdot \varphi\left(g'\right) \cdot \varphi(g)^{-1} = \alpha_{\varphi(g)}\left(\varphi\left(g'\right)\right) .$$

Hence $[\cdot,\cdot] \equiv 0$: the Lie algebra of an abelian Lie group is abelian.

Now let $G \subset \mathrm{GL}(n;\mathbb{K})$ be a matrix group. For any $X \in \mathrm{Mat}(n \times n;\mathbb{K})$, we set

$$e^X := \sum_{k=0}^{\infty} \frac{X^k}{k!} \,. \tag{1.7}$$

(Note that the series converges absolutely). Then we have $e^0 = \mathbb{1}_n$ and $\frac{d}{dt}\Big|_{t=0} e^{tX} = X$. We further compute (substituting $m = k - l$):

$$\begin{aligned} e^{(s+t)X} &= \sum_{k=0}^{\infty} \frac{(s+t)^k X^k}{k!} \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{s^{k-l}t^l}{(k-l)!l!} X^k \\ &= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} \frac{s^m}{m!} X^m \cdot \frac{t^l}{l!} X^l \\ &= e^{sX} \cdot e^{tX} \,. \end{aligned}$$

Thus $t \mapsto e^{tX}$ is a Lie group homomorphism from $\mathbb{R}$ to G. By Lemma 1.4.2, it is the integral curve to the left-invariant vector field on G defined by $X \in T_eG$. Hence $e^{tX} = \exp(tX)$ and $e^X = \exp(X)$, i.e. for a matrix Lie group G, the exponential map $\exp : \mathfrak{g} \to G$ coincides with the usual exponential map of matrices as defined by (1.7).

Example 1.4.12. For $G = \mathrm{SO}(2)$, we have $\mathfrak{g} = \mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \middle| \, \theta \in \mathbb{R} \right\}$. For $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \in \mathfrak{so}(2)$, we compute

$$\begin{aligned} A^2 &= \begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} \\ A^{2k} &= (-1)^k \theta^{2k} \cdot \mathbb{1}_n \\ A^{2k+1} &= (-1)^k \theta^{2k+1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,. \end{aligned}$$

Using the power series expansions of cos and sin respectively, we find:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} \mathbb{1}_2 = \cos(\theta)\mathbb{1}_2 \\ \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!} &= \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \sin(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \,. \end{aligned}$$

For the exponential of A we thus get:

$$e^A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} .$$

Obviously from this expression, the exponential map $\exp : \mathfrak{so}(2) \to \mathrm{SO}(2)$ is surjective but not injective.

Remark 1.4.13. If G is a compact, connected Lie group, then the exponential map $\exp : \mathfrak{g} \to G$ is surjective.

1.5 Group actions

Definition 1.5.1. Let G be a Lie group, M a smooth manifold. A smooth map $G \times M \to M$, $(g, x) \mapsto g \cdot x$, is called a ***left action*** (or ***action*** in short) of G on M iff

(i) $\forall\, x \in M$, $\forall\, g, h \in G$: $(g \cdot h) \cdot x = g \cdot (h \cdot x)$.

(ii) $\forall\, x \in M$: $e \cdot x = x$.

Remark 1.5.2. From (i) and (ii), we conclude that for any $g \in G$, the following holds:

$$\begin{aligned} x = e \cdot x &= \left(g \cdot g^{-1}\right) \cdot x = g \cdot \left(g^{-1} \cdot x\right) = L_g\left(L_{g^{-1}}(x)\right) \\ &= \left(g^{-1} \cdot g\right) \cdot x = g^{-1} \cdot (g \cdot x) = L_{g^{-1}}\left(L_g(x)\right) . \end{aligned}$$

Hence for any $g \in G$, the map $L_g : M \to M$, $L_g(x) := g \cdot x$, is a diffeomorphism with inverse $(L_g)^{-1} = L_{g^{-1}}$.
Condition (i) yields $L_g \circ L_h = L_{g \cdot h}$. Hence the map $g \mapsto L_g$ is a group homomorphism $G \to \mathrm{Diff}(M)$.

Example 1.5.3

1. Any Lie group acts on any manifold M in an uninteresting manner, namely by $g \cdot x := x$. This is called the ***trivial action***.
2. Associated to any representation $\varrho : G \to \mathrm{Aut}(V)$ is an action of G on V by $g \cdot v := \varrho(g)(v)$.
3. Any Lie group acts on itself by the following natural actions $G \times G \to G$, $(g, h) \mapsto g * h$:

- by the group multiplication: $g * h := g \cdot h$. In this case, condition (i) is equivalent to the associativity of the group multiplication $\cdot$, whereas (ii) is the definition of the neutral element $e \in G$.
- by conjugation: $g * h := \alpha_g(h)$.

Definition 1.5.4. A (left) action of G on M is called ***effective***, iff

$$\begin{aligned} & \forall\, g \in G : \big((\forall\, x \in M : g \cdot x = x) \implies g = e\big) \\ \Leftrightarrow\quad & \forall\, g \in G : \big(L_g = \mathrm{id}_M \implies g = e\big) \\ \Leftrightarrow\quad & \text{the group homomorphism } G \to \mathrm{Diff}(M) \text{ is injective}. \end{aligned}$$

It is called ***free***, iff $\forall\, g \in G : \big((\exists\, x \in M : g \cdot x = x) \implies g = e\big)$.
It is called ***transitive***, iff $\forall\, x, y \in M : \exists\, g \in G : g \cdot x = y$.

Remark 1.5.5. Every free action is effective (unless $M = \emptyset$).

Example 1.5.6

1. The trivial action is effective $\Leftrightarrow G = \{e\} \Leftrightarrow$ The trivial action is free.
2. The action given by a representation ϱ on V is never transitive (unless $V = \{0\}$), since $\forall\, g \in G$: $\varrho(g) \cdot 0 = 0$.
3. For the two natural actions of a Lie group G on itself, we find:
   - The action by left multiplication is free (hence also effective), since $g \cdot g' = g'$ implies $g = e$ (by right multiplication with $(g')^{-1}$). The action is transitive, since for given $x, y \in G$, the equation $g \cdot x = y$ is solved by $g = y \cdot x^{-1}$.
   - For the action by conjugation, we have:

     $$\forall\, x \in G : gxg^{-1} = x \;\Leftrightarrow\; \forall\, x \in G : xg = gx \;\Leftrightarrow\; g \in Z(G)\,,$$

     where $Z(G) := \{g \in G \,|\, \forall\, h \in G : gh = hg\}$ is the ***center*** of G. Thus the action by conjugation is not effective iff $Z(G) \neq \{e\}$. In general, the action is not transitive either, unless the group has only one conjugacy class.

Example 1.5.7. Now we consider two more concrete examples:

1. $G = \mathrm{SO}(2)$ acts on $M = S^2$ by rotations around the z axis, i.e.:
$$g \cdot x := \begin{pmatrix} \begin{matrix} & \\ & g & \\ & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \cdot x \, .$$
The action is effective: if $\forall\, x \in S^2$: $g \cdot x = x$, then $g = \mathbb{1}_2$.
The action is not free, since $\forall\, g \in \mathrm{SO}(2)$: $g \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $g \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.
The action is not transitive, since the circles of latitude are invariant under rotation about the z-axis.

2. $G = \mathrm{U}(1)$ acts on $M = S^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$ by scalar multiplication on the complex coordinates, i.e. $(z, x) \mapsto z \cdot x$.
The action is free, since for $w \neq 0$, $z \cdot w = w$ implies $z = 1$.
The action is not transitive, unless $n = 1$. In this case, the action is just left multiplication on the Lie group $\mathrm{U}(1)$.
In any case, $z \cdot x = y$ for $x, y \in S^{2n-1} \subset \mathbb{C}^n$ implies that x, y are linearly dependent in the complex vector space $\mathbb{C}^n$. Thus the action is not transitive iff $n > 1$.

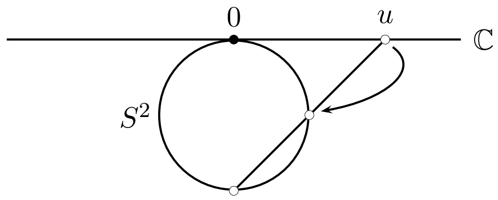
Definition 1.5.8. Let G act on M. Then for any $x \in M$,
$$G \cdot x := \{g \cdot x \,|\, g \in G\}$$
is called the ***orbit*** of x under this action. G acts transitively on M, iff $G \cdot x = M$. The set
$$G \backslash M := \{G \cdot x \,|\, x \in M\}$$
is called the ***orbit space*** of the action.

Example 1.5.9. For the rotation action of $G = \mathrm{U}(1)$ on $M = S^2$, the orbits are the circles of latitude, including the north and south pole. Hence the orbits are naturally parametrised by the z-coordinate, and the orbit space is thus identified with the interval $[-1, 1]$.

Example 1.5.10. We consider the action of $G = \mathrm{U}(1)$ on $M = S^3 \subset \mathbb{C}^2$ by scalar multiplication in more detail. Given $w = (w_1, w_2)$, $w' = (w_1', w_2')$, they lie in the same orbit iff $\frac{w_1}{w_2} = \frac{w_1'}{w_2'} \in \mathbb{C} \cup \infty =: \hat{\mathbb{C}}$. So the orbit space $\mathrm{U}(1)\backslash S^3$ is naturally identified with the Riemann sphere $\hat{\mathbb{C}}$.

By stereographic projection

$$u \mapsto \frac{1}{4+|u|^2} \cdot \left(4u, 4-|u|^2\right).$$



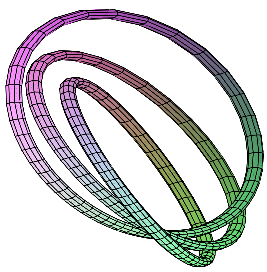
Hence for $u = \frac{w_1}{w_2}$ we get

$$\begin{aligned} \frac{1}{4+\left|\frac{w_1}{w_2}\right|^2} \cdot \left(4\frac{w_1}{w_2}, 4-\left|\frac{w_1}{w_2}\right|^2\right) &= \frac{|w_2|^2}{4|w_2|^2+|w_1|^2} \cdot \left(4\frac{w_1}{w_2}, 4-\left|\frac{w_1}{w_2}\right|^2\right) \\ &= \frac{1}{4|w_2|^2+|w_1|^2} \cdot \left(4w_1\bar{w}_2, 4|w_2|^2-|w_1|^2\right) . \end{aligned}$$

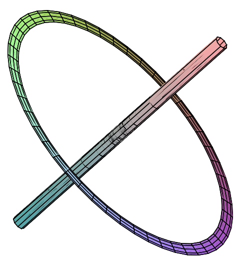
We thus found the so called ***Hopf map***

$$\mathrm{Hopf} : S^3 \to S^2 \, , w \mapsto \frac{1}{4|w_2|^2+|w_1|^2} \cdot \left(4w_1\bar{w}_2, 4|w_2|^2-|w_1|^2\right) .$$

This map is smooth and its pre-images $\mathrm{Hopf}^{-1}(p)$ are the orbits of the $\mathrm{U}(1)$-action on S^3. The $\mathrm{U}(1)$-orbits can be visualized by mapping S^3 minus one point to $\mathbb{R}^3$ via a stereographic projection just as we did for S^2. Then R^3 becomes a disjoint union of circles and one straight line corresponding to the orbit through the exceptional point of S^3.



Three nearby Hopf circles after stereographic projection to $\mathbb{R}^3$



One regular and the exceptional Hopf circle

It turns out that any two of the Hopf circles in $\mathbb{R}^3$ are linked, they form a *Hopf link*.

Theorem 1.5.11
Let G be a compact Lie group acting freely on a manifold M. Then $G\backslash M$ carries the structure of a smooth manifold such that

(i) the map
$$M \to G\backslash M, \quad x \mapsto G \cdot x\,,$$
is smooth and its differential has maximal rank at each point.

(ii)
$$\dim(G\backslash M) = \dim(M) - \dim(G)\,.$$

(iii) $G\backslash M$ has the following universal property: for every differentiable manifold N and every smooth map $f : M \to N$, which is constant along the orbits of the action there exists a unique smooth map $\tilde{f} : G\backslash M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \\ G\backslash M \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Idea of proof. For $x \in M$, choose a small embedded disc D of maximal dimension intersecting $G \cdot x$ transversely at x. Then to any y in the disc, there corresponds an orbit $G \cdot y$ near the orbit $G \cdot x$ through x. Check that (after possibly shrinking the disc) the map $G \times D \to M$, $(g, y) \mapsto g \cdot y$, is a diffeomorphism onto its image. This yields a local chart of the orbit space.
The compactness of G is needed to ensure that the points in a sufficiently small disc are in $1:1$ correspondence to the orbits and that the quotient topology is Hausdorff. $\square$

Example 1.5.12. We consider again the action of $\mathrm{U}(1)$ on $S^{2n-1} \subset \mathbb{C}^n$ by complex scalar multiplication. Since the action is free $\mathrm{U}(1)\backslash S^{2n-1}$ is a smooth manifold. Two points $w, w' \in S^{2n-1}$ lie in the same orbit iff for some $Z \in \mathbb{C}$ with $|z| = 1$: $w' = z \cdot w$, i.e. iff w, w' are linearly dependent, i.e. the complex lines panned by w, w coincide: $\mathbb{C} \cdot w = \mathbb{C} \cdot w'$. Hence the orbit space is identified with
$$\mathbb{C}P^{n-1} := \mathrm{U}(1)\backslash S^{2n-1} \cong \{\text{1-dim}_{\mathbb{C}} \text{ vector subspaces of } \mathbb{C}^n\}\,.$$

Definition 1.5.13. $\mathbb{C}P^{n-1}$ is called the $(n-1)$-dimensional ***complex projective space***.

Remark 1.5.14.

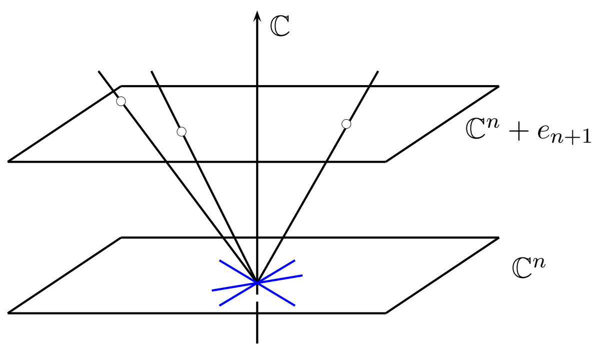
$$\dim_{\mathbb{R}}\left(\mathbb{C}P^{n-1}\right)=\dim_{\mathbb{R}}\left(S^{2n-1}\right)-\dim_{\mathbb{R}}\left(\mathrm{U}(1)\right)=(2n-1)-1=2(n-1)\,.$$

$\mathbb{C}P^{n-1}$ is compact and connected, because it is the image of S^{2n-1} under a continuous map. For $n=2$, we have

$$\begin{tikzcd}
S^3 \arrow[r, "\mathrm{Hopf}"] \arrow[d] & S^2 \\
\mathbb{C}P^1 \arrow[ur, "\widetilde{\mathrm{Hopf}}"'] &
\end{tikzcd}$$

The map $\widetilde{\mathrm{Hopf}}:\mathbb{C}P^1\to S^2$ is smooth and bijective. By explicit computation, we find that the differential of the Hopf map has maximal rank everywhere. Hence by the commutativity of the diagram the same holds for $\widetilde{\mathrm{Hopf}}$. Thus $\widetilde{\mathrm{Hopf}}:\mathbb{C}P^1\to S^2$ is a diffeomorphism.

Remark 1.5.15. The sequence of natural linear embeddings $\mathbb{C}\subset\mathbb{C}^2\subset\mathbb{C}^3\subset\cdots\subset\mathbb{C}^n$ yields a sequence of embeddings $S^1\subset S^3\subset S^5\subset\cdots\subset S^{2n-1}$ and by taking the quotient of the $\mathrm{U}(1)$ action the sequence $\{*\}\subset\mathbb{C}P^1\subset\mathbb{C}P^2\subset\cdots\subset\mathbb{C}P^{n-1}$. There is a natural inductive procedure to construct $\mathbb{C}P^n$ from $\mathbb{C}P^{n-1}$ by attaching $\mathbb{C}^n$:

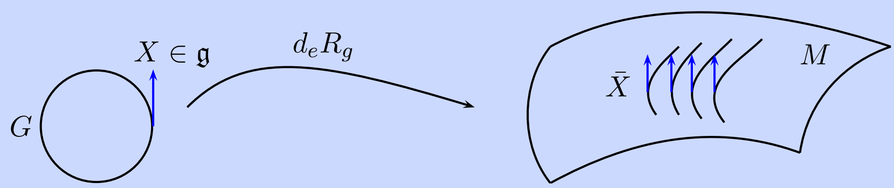


There are two types of 1-dimensional subspaces of $\mathbb{C}^{n+1}=\mathbb{C}^n\oplus\mathbb{C}$: the lines contained in the hyperplane $\mathbb{C}^n$ (they form $\mathbb{C}P^{n-1}$) and the ones intersecting the hyperplane only at 0. The latter ones intersect the parallel affine hyperplane $\mathbb{C}^n+e_{n+1}$ at exactly one point. Therefore they are in 1-1-correspondence with points in that affine hyperplane.

Hence we have decomposed $\mathbb{C}P^n$ disjointly into two subsets, one which is naturally identified with $\mathbb{C}P^{n-1}$ and one which is naturally identified with $\mathbb{C}^n + e_{n+1}$ (which we can in turn identify with $\mathbb{C}^n$),

$$\mathbb{C}P^n = \mathbb{C}P^{n-1} \sqcup \mathbb{C}^n.$$

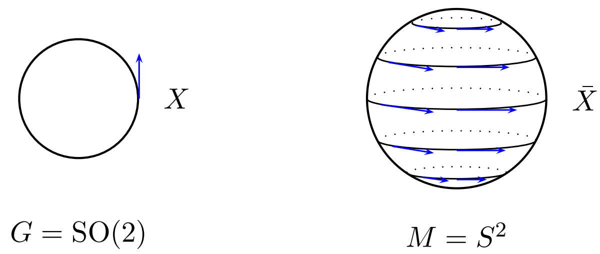
Definition 1.5.16. Let G be a Lie group acting on a differentiable manifold M. For $p \in M$ let $R_p : G \to M$ be the map $R_p(g) := g \cdot p$. The differential of R_p is a linear map $d_e R_p : \mathfrak{g} \cong T_e G \to T_p M$. For $X \in \mathfrak{g}$ we set $\bar{X}(p) := d_e R_p(X)$ and we obtain a vector field $\bar{X} \in \mathfrak{X}(M)$.



$\bar{X}$ is called the ***fundamental vector field*** associated with $X \in \mathfrak{g}$.

Remark 1.5.17. As remarked above a Lie group action can be thought of as a (Lie) group homomorphism $G \to \mathrm{Diff}(M)$. The map $\mathfrak{g} \ni X \mapsto \bar{X} \in \mathfrak{X}(M)$ is the corresponding Lie algebra homomorphism.

Example 1.5.18. For the action of $G = \mathrm{SO}(2)$ on $M = S^2 \subset \mathbb{R}^3$, the fundamental vector fields are tangent to the circles of latitude:



Remark 1.5.19. For $X \in \mathfrak{g}$, we compute:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} L_{\exp(tX)}(p) &= \left.\frac{d}{dt}\right|_{t=t_0} \exp(tX) \cdot p \\
&= \left.\frac{d}{ds}\right|_{s=0} \exp\left(t_0 + s\right) \cdot p \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(\exp(sX) \cdot \exp\left(t_0 X\right) \cdot p\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} R_{\exp(t_0 X)\cdot p}(\exp(sX)) \\
&= d_e R_{\exp(t_0 X)\cdot p}(X) \\
&= \bar{X}\left(\exp\left(t_0 X\right) \cdot p\right) .
\end{aligned}$$

Hence $L_{\exp(tX)}$ is the flow of the fundamental vexctor field $\bar{X}$. In particular if $\bar{X}(p) = 0$, then $\exp(tX) \cdot p = p$ for all $t \in \mathbb{R}$.
This observation yields an obstruction for the existence of free group actions: namely if M is acted upon freely by a Lie group G with $\dim(G) \geq 1$ (i.e. $\mathfrak{g} \neq \{0\}$) then M must have smooth nowhere vanishing vector fields. In particular, $\chi(M) = 0$, e.g. $M \ncong S^{2n}$.

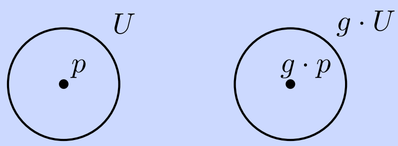
Definition 1.5.20. A zero-dimensional Lie group is called a ***discrete group***.

Remark 1.5.21. A discrete group is compact iff it is finite. If a compact group acts freely on a manifold M then $G \backslash M$ is again a manifold. We are now looking for a similar criterion for discrete group actions.

Definition 1.5.22. An action of a discrete group on a differentiable manifold M is called ***properly discontinuous*** iff

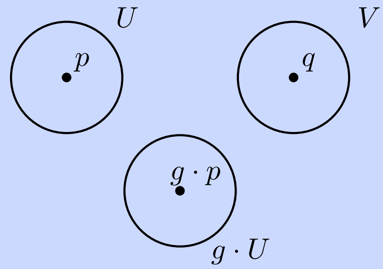
(i) $\forall\, p \in M$ there exists a neighborhood U of p in M such that

$$g \cdot U \cap U \neq \emptyset \implies g = e.$$



(ii) $\forall\, p, q \in M$ with $G \cdot p \neq G \cdot q$ there exist neighborhoods U of p and V of q in M such that

$$\forall\, g \in G : g \cdot U \cap V = \emptyset.$$



Theorem 1.5.23
If a discrete group G acts properly discontinuously on a manifold M then $G\backslash M$ carries a unique differentiable manifold structure such that the projection map $M \to G\backslash M$, $p \mapsto G \cdot p$, is smooth and a covering map (in particular a local diffeomorphism).
Furthermore, the quotient $G\backslash M$ has the following universal property: for any differentiable manifold N and any smooth map $f : M \to N$, which is constant along the orbits of G, there is a unique smooth map $\tilde{f} : G\backslash M \to N$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \\ G\backslash M \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

Idea of proof. Use the open neighborhoods U as in (i) as charts for the quotient $G\backslash M$. Then (ii) ensures that $G\backslash M$ is Hausdorff. $\square$

Example 1.5.24. Let $G = (\mathbb{Z}, +)$ act on $M = \mathbb{R}$ by $\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $(k, t) \mapsto k + t$. This action is properly discontinuous:
(i) is satisfied by chosing for a $t \in \mathbb{R}$ the neighborhood $U := (t - \frac{1}{2}, t + \frac{1}{2})$.
For $s, t \in \mathbb{R}$ in different $\mathbb{Z}$ orbits, i.e. with $t - s \notin \mathbb{Z}$, we put $\epsilon := \min\{|t - (s+k)| \mid k \in \mathbb{Z}\}$. Then setting $U := (s - \frac{\epsilon}{2}, s + \frac{\epsilon}{2})$, $V := (t - \frac{\epsilon}{2}, t + \frac{\epsilon}{2})$ yields separating neighborhoods as required for (ii) to hold.

The quotient $\mathbb{Z}\backslash\mathbb{R}$ is identified as follows: the map $f : \mathbb{R} \to S^1$, $f(t) := \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$ is smooth and constant along the $\mathbb{Z}$ orbits. By the universal property of the quotient, we have another smooth surjective map $\tilde{f} : \mathbb{Z}\backslash\mathbb{R} \to S^1$. It is then also injective. Furthermore, $df \neq 0$ everywhere and thus same holds for $d\tilde{f}$. Hence $\tilde{f} : \mathbb{Z}\backslash\mathbb{R} \to S^1$ is a diffeomorphism.

Example 1.5.25. Let $G = (\mathbb{Q}, +)$ with the discrete topology act on $M = \mathbb{R}$ by $(q, t) \mapsto q + t$. This map is not properly discontinous, since any two different orbits are arbitrarily close: if $x := s - t \notin \mathbb{Q}$, then approximating x by rationals yields points in the orbit of s arbitrarily close to t. Indeed the quotient $\mathbb{Q}\backslash\mathbb{R}$ is not Hausdorff.

Definition 1.5.26. Let G be a Lie group and let M be a differentiable manifold. A ***right action*** of G on M is a smooth map $M \times G \to M$, $(x, g) \mapsto x \cdot g$, satisfying:

(i) $\forall\, x \in M$, $\forall\, g, h \in G$: $x \cdot (g \cdot h) = (x \cdot g) \cdot h$.

(ii) $\forall\, x \in M$: $x \cdot e = x$.

Remark 1.5.27. Note that if we set $g * p := p \cdot g$, then (i) says: $(g \cdot h) * p = h * (g * p)$. Thus if $G \times M \to M$, $(g, p) \mapsto g \cdot p$, is a left action, then

$$\begin{array}{rcl} M \times G & \to & M, \\ (p, g) & \mapsto & p * g := g^{-1} \cdot p, \end{array}$$

defines a right action.
Conversely, if $M \times G \to M$, $(p, g) \mapsto p \cdot g$ is a right action, then

$$\begin{array}{rcl} G \times M & \to & M, \\ (g, p) & \mapsto & g * p := p \cdot g^{-1} \end{array}$$

defines a left action.

2 Bundle theory

2.1 Fiber bundles

Definition 2.1.1. Let E, B, F be differentiable manifolds, let $\pi : E \to B$ be a surjective smooth map. Then (E, π, B) is called a ***fiber bundle*** with ***typical fiber*** F iff each point $x \in B$ has an open neighborhood $U \subset B$ such that there exists a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

B is called the ***base*** and E is called the ***total space*** of the fiber bundle. ψ_U is called a ***local trivialization*** over U.

Remark 2.1.2. For any $U \ni x \in B$, a local trivialization ψ_U yields a diffeomorphism $\psi_U|_{\pi^{-1}(x)} : \pi^{-1}(x) := \{x\} \times F \cong F$. Thus the fibers $E_x := \pi^{-1}(x)$ (of the projection map π) of a fiber bundle are diffeomorphic to the typical fiber F (thus the name).

Example 2.1.3

1. The trivial fiber bundle is the cartesian product $(B \times F, \mathrm{pr}_1, B)$.

2. Let (B, g) be a Riemannian manifold of dimension n. The ***unit sphere bundle*** of (B, g) is defined as $E := \{X \in TB \,|\, \|X\|_g = 1\}$. The projection map is the restriction of the foot point projection of the tangent bundle TB. For $p \in B$, $\pi^{-1}(p) = \{X \in T_pB \,|\, \|X\|_g = 1\}$, so $F = S^{n-1}$. Local trivializations of E are obtained from local trivializations of the tangent bundle.

Remark 2.1.4. Let F be a smooth manifold and $\varphi : F \to F$ be a diffeomorphism. Then $\mathbb{Z}$ acts properly discontinuously on $\mathbb{R} \times F$ by $(k, (t, f)) \mapsto (t + k, \varphi^k(f))$. Set $E := \mathbb{Z}\backslash(\mathbb{R} \times F)$. The projection pr_1 onto the first factor induces a projection map

$\pi : E \to \mathbb{Z}\backslash\mathbb{R} \cong S^1$. Then (E, π, B) is a fiber bundle with typical fiber F. To construct local trivializations, use the (global) triviality of the bundle $\mathbb{R} \times F \to \mathbb{R}$ together with the proper discontinuity of the action. Geometrically, the total space E is constructed from the trivial bundle $[0,1] \times F \to [0,1]$ by glueing the fibers over $0, 1$ by the diffeomorphism φ.

Example 2.1.5. For $F = (-1, 1)$, $\varphi : F \to F$, $x \mapsto -x$, the construction yields the Möbius strip.

Definition 2.1.6. Two fiber bundles (E, π, B) and (E', π', B') are called ***isomorphic*** iff there exists a diffeomorphism $\psi : E \to E'$ such that the following diagram commutes:

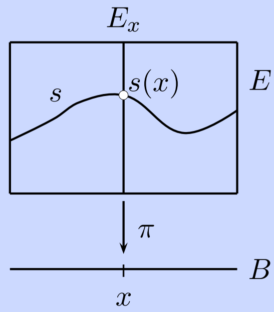
$$\begin{tikzcd} E \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$

A fiber bundle is called ***trivial*** iff it is isomorphic to the trivial fiber bundle $B \times F \to B$. (This is equivalent to the existence of a global trivialization, i.e. a local trivialization ψ_U defined on $U = B$).

Definition 2.1.7. A fiber bundle (E, π, B) is with typical fiber $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called a ***(real oder complex) vector (space) bundle*** of ***rank n*** iff each fiber E_x carries the structure of a $\mathbb{K}$ vector space and the local trivializations ψ_U can be chosen such that $\psi_U|_{\pi^{-1}(x)} : E_x \to \{x\} \times \mathbb{K}^n \cong \mathbb{K}^n$ is a linear isomorphism (i.e. the local trivializations are linear along each fiber).

Example 2.1.8. If B is a smooth manifold, then the tangent TB is a vector bundle, and so are all bundles constructed from TB by linear algebra (applied fiberwise), such as T^*B, $\Lambda^k T^*B$, $\odot^k T^*B$ etc.

Definition 2.1.9
Let (E, π, B) be a fiber bundle. A ***section*** is a map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$.



Convention: From now on sections will be assumed smooth unless specified otherwise.

Remark 2.1.10. Any vector bundle (V, π, B) has (smooth) sections, e.g. the zero section defined by $s(x) := 0_x \in V_x$ for every $x \in B$. General fiber bundles need not have any continuous sections at all as we shall see.

Let B be a smooth manifold. The following tabular indicates how several well known objects from geometry can be considered as sections of vector bundles over the base B:

vector bundle	sections
TB	vector fields
T^*B	differential 1-forms
$\Lambda^k T^*M$	differential k-forms
$\bigotimes^k TB \otimes \bigotimes^l T^*B$	(k, l)-tensor fields

Let (E, π, B) be a fiber bundle with typical fiber F. Let $\lambda : B \to B'$ be a smooth map. We want to construct a fiber bundle over B' with typical fiber F. To this end, put

$$\begin{aligned} E' &:= \left\{\left(b', p\right) \in B' \times E \mid \lambda\left(b'\right) = \pi(p)\right\} \quad \text{and} \\ \pi' &:= \mathrm{pr}_1|_{E'} : E' \to B'. \end{aligned}$$

To construct a local trivialization for (E', π', B') in a neighborhood of $b'_0 \in B'$, choose an open neighborhood U of $\lambda(b'_0)$ in B and a local trivialization $\psi_U : \pi^{-1}(U) \to U \times F$. Then take $U' := \lambda^{-1}(U)$ as a neighborhood of b'_0 in B' and compute:

$$\begin{aligned} \left(\pi'\right)^{-1}\left(U'\right) &= \mathrm{pr}_1^{-1}\left(U'\right) \\ &= \left\{\left(b', p\right) \in U' \times E \mid \lambda\left(b'\right) = \pi(p)\right\} \\ &\cong \left\{\left(b', u, f\right) \in U' \times U \times F \mid \lambda\left(b'\right) = u\right\} \\ &\cong U' \times F\,. \end{aligned}$$

This identification shows that $E' \subset B' \times E$ is a smooth submanifold and that (E', π', B') is a fiber bundle.

Definition 2.1.11. The fiber bundle $\lambda^*(E, \pi, B) := (E', \pi', B')$ as constructed above is called the ***pull-back*** of (E, π, B) along λ.

By construction, the following diagram commutes:

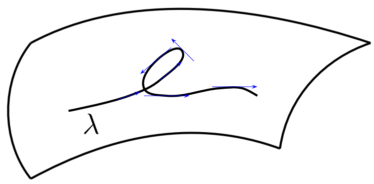
$$\begin{tikzcd} \lambda^* E \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & E \arrow[d, "\pi"] \\ B' \arrow[r, "\lambda"'] & B \end{tikzcd}$$

Now for $b_0' \in B'$, the fiber $E'_{b_0'}$ of the pull-back bundle is identified as follows:

$$\begin{aligned} E'_{b_0'} &= \mathrm{pr}_1^{-1}\left(b_0'\right) \\ &= \left\{\left(b', p\right) \in B' \times E \mid \lambda\left(b'\right) = \pi(p),\ \mathrm{pr}_1\left(b', p\right) = b_0'\right\} \\ &= \left\{\left(b_0', p\right) \in B' \times E \mid \lambda\left(b_0'\right) = \pi(p)\right\} \\ &= \left\{b_0'\right\} \times E_{\lambda(b_0')} \\ &\overset{\mathrm{pr}_2}{\cong} E_{\lambda(b_0')} . \end{aligned}$$

Thus pr_2 identifies the fiber of the pull-back bundle at b_0' with the fiber of E at $\lambda(b_0')$.

Example 2.1.12. Let $E = TB$ be the tangent bundle of a smooth manifold B and let $\lambda : (-\epsilon, \epsilon) \to B$ be a smooth curve in B. Then sections of λ^*TB are vector fields along λ. The velocity field $\dot{\lambda}$ of the curve is one particular such vector field along λ.



The velocity field of λ is tangent to the image of λ in B. More general vector fields along curves naturally occur in Riemannian geometry e.g. as variational fields of variations of the curve λ (Jacobi fields for geodesic variations etc).

2.2 Principal bundles

Definition 2.2.1. A fiber bundle (P, π, B) together with a right action of a Lie group G on P is called a ***G-principal bundle*** iff

(i) The group action is free.

(ii) The group actions is transitive on the fibers of the bundle, i.e. for any $p \in P$ we have $p \cdot G = P_{\pi(p)}$.

(iii) The local trivializations $\psi_U : \pi^{-1}(U) \to U \times G$ can be chosen such that the following diagram commutes[1] (where $\mu_G : G \times G \to G$ is thne multiplication in the Lie group G):

$$\begin{tikzcd} \pi^{-1}(U) \times G \arrow[r, "\psi_U \times id_G"] \arrow[d, "\psi_U \times \mathrm{id}_G"'] & \pi^{-1}(U) \arrow[d, "\psi_U"] \\ U \times G \times G \arrow[r, "\mathrm{id}_U \times \mu_G"'] & U \times G \end{tikzcd}$$

G is called the ***structure group*** of the principal bundle.

Remark 2.2.2. For a fixed $p \in P$ look at the map $L_p : G \to P_{\pi(p)}$, $g \mapsto p \cdot g$. Then L_p is a smooth map by definition, moreover it is injective by (i) and surjective by (ii). The differential $d_e L_p$ has maximal rank, since for a free group action, $d_e L_p : \mathfrak{g} \cong T_e G \ni X \mapsto \bar{X}(p)$ is injective by Remark 1.5.19. Moreover, for $g \in G$, we have $L_p = L_{p\cdot g} \circ L_{g^{-1}}$, so

$$d_g L_p = \underbrace{d_e L_{p\cdot g}}_{\text{injective}} \circ \underbrace{d_g L_{g^{-1}}}_{\text{bijective}},$$

since $L_{g^{-1}} : G \to G$ is a diffeomorphism. Therefore, $L_p : G \to P_{\pi(p)}$ is a diffeomorphism, and the typical fiber of a G-principal bundle is the Lie group G.
Note that the fibers of a G-principal bundle are naturally diffeomorphic to G but they do not carry a natural group structure!

Remark 2.2.3. As an extension of Theorem 1.5.11, we have the following statement: If a compact group acts freely (from the right) on a manifold P, then $(P, \pi, G \backslash P)$ is a G-principal bundle. Here $\pi : P \to G \backslash P$ is the natural projection to the orbits, i.e. $\pi(p) := p \cdot G$. Properties (i) and (ii) are obviously satisfied, (iii) follows from the sketch of proof of Theorem 1.5.11.

Example 2.2.4. The ***Hopf fibration*** $\pi : S^{2n-1} \to \mathbb{C}P^n$ from Example 1.5.12 is a $\mathrm{U}(1)$-principal bundle.

Example 2.2.5. Let $V \to B$ be a $\mathbb{K}$-vector bundle of rank n. For any $b \in B$, the fiber V_b is an n-dimensional $\mathbb{K}$-vector space. Set $P_b := \{(\text{ordered}) \text{ bases } (b_1, \dots, b_n) \text{ of } V_b\}$. Then $G = \mathrm{GL}(n; \mathbb{K})$ acts freely and transitively on P_b from the right by:

$$(b_1, \dots, b_n) \cdot A := \left(\sum_{i=1}^{n} A_{i1} b_i, \dots, \sum_{i=1}^{n} A_{in} b_i \right),$$

where $A = (A_{ij})_{i,j=1\dots n}$. Then $P := \bigsqcup_{b \in B} P_b$ together with the projection $\pi : P \to B$ defined such that $\pi|_{P_b} \equiv b$ is a $\mathrm{GL}(n; \mathbb{K})$-principal bundle.

Example 2.2.6. Similarly, we can construct principal bundles for different structure groups by considering the bundles of bases of vector bundles with further structures: Let $V \to B$ be a $\mathbb{K}$-vector bundle of rank n with a Riemannian or Hermitian metric resp. For $b \in B$, set $P_b := \{(\text{ordered}) \text{ orthonormal bases } (b_1, \dots, b_n) \text{ of } V_b\}$. Then $G = \mathrm{O}(n)$ resp. $G = \mathrm{U}(n)$ acts on P_b freely and transitively. We thus obtain an $\mathrm{O}(n)$- or $\mathrm{U}(n)$-principal bundle resp.

Definition 2.2.7. Let B be a smooth n-manifold and let $V = TB \to B$ be the tangent bundle. Then the $\mathrm{GL}(n; \mathbb{K})$-principal bundle (P, π, B) constructed in Example 2.2.5 is called the ***frame bundle*** of B. Let (B, g) be a Riemannian manifold, and let $V \to B$ be the tangent bundle. Then the $\mathrm{O}(n)$-principal bundle from Example 2.2.6 is called the ***orthonormal frame bundle***.

By considering several structures on $\mathbb{K}$-vector bundles, we naturally obtain the following G-principal bundles as bundles of (ordered) bases respecting the given structure:

$\mathbb{K}$	vector bundle	ordered bases	G
$\mathbb{R}$, $\mathbb{C}$	any	all	$\mathrm{GL}(n; \mathbb{K})$
$\mathbb{R}$	Riemannian	orthonormal	$\mathrm{O}(n)$
$\mathbb{C}$	Hermitian	orthonormal	$\mathrm{U}(n)$
$\mathbb{R}$	oriented	positively oriented	$\mathrm{GL}^+(n; \mathbb{R})$
$\mathbb{R}$	Riemannian, oriented	orthonormal, oriented	$\mathrm{SO}(n)$

(Here $\mathrm{GL}^+(n;\mathbb{R}) := \{A \in \mathrm{GL}(n;\mathbb{R}) \mid \det(A) > 0\}$).

Remark 2.2.8. Let (P, π, B) be a G-principal bundle. If $\lambda : B \to B'$ is a smooth map, then $\lambda^* P \to B'$ is again a G-principal bundle. The pull-back bundle is given by $\lambda^* P := \{(b', p) \in B' \times P \mid \lambda(b') = \pi(p)\}$ with the right action given by $(b', p)\cdot g := (b', p\cdot g)$.

Next we want to replace the structure group G of a principal bundle: So let $P \to B$ be a G-principal bundle and let $\varphi : G \to H$ be a Lie group homomorphism. Now G acts from the right on $P \times H$ by $(p, h) \cdot g := (p \cdot g, \varphi(g^{-1}) \cdot h)$ (this follows from Remark 1.5.27). The action is free, since $(p, h) \cdot g = (p, h)$ implies $p \cdot g = p$ and thus $g = e$, because the action of G on P is free.
Now if G is compact, then we know from Remark 2.2.3 that $P' := (P \times H)/G$ is a smooth manifold. Actually, this also holds for non-compact Lie groups G. Consider the diagram

$$\begin{tikzcd} P \times H \arrow[r] \arrow[d, "\pi \circ pr_1"'] & P \times_\varphi H \arrow[dl, "\exists ! \pi'"] \\ B & \end{tikzcd}$$

Since $\pi \circ pr_1$ is constant along the G-orbits, there is a unique map π' making the diagram commutative. It is smooth by the general theory of group actions.
In fact, P' is the total space of an H-principal bundle over B. H acts from the right on $P \times H$ by $(p, h) \cdot h' = (p, hh')$. Since this H-action commutes with the G-action, it descends to an action on $P \times H/G$ by

$$[p, h] \cdot h' = \left[p, hh'\right]$$

This H-action on $P \times_\varphi G$ is free:

$$\begin{aligned}
\left[p, hh'\right] = [p, h] \cdot h' = [p, h] &\Rightarrow \exists\, g \in G : \left(p, hh'\right) = \left(p \cdot g, \varphi(g^{-1})h\right) \\
&\Rightarrow p = p \cdot g \\
&\Rightarrow g = e \\
&\Rightarrow hh' = \varphi\left(e^{-1}\right) h = h \\
&\Rightarrow h' = e
\end{aligned}$$

Conclusion 2.2.9
$(P \times_\varphi H, \pi', B)$ is an H-principal bundle.

We proved this for the case that H is compact but it is also true for the general case.

Definition 2.2.10. $(P \times_\varphi H, \pi', B)$ is called the **H-*principal bundle associated to*** (P, π, B) ***with respect to*** φ.

If $\varphi : G \to H$ is an embedding of a subgroup, then one says that $(P \times_\varphi H, \pi', B)$ is obtained from (P, π, B) by ***extension of the structure group***. Conversely, given an H-principal bundle $Q \to B$, a G-principal bundle $P \to B$ such that its extension to an H-principle bundle is isomorphic to $Q \to B$ is called a ***reduction to the structure group*** G.

Example 2.2.11. An H-principal bundle can be reduced to the trivial group $G = \{e\}$ if and only if it is trivial.

Now let $P \xrightarrow{\pi} B$ be a G-principal bundle and let $\rho : G \to \mathrm{Aut}(V)$ be a representation. We define

$$P \times_\rho V := P \times V/G$$

where G acts as before: $(p, v) \cdot g = (p \cdot g, \rho(g^{-1})v)$. The same construction now yields a vector bundle

$$P \times_\rho V \to B$$

Definition 2.2.12. $P \times_\rho V$ is called the ***associated vector bundle***.

Example 2.2.13. If E is a $\mathbb{K}$-vector bundle, let P be its frame bundle with structure group $G = GL(n; \mathbb{K})$. If ρ_{std} is the standard representation of G on $\mathbb{K}^n$, then we have the following isomorphism of vector bundles:

$$P \times_{\rho_{std}} \mathbb{K}^n \cong E,$$
$$[(b_1, \dots, b_n), (x_1, \dots, x_n)] \mapsto \sum_{j=1}^n x_j b_j.$$

This map is well-defined because

$$\begin{aligned}[b,x] = [b',x'] &\Rightarrow \exists\, g \in GL(n;\mathbb{K}) : (b',x') = \left(b \cdot g, \rho_{std}\left(g^{-1}\right) x\right) = \left(b \cdot g, g^{-1}x\right) \\ &\Rightarrow [b',x'] \mapsto b \cdot g \cdot g^{-1}x = b \cdot x\end{aligned}$$

Remark 2.2.14. Let $E \to B$ be a vector bundle and P its frame bundle (again, $G = GL(n;\mathbb{K})$). Taking the k-th exterior power of the standard representation, $\rho := \wedge^k \rho_{std}$, we have

$$P \times_\rho \Lambda^k \mathbb{K}^n \cong \Lambda^k E$$

This works as well for tensor products, direct sums, dual representations etc.

In the following, we are going to discuss the local description of principal bundles. Let $P \xrightarrow{\pi} B$ be a G-principal bundle and $U \subset B$ an open set such that there is a local trivialization

$$\psi_U : \pi^{-1}(U) =: P|_U \to U \times G,$$

that is, the following diagram commutes

$$\begin{tikzcd} P|_U \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "pr_1"'] \\ U & \end{tikzcd}$$

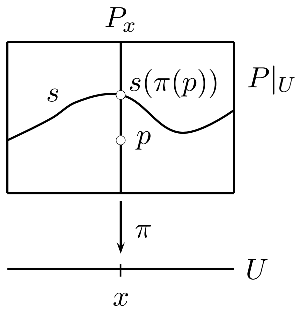
Define a local section $s : U \to P|_U$ by $s(x) := \psi^{-1}(x,e)$, which is obviously smooth. By the definition of principal fibre bundle, we have

$$\begin{tikzcd} \pi^{-1}(U) \times G \arrow[r, "\psi_U \times id_G"] \arrow[d, "\psi_U \times \mathrm{id}_G"] & \pi^{-1}(U) \arrow[d, "\psi_U"] \\ U \times G \times G \arrow[r, "\mathrm{id}_U \times \mu_G"'] & U \times G \end{tikzcd}$$

This implies, that the local trivialization may be expressed using the section s:

$$\psi_U^{-1}(u,g) = \psi_U^{-1}(u,eg) = \psi_U^{-1}(u,e) \cdot g = s(x) \cdot g$$

Conversely, let $s : U \to P|_U$ be a smooth section. For any $p \in P_x$, there exists a unique $g(p) \in G$ such that $p = s(\pi(p)) \cdot g(p)$, because the group action is free and transitive.
Define $\psi_U(p) := (\pi(p), g(p)) \in U \times G$. This is a local trivialization.



Conclusion 2.2.15
There is a 1-1 correspondence

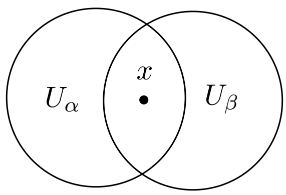
$$\textit{local trivializations} \;\leftrightarrow\; \textit{local sections}$$

In particular, a principal bundle has global sections if and only if it is trivial.

Example 2.2.16. The Hopf bundle $S^3 \to S^2$, $G = U(1)$ has no global section because otherwise, it would be trivial. So in particular $S^3 \cong S^2 \times S^1$. But this would imply

$$\{e\} \cong \pi_1\left(S^3\right) \cong \pi_1\left(S^2 \times S^1\right) \cong \pi_1\left(S^2\right) \times \pi_1\left(S^1\right) \cong \{e\} \times \mathbb{Z} = \mathbb{Z} \qquad ↯$$

Now cover B by open sets U_α, $\alpha \in A$, such that $P|_{U_\alpha}$ is trivial and we can choose sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$. We consider the intersection $U_\alpha \cap U_\beta$:



For $x \in U_\alpha \cap U_\beta =: U_{\alpha\beta}$:

$$\exists!\; g_{\alpha\beta}(x) \in G \text{ such that } s_\beta(x) = s_\alpha(x) \cdot g_{\alpha\beta}(x)$$

This yields smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ (so called ***transition functions***) satisfying the ***cocycle conditions***:

1. $g_{\alpha\alpha} = e$
2. $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$
3. $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = e$

The first identity is trivial. By the definition of $g_{\alpha\beta}$ and $g_{\beta\alpha}$, we have

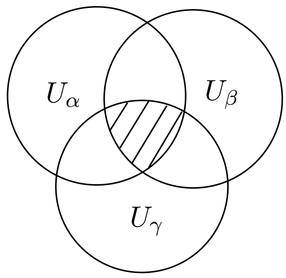
$$s_\beta = s_\alpha\, g_{\alpha\beta} = s_\beta\, g_{\beta\alpha}\, g_{\alpha\beta}.$$

This implies the second identity because the group action is free.

To prove the third equation, let $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Similar to the second part, we have

$$s_\alpha = s_\beta\, g_{\beta\alpha} = s_\gamma\, g_{\gamma\beta}\, g_{\beta\alpha} = s_\alpha\, g_{\alpha\gamma}\, g_{\gamma\beta}\, g_{\beta\alpha}$$

which implies the claim since G acts freely.



If we choose a second set of local sections, $\tilde{s}_\alpha : U_\alpha \to P|_{U_\alpha}$, we obtain corresponding transition functions $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to G$. We may now ask: What is the relation between $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$?

For any $x \in U_\alpha$, we have:

$$\begin{aligned} &\exists!\, h_\alpha(x) \in G : \tilde{s}_\alpha(x) = s_\alpha(x)\, h_\alpha(x) \\ \Rightarrow\ & s_\beta\, h_\beta = \tilde{s}_\beta = \tilde{s}_\alpha\, \tilde{g}_{\alpha\beta} = s_\alpha\, h_\alpha\, \tilde{g}_{\alpha\beta} = s_\beta\, g_{\beta\alpha}\, h_\alpha\, \tilde{g}_{\alpha\beta} \\ \Rightarrow\ & h_\beta = g_{\beta\alpha}\, h_\alpha\, \tilde{g}_{\alpha\beta} \end{aligned}$$

Hence, we obtain the ***coboundary condition***:

$$g_{\alpha\beta} = h_\alpha\, \tilde{g}_{\alpha\beta}\, h_\beta^{-1} \tag{2.1}$$

We will now discuss the construction of a principal fibre bundle out of prescribed local data. Let B be a smooth manifold and $\{U_\alpha\}$ an open covering. Let $g_{\alpha\beta} : U_{\alpha\beta} \to G$ be smooth maps such that the cocycle conditions 1., 2., and 3. hold. We construct the total space of the bundle a follows:

$$P := \bigsqcup_\alpha U_\alpha \times G \Big/ \sim$$

Here, the equivalence relation $\sim$ is given by

$$\underbrace{(x, g)}_{\in U_\alpha \times G} \sim \underbrace{(x', g')}_{\in U_\beta \times G} \quad :\Leftrightarrow \quad \begin{array}{rrcll} i) & x & = & x' & \text{and} \\ ii) & g & = & g_{\alpha\beta}(x) \cdot g' & \end{array}$$

Since $s_\alpha(x)\cdot g$ and $s_\beta(x')\cdot g'$ are supposed to represent the same element in the resulting bundle, property ii) of the preceeding definition is motivated by the following computation:

$$s_\beta(x)g_{\beta\alpha}(x)\cdot g = s_\alpha(x)\cdot g \overset{!}{=} s_\beta\left(x'\right)\cdot g' = s_\beta(x)\cdot g'$$

The definition yields a G-principal bundle $P\to B$. This way, one reconstructs a given G-principle bundle P from a system of transition functions (up to isomorphism). The cocycle conditions are needed to ensure that $\sim$ indeed defines an equivalence relation. If we have two systems of transition functions $\{g_{\alpha\beta}\}$, $\{\tilde{g}_{\alpha\beta}\}$ and a system of maps $\{h_\alpha : U_\alpha \to G\}$ such that the coboundary conditions (2.1) hold, then the corresponding G-principal bundles P and $\tilde{P}$ are isomorphic. An isomorphism is given by the map

$$\begin{array}{rcl} P := \coprod_\alpha U_\alpha \times G/\sim & \longrightarrow & \tilde{P} := \coprod_\alpha U_\alpha \times G/\approx \\ {[x,g]} & \mapsto & [x, h_\alpha^{-1}(x)g] \end{array}$$

It is well-defined: If $[x,g]\sim[x,g']$, that is $g = g_{\alpha\beta}g' = h_\alpha \tilde{g}_{\alpha\beta}h_\beta^{-1}g'$ by using (2.1), we see that $\tilde{g} = h_\alpha^{-1}g$ and $\tilde{g}' = h_\beta^{-1}g'$ are related by $\tilde{g} = \tilde{g}_{\alpha\beta}\tilde{g}'$ and thus, $[x,\tilde{g}]\approx[x,\tilde{g}]'$.

Example 2.2.17. We are going to determine transition functions for the Hopf bundle:

$$\begin{aligned} \mathrm{Hopf} : S^3 \subset \mathbb{C}^2 &\longrightarrow S^2 \subset \mathbb{C}\times\mathbb{R} \\ (w_1,w_2) &\mapsto \frac{\left(4w_1\overline{w}_2, 4|w_2|^2 - |w_1|^2\right)}{4|w_2|^2+|w_1|^2} \end{aligned}$$

For $(z,t)\in S^2$, that is $|z|^2+t^2=1$, we define

$$s_1(z,t) := \left(\frac{4|z|^2}{(1+t)^2}+1\right)^{-1/2}\left(\frac{2z}{1+t},1\right)$$

s_1 is a smooth and defined on $U_1 := S^2\setminus\{(0,-1)\}$. Furthermore, s_1 is a section because

$$\begin{aligned} \mathrm{Hopf}(s_1(z,t)) &= \frac{\left(4\cdot\frac{2z}{1+t}\cdot\overline{1}, 4\cdot 1^2 - \left|\frac{2z}{1+t}\right|^2\right)}{4\cdot 1^2 + \left|\frac{2z}{1+t}\right|^2} &&= \frac{(2z(1+t),(1+t)^2-|z|^2)}{(1+t)^2+|z|^2} \\ &= \frac{(2z(1+t),1+2t+t^2-|z|^2)}{1+2t+t^2+|z|^2} &&= (z,t) \end{aligned}$$

In the last step, we used $|z|^2+t^2=1$. Analogously, we define a smooth section on $U_2 := S^2\setminus\{(0,1)\}$:

$$s_2(z,t) := \left(1+\frac{|z|^2/4}{(1-t)^2}\right)^{-1/2}\left(1,\frac{\overline{z}/2}{1-t}\right)$$

We calculate the transition function g_{12} with respect to these sections using $(1-t)(1+t) = 1-t^2 = |z^2|$ and $(1+t)^2 = |z|^4/(1-t)^2$:

$$\begin{aligned}
s_1(z,t)\cdot\frac{\overline{z}}{|z|} &= \left(\frac{4|z|^2}{(1+t)^2}+1\right)^{-1/2}\left(\frac{2|z|}{1+t},\frac{z}{|z|}\right)\\
&= \left(\frac{4|z|^2}{(1+t)^2}+1\right)^{-1/2}\frac{2|z|}{1+t}\left(1,\frac{(1+t)\overline{z}}{2|z|^2}\right)\\
&= \frac{2|z|}{\sqrt{4|z|^2+(1+t)^2}}\left(1,\frac{\overline{z}/2}{1-t}\right)\\
&= \left(1+\frac{(1+t)^2}{4|z|^2}\right)^{-1/2}\left(1,\frac{\overline{z}/2}{1-t}\right)\\
&= \left(1+\frac{|z|^2/4}{(1-t)^2}\right)^{-1/2}\left(1,\frac{\overline{z}/2}{1-t}\right)\\
&= s_2(z,t)
\end{aligned}$$

Hence, the transition function is given by

$$g_{12} : U_{12} = S^2 \setminus \{(0,1),(0,-1)\} \to \mathrm{U}(1), \qquad g_{12}(z,t) = \frac{\overline{z}}{|z|} = \frac{|z|}{z}.$$

Remark 2.2.18. Let $P \to B$ be a G-principal bundle, $\varphi : G \to H$ a Lie group homomorphism and $P' := P \times H/G$ the associated H-principal bundle. Let $s_\alpha : U_\alpha \to P|_{U_\alpha}$ be local sections with corresponding transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$. Then we have induced sections of the associated bundle:

$$\begin{aligned}
s'_\alpha &: U_\alpha \to P'|_{U_\alpha}\\
s'_\alpha(u) &:= [s_\alpha(u), e]
\end{aligned}$$

On one hand we have

$$s'_\beta = s'_\alpha g'_{\alpha\beta} = [s_\alpha, e]\, g'_{\alpha\beta} = \left[s_\alpha, g'_{\alpha\beta}\right]$$

and on the other hand:

$$s'_\beta = [s_\beta, e] = [s_\alpha g_{\alpha\beta}, e] = \left[s_\alpha g_{\alpha\beta} g_{\beta\alpha}, \varphi(g_{\beta\alpha}^{-1})e\right] = [s_\alpha, \varphi(g_{\alpha\beta})]$$

Since the action of G on P is free, we conclude

$$g'_{\alpha\beta} = \varphi \circ g_{\alpha\beta}$$

Thus forming the associated bundle with respect to φ amounts to composing the transition functions with φ.

Remark 2.2.19. We can think of local sections (or equivalently, local trivializations) as *local gauges*. For example, if $G = \mathbb{R}$, then a local section identifies points in a fibre P_b with real numbers. If elements of fibres P_b are results of measurements, then the choice of local sections correspond to the choice of a system of units.

2.3 Connections

Definition 2.3.1. Let $P \to B$ be a G-principal bundel. A 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ (i.e. a section of $T^*P \otimes \mathfrak{g}$) is called a ***connection 1-form*** iff

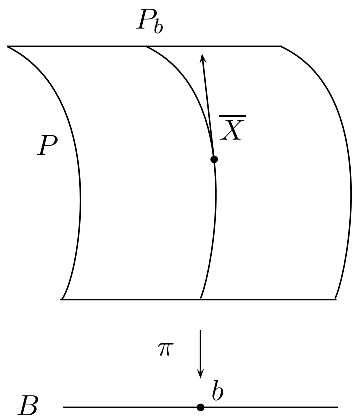
1. $R_g^* = \mathrm{Ad}_{g^{-1}} \circ \omega \quad \forall\, g \in G$ or equivalently, the following diagram commutes:

$$\begin{tikzcd} T_pP \arrow[r, "\omega_p"] \arrow[d, "dR_g"'] & \mathfrak{g} \arrow[d, "\mathrm{Ad}_{g^{-1}}"] \\ T_{p\cdot g}P \arrow[r, "\omega_{p\cdot g}"] & \mathfrak{g} \end{tikzcd}$$

2. For any $X \in \mathfrak{g}$, let $\overline{X}$ be the corresponding fundamental vector field on P. Then:

$$\omega(\overline{X}(p)) = X$$

All the vectors tangent to the fibres P_b are given in the form $\overline{X}$ for a suitible $X \in \mathfrak{g}$. Thus, the second condition determines the values of a connection 1-form ω for the vectors tangent to the fibers.



Remark 2.3.2. The two conditions 1. and 2. are compatible. On one hand, we have

$$\omega\left(dR_g\left(\overline{X}(p)\right)\right) \overset{(1)}{=} \mathrm{Ad}_{g^{-1}}(\omega(\overline{X}(p))) \overset{(2)}{=} \mathrm{Ad}_{g^{-1}}(X)$$

On the other hand, we have

$$dR_g\left(dL_p(X)\right) = \frac{d}{dt}\bigg|_0 (p \cdot \exp(tX)g) = \frac{d}{dt}\bigg|_0 \left(p \cdot gg^{-1}exp(tX)g\right) = dL_{pg}\left(\mathrm{Ad}_{g^{-1}}(X)\right)$$

which implies

$$\begin{aligned}\omega\left(dR_g(\overline{X}(p))\right) &= \omega\left(dR_g\left(dL_p(X)\right)\right) = \omega\left(dL_{pg}\left(\mathrm{Ad}_{g^{-1}}(X)\right)\right)\\ &= \omega\left(\overline{\mathrm{Ad}_{g^{-1}}(X)}(pg)\right) = \mathrm{Ad}_{g^{-1}}(X)\end{aligned}$$

Hence, 1. and 2. are consistent. Changing $\mathrm{Ad}_{g^{-1}}$ to Ad_g would lead to inconsistent conditions.

Example 2.3.3 (Fundamental example). Let $V \to B$ be a vector bundle with a connection (i.e. covariant derivative) ∇. Let $P \xrightarrow{\pi} B$ be the frame bundle of V, $G = GL(n;\mathbb{K})$ and $\mathfrak{g} = Mat(n \times n;\mathbb{K})$. Let $X \in T_pP$ be a vector tangent to P and choose a curve $t \mapsto p(t) = (p_1(t), \ldots, p_n(t))$ such that $\dot{p}(0) = X$ and $p(0) = p$. Putting $c(t) := \pi(p(t))$, we have $p_j(t) \in V_{c(t)}$. Hence p_j is a section in c^*V and its covariant derivative along c can again be expressed in terms of the basis p:

$$\frac{\nabla}{dt}\bigg|_0 p_j(t) =: \sum_{i=1}^{n} \Gamma^i_j(x) \cdot p_i(0) =: (p(0) \cdot \omega(X))_j$$

Here, $\omega(X) \in Mat(n \times n;\mathbb{K})$ is defined by its action on the basis $p(0)$ of $V_{c(0)}$. Since $\frac{\nabla}{dt}\big|_0 p_j(t)$ depends only on X and not on the particular choice of the curve $p(t)$, also $\omega(X)$ is independent of this choice.

Check of property 2: Let $X \in \mathfrak{g} = Mat(n \times n;\mathbb{K})$ and put $p(t) := p \cdot \exp(tX)$. Then we have $\dot{p}(0) = \overline{X}(p)$. Thus

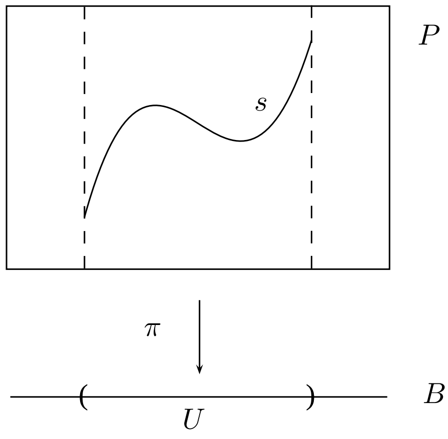
$$p(0) \cdot \omega(\overline{X}(p)) = \frac{\nabla}{dt}\bigg|_0 p(t) = \dot{p}(0) = p(0) \cdot X \quad \Rightarrow \quad \omega(\overline{X}(p)) = X$$

Here, we used, that $p(t)$ is a curve in a fixed fiber of P and therefore, the covariant derivate ist just an ordinary derivate.

Check of property 1: Let $X \in T_pP$ and choose a curve $p(t)$ such that $p(0) = p$ and $\dot{p}(0) = X$. We obtain

$$\begin{aligned} & & dR_g(X) &= \frac{d}{dt}\bigg|_0 R_g(p(t)) = \frac{d}{dt}\bigg|_0 (p(t) \cdot g) = \dot{p}(0) \cdot g\\ &\Rightarrow & p(0) \cdot g \cdot (dR_g(X)) &= \frac{\nabla}{dt}\bigg|_0 (p(t) \cdot g) = \frac{\nabla}{dt}\bigg|_0 p(t) \cdot g = p(0) \cdot \omega(X) \cdot g\\ &\Rightarrow & g \cdot \omega\left(dR_g(X)\right) &= \omega(X) \cdot g\\ &\Rightarrow & \omega\left(dR_g(X)\right) &= g^{-1}\omega(X)g = \mathrm{Ad}_{g^{-1}}(\omega(X))\end{aligned}$$

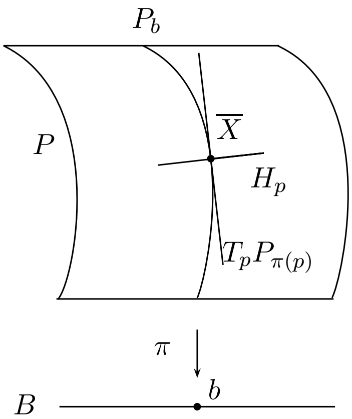
Remark 2.3.4
Let $P \xrightarrow{\pi} B$ be a G-principal bundle with connection 1-form ω. Let $s : U \subset B \to P|_U$ be a local section.
Then $s^*\omega \in \Omega(U, \mathfrak{g})$ is given by $s^*\omega(Y) = \omega(ds(Y))$. If $V = TB \to B$ and $x^1, \ldots, x^n$ are local coordinates of the manifold on U, then $s := (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$ is a local section of the frame bundle of TB.



The usual Christoffel symbols are then given by

$$\Gamma^j_{ik} := \Gamma^j_i \left(ds \left(\frac{\partial}{\partial x^k} \right) \right) = s^*\omega \left(\frac{\partial}{\partial x^k} \right)^j_i \in \mathfrak{g} = Mat(n \times n; \mathbb{K})$$

Let $\omega \in \Omega^1(P; \mathfrak{g})$ be a connection 1-form on $P \xrightarrow{\pi} B$. For a fixed $p \in P$, the restriction of the linear map $\omega_p : T_pP \to \mathfrak{g}$ to the tangent space of the fiber yields an isomorphism $\omega_p|_{T_pP_{\pi(p)}} : T_pP_{\pi(p)} \to \mathfrak{g}$.



Setting $H_p := \ker(\omega_p)$, we have the decomposition

$$H_p \oplus T_pP_{\pi(p)} = T_pP.$$

In particular, $\dim(H_p) = \dim(P) - \dim(P_{\pi(p)}) = \dim(B)$.
The subspace H_p is called the ***horizontal subspace*** .

For $X \in H_p$, we find

$$\omega_{pg} \left(dR_g(X) \right) = \left(R_g^* \omega \right) (X) \overset{1.}{=} Ad_{g^{-1}} (\underbrace{\omega(X)}_{=0}) = 0\,.$$

Hence $dR_g(H_p) \subset H_{pg}$. Since dR_g is a linear isomorphism and $\dim(H_p) = \dim(H_{pg}) = \dim(B)$, we conclude $dR_g(H_p) = H_{pg}$.

Example 2.3.5. The Hopf bundle $S^3 \to S^2$ has structure group $G = \mathrm{U}(1)$ with the Lie algebra $\mathfrak{g} = \mathfrak{u}(1) = i\mathbb{R}$. For a fixed $p \in S^3 \subset \mathbb{C}^2$ and $X = i \in \mathfrak{u}(1)$, the fundamental vector is given by $\bar{X}(p) = \frac{d}{dt}\Big|_{t=0}(p \cdot e^{it}) = p \cdot i$.
We denote by $\langle \cdot, \cdot \rangle$ the real scalar product on $\mathbb{C}^2 \cong \mathbb{R}^4$. For $p \in S^3$ and $Y \in T_pS^3 \subset \mathbb{R}^4$, set $\omega_p(Y) := i\langle Y, p \cdot i\rangle$. Then $\omega \in \Omega^1(S^3; i\mathbb{R})$ and property 2. from Definition 2.3.1 holds:

$$\begin{aligned} R_z^*\omega(Y) &= \omega_{pz}\left(dR_z(Y)\right) \\ &= \omega_{pz}(z \cdot Y) \\ &= i\langle z \cdot Y, p \cdot z \cdot i\rangle \\ &= i\langle Y, p \cdot i\rangle \qquad (z \text{ acts as isometry}) \\ &= \omega_p(Y)\,. \end{aligned}$$

Hence $R_z^*\omega = \omega = \mathrm{Ad}_{z^{-1}} \circ \omega$, since the adjoint representation is trivial. The horizontal space is given by $H_p = \ker \omega_p = (p \cdot i)^\perp$. Property 1 holds as well:

$$\begin{aligned} \omega(\bar{X}) &= \omega(p \cdot i) \\ &= i \cdot \langle p \cdot i, p \cdot i\rangle \\ &= i. \end{aligned}$$

Local description of connections

Let $P \to B$ be a G-principal bundle. Take an open cover $\{U_\alpha\}_{\alpha \in I}$ of B and choose local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$. We set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. The ***transition functions*** are the uniquely defined functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $s_\beta = s_\alpha \cdot g_{\alpha\beta}$. Let $\omega \in \Omega^1(P, \mathfrak{g})$ be a connection 1-form and set $\omega_\alpha := s_\alpha^*\omega \in \Omega^1(U_\alpha, \mathfrak{g})$. Then we have the following transformation formula:

$$s_\beta(u) = s_\alpha(u) \cdot g_{\alpha\beta}(u) = s_\alpha(u) \cdot g_{\alpha\beta}(u_0) \cdot g_{\alpha\beta}^{-1}(u_0) \cdot g_{\alpha\beta}(u)\,.$$

Differentiating at $u = u_0$ yields:

$$ds_\beta|_{(u_0)} = dR_{g_{\alpha\beta}(u_0)} \circ ds_\alpha|_{u_0} + dL_{s_\alpha(u_0)\cdot g_{\alpha\beta}(u_0)} \circ d\left(g_{\alpha\beta}^{-1}(u_0) \cdot g_{\alpha\beta}\right)|_{u_0}\,.$$

For the locally defined 1-forms we get:

$$\begin{aligned} \omega_\beta|_{u_0} &= s_\beta^*\omega|_{u_0} \\ &= \omega \circ ds_\beta|_{u_0} \qquad (2.2) \\ &= \omega \circ dR_{g_{\alpha\beta}(u_0)} \circ ds_\alpha|_{u_0} + \omega \circ dL_{s_\alpha(u_0)\cdot g_{\alpha\beta}(u_0)} \circ d\left(g_{\alpha\beta}^{-1}(u_0) \cdot g_{\alpha\beta}\right)|_{u_0} \\ &\overset{1.,2.}{=} \mathrm{Ad}_{g_{\alpha\beta}^{-1}(u_0)} \circ \omega \circ ds_\alpha|_{u_0} + d\left(g_{\alpha\beta}^{-1}(u_0) \cdot g_{\alpha\beta}\right)|_{u_0} \\ &= \mathrm{Ad}_{g_{\alpha\beta}^{-1}(u_0)} \circ \omega_\alpha|_{u_0} + d\left(g_{\alpha\beta}^{-1}(u_0) \cdot g_{\alpha\beta}\right)|_{u_0}\,. \qquad (2.3) \end{aligned}$$

Example 2.3.6. The connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ is uniquely defined by the collection of (locally defined) 1-forms $(\omega_\alpha)_{\alpha \in I}$.
By condition 2., ω_α determines ω on U_α at the points $s_\alpha(u)$, $u \in U_\alpha$. By condition 1., it already determines ω for all points of the corresponding fiber and hence on $P|_{U_\alpha}$.
If a collection $(\omega_\alpha)_{\alpha \in I}$, $\omega_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ is given such that (2.2) holds for all $\alpha, \beta \in I$, then this defines a unique connection 1-form $\omega \in \Omega^1(P; \mathfrak{g})$.

Example 2.3.7. Each principal bundle has connection 1-forms since one can use a partition of unity to construct them out of locally defined 1-forms. Our next question is: how many connection 1-forms are there on a given principal bundle?

Let $\omega, \tilde{\omega}$ be connection 1-forms on $P \to B$. Let $\omega_\alpha, \tilde{\omega}_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ be the corresponding locally defined 1-forms associated with the local sections s_α. Then

$$\omega_\beta - \tilde{\omega}_\beta = \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ (\omega_\alpha - \tilde{\omega}_\alpha).$$

For any vector field X on B, we look at the local section $[s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)(X)]$ of the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$. We observe that on $U_{\alpha\beta}$

$$[s_\beta, (\omega_\beta - \tilde{\omega}_\beta)(X)] = \left[s_\alpha g_{\alpha\beta}, \mathrm{Ad}_{g_{\alpha\beta}^{-1}} \circ (\omega_\alpha - \tilde{\omega}_\alpha)(X)\right] = [s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)(X)].$$

Hence $[s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)(X)]$ is the restriction of a globally defined section of $P \times_{\mathrm{Ad}} \mathfrak{g}$. Putting $[s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)](X) := [s_\alpha, (\omega_\alpha - \tilde{\omega}_\alpha)(X)]$ we get a globally well-defined 1-form on B with values in $P \times_{\mathrm{Ad}} \mathfrak{g}$, i.e., a section of $T^*B \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})$. Hence the differences $\omega - \tilde{\omega}$ of any two connection 1-forms on P correspond to elements of $\Omega^1(B; P \times_{\mathrm{Ad}} \mathfrak{g})$.

Remark 2.3.8. Note that the space $\mathcal{C}(P) := \{\text{connection 1-forms on } P \to B\}$ is *not* a vector space, because $0 \notin \mathcal{C}(P)$. We have thus found that $\mathcal{C}(P)$ is an affine space over the vector space $\Omega^1(B; P \times_{\mathrm{Ad}} \mathfrak{g})$. In particular, it is an infinite-dimensional affine space.

Remark 2.3.9. Let $P \xrightarrow{\pi} B$ be a G-principal bundle with connection 1-form ω. Let $\varrho : G \to \mathrm{Aut}(V)$ be a representation of G. Let $E := P \times_\varrho V \to B$ be the associated vector bundle. We construct a covariant derivative ∇ on E out of the connection 1-form ω as follows: For $X \in T_{u_0}B$, set

$$\nabla_X [p(u), v(u)] := [p(u_0), \partial_X v + \varrho_* (p^*\omega(X))\, v(u_0)].$$

A simple computation shows that this is well-defined. Indeed,

$$[p(u), v(u)] = \left[p(u) \cdot g(u), \varrho\left(g^{-1}(u)\right) v(u)\right]$$

yields

$$\begin{aligned} &[p(u_0), \partial_X v(u_0) + \varrho_* (p^*\omega(X))\, v(u_0)] \\ = \; &\left[p(u_0) \cdot g(u_0), \partial_X \left(\varrho\left(g^{-1}(u)\right) v(u)\right) + \varrho_* ((pg)^*\omega(X))\, \varrho\left(g^{-1}(u)\right) v(u)\right]. \end{aligned}$$

2.4 Curvature

Let $P \to B$ be a G-principal bundle with connection 1-form ω. For any $p \in P$, we have the decomposition $T_pP = T_pP_{\pi(p)} \oplus H_p$. Let $\pi_H : T_pP \to H_p$ be the horizontal projection.

Definition 2.4.1. The 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ given by $\Omega(X,Y) := d\omega(\pi_H(X), \pi_H(Y))$ is called the ***curvature form*** of ω.

Notation: For $\eta, \varphi \in \Omega^1(P, \mathfrak{g})$ we define $[\eta, \varphi](X,Y) := [\eta(X), \varphi(Y)] - [\eta(Y), \varphi(X)]$.

Proposition 2.4.2 (Structure equations)
Let $P \to B$ be a G-principal bundle with connection 1-form ω. Then the curvature form Ω satisfies

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]\,. \tag{2.4}$$

Proof. We check the formula by inserting $X, Y \in T_pP$, distinguishing the following different cases:

(i) If X, Y are fundamental vector fields, i.e. $X = \bar{X}'$ and $Y = \bar{Y}'$ for $X', Y' \in \mathfrak{g}$, then $\Omega(X,Y) = d\omega(\pi_H(X), \pi_H(Y)) = 0$. On the other hand, $[\omega,\omega](X,Y) = 2[\omega(X), \omega(Y)] = 2[X', Y']$ and (using that X' and Y' are constant)

$$\begin{aligned} d\omega(X,Y) &= \partial_X \omega(Y) - \partial_Y \omega(X) - \omega([X,Y]) \\ &= \partial_X Y' - \partial_Y X' - \left[X', Y'\right] \\ &= -\left[X', Y'\right] \\ &= -\frac{1}{2}[\omega,\omega](X,Y)\,. \end{aligned}$$

□

(ii) If X, Y are horizontal, then $[\omega,\omega](X,Y) = 2[\omega(X), \omega(Y)] = 0$ and $\Omega(X,Y) = d\omega(\pi_H(X), \pi_H(Y)) = d\omega(X,Y)$.

(iii) If X is a fundamental vector field, i.e. $X = \bar{X}'$, $X' \in \mathfrak{g}$ and Y is horizontal, then

$$[\omega,\omega](X,Y) = 2[\omega(X), \underbrace{\omega(Y)}_{=0}] = 0$$

and

$$d\omega(X,Y) \overset{2.4.3}{=} \partial_X(\underbrace{\omega(Y)}_{=0}) - \underbrace{\partial_Y(\omega(X))}_{=\partial_Y X'=0} - \underbrace{\omega(\underbrace{[X,Y]}_{\text{horizontal}})}_{=0} = 0\,.$$

For the curvature form Ω, we find

$$\Omega(X,Y) = d\omega\big(\underbrace{\pi_H(X)}_{=0}, \pi_H(Y)\big) = 0\,.$$

Lemma 2.4.3
Let $P \to B$ be a G-principal bundle with connection 1-form ω. Let $\bar{X}$ be a fundamental vector field ($X \in \mathfrak{g}$), let Y be a horizontal vector field. Then $[\bar{X}, Y]$ is horizontal.

Proof. The flow of $\bar{X}$ is given by $R_{\exp(tX)}$. Using the Lie derivative, we compute:

$$\begin{aligned}
\omega\left(\left[\bar{X}, Y\right]\right) &= \omega\left(\mathcal{L}_{\bar{X}} Y\right) \\
&= \mathcal{L}_{\bar{X}}\underbrace{(\omega(Y))}_{\equiv 0} - \left(\mathcal{L}_{\bar{X}}\omega\right)(Y) \\
&= -\frac{d}{dt}\Big|_{t=0} R^*_{\exp(tX)}\omega(Y) \\
&= -\frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)^{-1}} \circ \underbrace{\omega(Y)}_{\equiv 0} \\
&= 0\,.
\end{aligned}$$

□

Lemma 2.4.4
For any $g \in G$, we have:

$$R_g^*\Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega\,. \tag{2.5}$$

Proof. For any tangent vectors X, Y, we have:

$$\begin{aligned}
\left(R_g^*\Omega\right)(X,Y) &= \Omega\left(dR_g(X), dR_g(Y)\right) \\
&= d\omega\left(\pi_H \circ dR_g(X), \pi_H \circ dR_g(Y)\right) \\
&= d\omega\left(dR_g \circ \pi_H(X), dR_g \circ \pi_H(Y)\right) \\
&= \left(R_g^* d\omega\right)(\pi_H X, \pi_H Y) \\
&= d\left(R_g^*\omega\right)(\pi_H X, \pi_H Y) \\
&= d\left(\mathrm{Ad}_{g^{-1}} \circ \omega\right)(\pi_H X, \pi_H Y) \\
&= \mathrm{Ad}_{g^{-1}} d\omega\left(\pi_H X, \pi_H Y\right) \\
&= \mathrm{Ad}_{g^{-1}} \Omega(X,Y)\,.
\end{aligned}$$

Here we used that dR_g preserves the splitting of TP into the horizontal and vertical part, i.e. $dR_g \circ \pi_H = \pi_H \circ dR_g$. □

Proposition 2.4.5 (Bianchi identity)
Let ω be a connection 1-form on a G-principal bundle $P \to B$ and let Ω be the curvature of ω. Then $d\Omega$ vanishes on $H \times H \times H$.

Proof. Since $d\Omega = dd\omega + \frac{1}{2}d[\omega,\omega] = \frac{1}{2}d[\omega,\omega]$, we need to show that $\frac{1}{2}d[\omega,\omega]$ vanishes on $H \times H \times H$. For $\eta := [\omega,\omega] \in \Omega^2(P;\mathfrak{g})$, we know that $\eta(X,Y) = 0$ if X or Y is horizontal. For horizontal vectors X_1, X_2, X_3 we thus have:

$$\begin{aligned}
d\eta\left(X_1, X_2, X_3\right) &= \partial_{X_1}\eta\left(X_2, X_3\right) - \partial_{X_2}\eta\left(X_1, X_3\right) + \partial_{X_3}\eta\left(X_1, X_2\right) \\
&\quad -\eta\left(\left[X_1, X_2\right], X_3\right) + \eta\left(\left[X_1, X_3\right], X_2\right) - \eta\left(\left[X_2, X_3\right], X_1\right) \\
&= 0\,.
\end{aligned}$$

□

Remark 2.4.6. If G is abelian, then the structure equation yields $\Omega = d\omega$ and thus

$$d\Omega = 0\,. \tag{2.6}$$

Let us now consider the implications of the structure equation in terms of the local data describing the bundle and connection by transition functions $g_{\alpha\beta}$ and local 1-forms ω_α. So we cover B by $\{U_\alpha\}_{\alpha\in I}$, we choose local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$ which define the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ such that $s_\beta = s_\alpha \cdot g_{\alpha\beta}$. For the connection 1-form ω

and curvature Ω, we set $\omega_\alpha := s_\alpha^*\omega$ and $\Omega_\alpha := s_\alpha^*\Omega$. The structure equation in the local data reads:

$$\begin{aligned}\Omega_\alpha &= s_\alpha^*\Omega \\ &= s_\alpha^*\left(d\omega + \frac{1}{2}[\omega,\omega]\right) \\ &= ds_\alpha^*\omega + \frac{1}{2}\left[s_\alpha^*\omega, s_\alpha^*\omega\right] \\ &= d\omega_\alpha + \frac{1}{2}\left[\omega_\alpha,\omega_\alpha\right] . \end{aligned} \tag{2.7}$$

If G is abelian, then $\Omega_\alpha = d\omega_\alpha$ depends *linearly* on ω_α. In general, (2.7) is a *semilinear* partial differential equation of first order for ω_α.
Now let us compute the transformation behaviour of the local curvature forms Ω_α under the transitions between the open sets from the cover $\{U_\alpha\}_{\alpha\in I}$. Differentiating $s_\beta = s_\alpha \cdot g_{\alpha\beta}$ at $u_0 \in U_{\alpha\beta}$, we get:

$$\begin{aligned} ds_\beta|_{u_0} &= dR_{g_{\alpha\beta}(u_0)} \circ ds_\alpha + dL_{s_\alpha(u_0)} \circ d\left(g_{\alpha\beta}(u_0)\cdot g_{\alpha\beta}(u_0)^{-1}\cdot g_{\alpha\beta}\right) \\ &= dR_{g_{\alpha\beta}(u_0)} \circ ds_\alpha + dL_{s\alpha(u_0)\cdot g_{\alpha\beta}(u_0)} \circ d\left(g_{\alpha\beta}(u_0)^{-1}\cdot g_{\alpha\beta}\right) . \end{aligned}$$

This yields for the transformation of the local curvature forms:

$$\begin{aligned} \Omega_\beta &= s_\beta^*\Omega \\ &= \Omega \circ ds_\beta \\ &= \Omega \circ dR_{g_{\alpha\beta}(u_0)} \circ ds_\alpha + \Omega \circ \underbrace{dL_{s\alpha(u_0)\cdot g_{\alpha\beta}(u_0)} \circ d\left(g_{\alpha\beta}^{-1}(u_0)\cdot g_{\alpha\beta}\right)}_{\text{vertical}} \\ &= \left(R_{g_{\alpha\beta}(u_0)}^*\Omega\right) \circ ds_\alpha \\ &= \mathrm{Ad}_{g_{\alpha\beta}^{-1}(u_0)} \circ \Omega \circ ds_\alpha \\ &= \mathrm{Ad}_{g_{\alpha\beta}^{-1}(u_0)} \circ \Omega_\alpha . \end{aligned}$$

Hence if G is abelian, then $\Omega_\beta = \Omega_\alpha$ on $U_{\alpha\beta}$. Thus the Ω_α are restrictions of a globally defined 2-form $\bar\Omega \in \Omega^2(B;\mathfrak{g})$, i.e., $\bar\Omega|_{U_\alpha} = \Omega_\alpha$.
In general, the transformation behaviour for the local curvature forms implies that s_α and Ω_α together yield well-defined global sections of the bundle $P\times_{\mathrm{Ad}}\mathfrak{g}$. Indeed for any $X, Y \in T_uB$, we have:

$$[s_\beta, \Omega_\beta(X,Y)] = \left[s_\alpha\cdot g_{\alpha\beta}, \mathrm{Ad}_{g_{\alpha\beta}^{-1}}\Omega_\alpha(X,Y)\right] = [s_\alpha, \Omega_\alpha(X,Y)] .$$

Thus for fixed $X, Y \in \mathfrak{X}(B)$, $[s_\alpha, \Omega_\alpha(X,Y)]$ is the restriction of a globally defined section of $P\times_{\mathrm{Ad}}\mathfrak{g}$. Hence the local 2-forms $[s_\alpha,\Omega_\alpha]$ defined by $[s_\alpha,\Omega_\alpha](X,Y) := [s_\alpha,\Omega_\alpha(X,Y)]$ are the restrictions to U_α of a globally defined 2-form $\bar\Omega$ on B with values in the bundle $P\times_{\mathrm{Ad}}\mathfrak{g}$, i.e. a section of $\Lambda^2T^*B\otimes(P\times_{\mathrm{Ad}}\mathfrak{g})$.

Example 2.4.7. For the Hopf bundle $S^3 \to S^2$, we have the connection 1-form $\omega_p(Y) := i \cdot \langle ip, Y\rangle$. For the curvature, we obtain:

$$\begin{aligned}
\Omega_p(X,Y) &= d\omega_p(X,Y) \\
&= \partial_X\omega(Y) - \partial_Y\omega(X) - \omega([X,Y]) \\
&= i\cdot\langle ip, \partial_X Y\rangle - i\cdot\langle iX, Y\rangle - i\cdot\langle ip, \partial_Y X\rangle + i\cdot\langle iY, X\rangle - i\cdot\langle ip, [X,Y]\rangle \\
&= i\cdot\big(\langle iX, Y\rangle - \underbrace{\langle iY, X\rangle}_{=\langle X, iY\rangle}\big) \\
&= i\cdot\big(\langle iX, Y\rangle - \langle iX, i^2 Y\rangle\big) \qquad (i \text{ acts as isometry}) \\
&= 2i\cdot\langle iX, Y\rangle\,.
\end{aligned}$$

It is easy to see, that Ω_p indeed vanishes on vertical vectors $X = ip$, since $\Omega_p(ip, Y) = 2i\cdot\langle -p, Y\rangle = 0$, because $Y \in T_pS^3 = p^\perp$.

2.5 Characteristic classes

Definition 2.5.1. A multilinear symmetric function $\lambda : \mathfrak{g}\times\cdots\times\mathfrak{g} \to \mathbb{K}$ is called ***invariant*** iff for all $g \in G$ and all $X_1, \ldots, X_k \in \mathfrak{g}$:

$$\lambda\left(\mathrm{Ad}_g(X_1), \ldots, \mathrm{Ad}_g(X_k)\right) = \lambda(X_1, \ldots, X_k)\,.$$

Let $P \to B$ be a G-principal bundle and let $\lambda : \mathfrak{g}\times\cdots\times\mathfrak{g} \to \mathbb{K}$ be an invariant multilinear symmetric function of degree k. Then λ induces a well-defined symmetric multilinear map on each fiber of the bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$

$$([p, X_1], \ldots, [p, X_k]) \mapsto \lambda(X_1, \ldots, X_k)\,.$$

This is well-defined because

$$\begin{aligned}
&\big(\left[pg, \mathrm{Ad}_{g^{-1}}(X_1)\right], \ldots, \left[pg, \mathrm{Ad}_{g^{-1}}(X_k)\right]\big) \\
&\qquad\qquad \mapsto \lambda\left(\mathrm{Ad}_{g^{-1}}(X_1), \ldots, \mathrm{Ad}_{g^{-1}}(X_k)\right) = \lambda(X_1, \ldots, X_k)\,.
\end{aligned}$$

We choose a connection 1-form ω on P. Let $\bar\Omega \in \Omega^2(B; P\times_{\mathrm{Ad}}\mathfrak{g})$ be the corresponding curvature 2-form on B. We then set $\lambda\circ\bar\Omega \in \Omega^{2k}(B, \mathbb{K})$, where

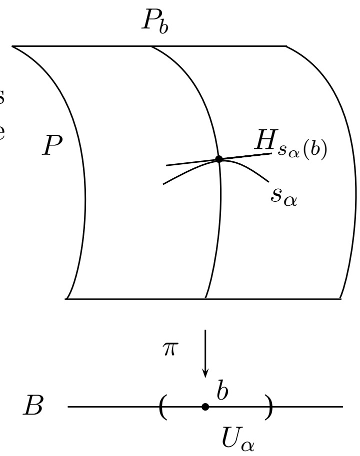
$$(\lambda\circ\bar\Omega)(X_1, \ldots, X_{2k}) := \frac{1}{k!}\sum_{\sigma\in S_{2k}} \mathrm{sign}(\sigma)\cdot\lambda\left(\bar\Omega\left(X_{\sigma(1)}, X_{\sigma(2)}\right), \ldots, \bar\Omega\left(X_{\sigma(2k-1)}, X_{\sigma(2k)}\right)\right)\,.$$

We have the following two important Lemmas:

Lemma 2.5.2
Let $P \xrightarrow{\pi} B$ be a G-principal bundle with connection 1-form ω and curvature form $\bar{\Omega} \in \Omega^2(B; P \times_{\mathrm{Ad}} \mathfrak{g})$. Let $\lambda : \underbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}_{k} \to \mathbb{K}$ be an invariant, symmetric multilinear map. Then

$$d(\lambda \circ \bar{\Omega}) = 0\,.$$

Proof.



Let $\{u_\alpha\}_{\alpha \in I}$ be an open cover of B with local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$. For a given $b \in U_\alpha \subset B$ we can choose the local section s_α such that $ds_\alpha(T_bB) = H_{s_\alpha(b)}$. Then for any $X, Y, Z \in T_bB$, we find:

$$\begin{aligned}
d\Omega_\alpha(X,Y,Z) \quad &= \quad d\left(s_\alpha^* \Omega\right)(X,Y,Z) \\
&= \quad s_\alpha^*(d\Omega)(X,Y,Z) \\
&= \quad (d\Omega)\left(ds_\alpha(X), ds_\alpha(Y), ds_\alpha(Z)\right) \\
&\overset{2.4.5}{=} \quad 0\,.
\end{aligned}$$

Thus $d\Omega_\alpha$ vanishes at b.

Now we have:

$$\begin{aligned}
\lambda\left(\bar{\Omega}\right)(X_1, \ldots, X_{2k}) &= \frac{1}{k!} \sum_{\sigma \in S_{2k}} \mathrm{sign}(\sigma)\, \lambda\left(\bar{\Omega}\left(X_{\sigma(1)}, X_{\sigma(2)}\right), \ldots, \bar{\Omega}\left(X_{\sigma(2k-1)}, X_{\sigma(2k)}\right)\right) \\
&= \frac{1}{k!} \sum_{\sigma \in S_{2k}} \mathrm{sign}(\sigma)\, \lambda\left(\Omega_\alpha\left(X_{\sigma(1)}, X_{\sigma(2)}\right), \ldots, \Omega_\alpha\left(X_{\sigma(2k-1)}, X_{\sigma(2k)}\right)\right).
\end{aligned}$$

Choosing a basis $Y_1, \ldots, Y_N$ of $\mathfrak{g}$ and writing $\Omega_\alpha = \sum_{j=1}^N \Omega_\alpha^j \cdot Y_j$ with $\Omega_\alpha^j \in \Omega^2(U_\alpha; \mathbb{R})$, we obtain:

$$\begin{aligned}
&\lambda\left(\bar{\Omega}\right)(X_1, \ldots, X_{2k}) \\
&= \frac{1}{k!} \sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) \sum_{j_1, \ldots, j_k = 1}^{N} \Omega_\alpha^{j_1}\left(X_{\sigma(1)}, X_{\sigma(2)}\right) \cdots \Omega_\alpha^{j_k}\left(X_{\sigma(2k-1)}, X_{\sigma(2k)}\right) \cdot \lambda\left(Y_{j_1}, \ldots, Y_{j_k}\right) \\
&= \sum_{j_1, \ldots, j_k = 1}^{N} \left(\Omega_\alpha^{j_1} \wedge \ldots \wedge \Omega_\alpha^{j_k}\right)(X_1, \ldots, X_{2k}) \cdot \lambda\left(Y_{j_1}, \ldots, Y_{j_k}\right).
\end{aligned}$$

At the point $b \in U_\alpha$ with the section s_α chosen above, we thus obtain:

$$\begin{aligned}
d\lambda\left(\bar{\Omega}\right) &= d \sum_{j_1,\dots,j_k=1}^{N} \left(\Omega_\alpha^{j_1} \wedge \ldots \wedge \Omega_\alpha^{j_k}\right) \cdot \lambda(Y_{j_1}, \ldots, Y_{j_k}) \\
&= \sum_{j_1,\dots,j_k=1}^{N} \left\{ d\Omega_\alpha^{j_1} \wedge \Omega_\alpha^{j_2} \wedge \ldots \wedge \Omega_\alpha^{j_k} + \ldots + \Omega_\alpha^{j_1} \wedge \ldots \wedge \Omega_\alpha^{j_{k-1}} \wedge d\Omega_\alpha^{j_k} \right\} \lambda(Y_{j_1}, \ldots, Y_{j_k}) \\
&= 0\,.
\end{aligned}$$

Since $b \in B$ was arbitrary, this shows that $d\lambda(\bar{\Omega}) = 0$. □

Lemma 2.5.3
Let ω' be another connection with curvature 2-form $\bar{\Omega}' \in \Omega^2(B; P \times_{\mathrm{Ad}} \mathfrak{g})$. Then $\lambda \circ \bar{\Omega} - \lambda \circ \bar{\Omega}'$ is exact.

Definition 2.5.4. The ***k-th de Rham cohomology*** of M is defined as

$$H^k_{\mathrm{dR}}(M; \mathbb{K}) := \frac{\ker\left(d : \Omega^k(M; \mathbb{K}) \to \Omega^{k+1}(M; \mathbb{K})\right)}{\operatorname{im}\left(d : \Omega^{k-1}(M; \mathbb{K}) \to \Omega^k(M; \mathbb{K})\right)}\,.$$

The number $b_k(M) := \dim_{\mathbb{R}}(H^k; \mathbb{R})$ is called the ***k-th Betti number*** of M.

Note that since $d \circ d \equiv 0$, we have

$$\operatorname{im}\Big(d : \Omega^{k-1}(M; \mathbb{K}) \to \Omega^k(M; \mathbb{K})\Big) \subset \ker\Big(d : \Omega^k(M; \mathbb{K}) \to \Omega^{k+1}(M; \mathbb{K})\Big)\,,$$

so the quotient is well-defined.

Definition 2.5.5. $c_\lambda(P) := [\lambda \circ \bar{\Omega}] \in H^{2k}(B; \mathbb{K})$ is called the ***characteristic class*** of the bundle $P \to B$ associated with λ.

Remark 2.5.6. Lemma 2.5.2 says that $\lambda\circ\bar{\Omega}$ indeed represents a de Rham class. Lemma 2.5.3 in turn says that this class is independent of the choice of connection on $P \to B$.

Remark 2.5.7. Let $P \xrightarrow{\pi} B$ be a G-principal bundle and let $f : M \to B$ be a smooth map. Then the pull-back bundle $f^*P \xrightarrow{\mathrm{pr}_2} P$ fits into the following commutative diagram:

$$\begin{tikzcd} f^*P \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d, "\pi"] \\ M \arrow[r, "f"] & B \end{tikzcd}$$

Let ω be a connection 1-form on P with curvature form Ω. Then $\mathrm{pr}_2^*\omega \in \Omega^1(f^*P;\mathfrak{g})$ is a connection 1-form on the pull-back bundle f^*P: To check property 1. from the definition (i.e. equivariance), we compute:

$$\begin{aligned} R_g^*\,\mathrm{pr}_2^*\omega &= (\mathrm{pr}_2\circ R_g)^*\,\omega \\ &= (R_g\circ \mathrm{pr}_2)^*\,\omega \\ &= \mathrm{pr}_2^*\,R_g^*\omega \\ &= \mathrm{pr}_2^*\,\mathrm{Ad}_{g^{-1}}\omega \\ &= \mathrm{Ad}_{g^{-1}}\,\mathrm{pr}_2^*\omega\,. \end{aligned}$$

As to property 2. (i.e. the evaluation on fundamental vector fields), for any $X \in \mathfrak{g}$, we have:

$$(\mathrm{pr}_2^*\omega)\,(\underbrace{\bar{X}}_{\in\mathfrak{X}(f^*P)}) = \omega(\underbrace{\bar{X}}_{\in\mathfrak{X}(P)}) = X\,.$$

For the curvature form Ω' of $\mathrm{pr}_2^*\omega$, we obtain:

$$\Omega' = d\,(\mathrm{pr}_2^*\omega) + \frac{1}{2}\,[\mathrm{pr}_2^*\omega, \mathrm{pr}_2^*\omega] = \mathrm{pr}_2^*\left(d\omega + \frac{1}{2}[\omega,\omega]\right) = \mathrm{pr}_2^*\Omega\,.$$

Now let $\{U_\alpha\}_{\alpha\in I}$ be an open cover with local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$ for the bundle $P \to B$. Then setting $V_\alpha := f^{-1}(U_\alpha)$ and $s'_\alpha := \mathrm{pr}_2^{-1}\circ s_\alpha\circ f$, we obtain an open cover with local sections for the pull-back bundle $f^*P \to M$. The local curvature forms Ω'_α for f^*P now read:

$$\Omega'_\alpha = (s'_\alpha)^*\,\Omega' = f^*\circ s_\alpha^*\circ(\mathrm{pr}_2^{-1})^*\,\mathrm{pr}_2^*\,\Omega = f^*\,\Omega_\alpha\,.$$

Thus pulling back the form $\lambda(\bar{\Omega}) \in \Omega^{2k}(B;\mathbb{K})$ along f to $\Omega^{2k}(M;\mathbb{K})$, we obtain:

$$\begin{aligned}
f^*\left(\lambda(\bar{\Omega})\right) &= f^* \sum_{j_1,\dots,j_k=1}^{N} \Omega_\alpha^{j_1} \wedge \ldots \wedge \Omega_\alpha^{j_k} \cdot \lambda\left(Y_1,\ldots,Y_k\right) \\
&= \sum_{j_1,\dots,j_k=1}^{N} f^*\Omega_\alpha^{j_1} \wedge \ldots \wedge f^*\Omega_\alpha^{j_k} \cdot \lambda\left(Y_1,\ldots,Y_k\right) \\
&= \sum_{j_1,\dots,j_k=1}^{N} \left(\Omega'\right)_\alpha^{j_1} \wedge \ldots \wedge \left(\Omega'\right)_\alpha^{j_k} \cdot \lambda\left(Y_1,\ldots,Y_k\right) \\
&= \lambda\left(\bar{\Omega}'\right) .
\end{aligned}$$

Hence we have shown that the assignment of de Rham cohomomology classes to G-principal bundles by means of the construction above is *natural*, i.e. :

$$H^{2k}(M;\mathbb{K}) \ni c_\lambda\left(f^*P\right) = f^*\Big(\underbrace{c_\lambda(P)}_{\in H^{2k}(B;\mathbb{K})} \Big) . \tag{2.8}$$

Remark 2.5.8. Let $\lambda : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{K}$ be a multilinear, symmetric, invariant map and let $P, P' \to B$ be isomorphic G-principal bundles. Then we have $c_\lambda(P) = c_\lambda(P')$:
If $\varphi : P \to P'$ is an isomorphism and ω is a connection 1-form on P, then $\omega' := \varphi^*\omega$ is a connection 1-form on P'. Further, the corresponding curvatures are related by $\Omega' = \varphi^*\Omega$. Given an open cover $\{U_\alpha\}_{\alpha\in I}$ with local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$, then $s'_\alpha := \varphi^{-1} \circ s_\alpha$ define local sections for P' on the same cover. The local curvature forms are related by

$$\Omega'_\alpha = (s'_\alpha)^*\Omega' = s_\alpha^* \circ (\varphi^{-1})^*\varphi^*\Omega = s_\alpha^*\Omega = \Omega_\alpha .$$

This implies $\lambda(\bar{\Omega}') = \lambda(\bar{\Omega})$ and hence $c_\lambda(P) = c_\lambda(P')$.

Remark 2.5.9. Let $\lambda : \overbrace{\mathfrak{g} \times \ldots \times \mathfrak{g}}^{k} \to \mathbb{K}$ be a multilinear, symmetric, invariant map, $k \geq 1$. If P is a trivial G-principal bundle, then $c_\lambda(P) = 0 \in H^{2k}(B;\mathbb{K})$:
By Remark 2.5.8, it suffices to prove that $c_\lambda(B \times G)$, i.e. we may replace the trivial bundle P by the product $B \times G$. Let $\varphi \in \Omega^1(G;\mathfrak{g})$ be given by $\varphi_g := dL_{g^{-1}}$. Then $\omega := \mathrm{pr}_2^*\varphi \in \Omega^1(B \times G;\mathfrak{g})$ is a connection 1-form: To show property 1., we compute:

$$\begin{aligned}
R_g^*\omega_{(b,g')} &= R_g^*\,\mathrm{pr}_2^*\,\varphi_{g'} \\
&= \left(\mathrm{pr}_2 \circ R_g\right)^*\varphi_{g'} \\
&= \left(R_g \circ \mathrm{pr}_2\right)^*\varphi_{g'} \\
&= \mathrm{pr}_2^*\,R_g^*\varphi_{g'} \\
&= \mathrm{pr}_2^*\left(dL_{(g')^{-1}} \circ dR_g\right) \\
&= \mathrm{pr}_2^*\left(\mathrm{Ad}_{g^{-1}} \circ dL_{g\cdot(g')^{-1}}\right) \\
&= \mathrm{Ad}_{g^{-1}} \circ \mathrm{pr}_2^*\,\varphi_{g'\cdot g^{-1}} \\
&= \mathrm{Ad}_{g^{-1}} \circ \omega_{(b,g'\cdot g^{-1})} .
\end{aligned}$$

As to property 2., for any $X \in \mathfrak{g}$, we have:

$$\omega\left(\bar{X}\right) = (\mathrm{pr}_2^*\varphi)\left(\bar{X}\right) = \varphi\left(\bar{X}\right) = \varphi\,.$$

For the curvature form of the connection ω, we obtain:

$$\Omega = d\omega + \frac{1}{2}[\omega,\omega] = d\mathrm{pr}_2^*\varphi + \frac{1}{2}\left[\mathrm{pr}_2^*\varphi, \mathrm{pr}_2^*\varphi\right] = \mathrm{pr}_2^*\left(d\varphi + \frac{1}{2}[\varphi,\varphi]\right)\,.$$

Now let $X, Y \in \mathfrak{g}$ be left-invariant vector fields on G. We then have:

$$\begin{aligned}
d\varphi(X,Y) &= \partial_X\varphi(Y) - \partial_Y\varphi(X) - \varphi([X,Y]) \\
&= \partial_X(Y(e)) - \partial_Y(X(e)) - [X,Y](e) \\
&= [X,Y](e) \\
[\varphi,\varphi](X,Y) &= [\varphi(X),\varphi(Y)] - [\varphi(Y,\varphi(X)] \\
&= 2[X,Y]
\end{aligned}$$

Hence $d\varphi + \frac{1}{2}[\varphi,\varphi] = 0$ and thus $\Omega = 0$. Consequently, $\lambda(\bar{\Omega}) = 0$ and $c_\lambda(B \times G) = 0$.

Corollary 2.5.10
Let $\lambda : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{K}$ be a multilinear, symmetric, invariant map, $k \geq 1$. If $c_\lambda(P) \neq 0 \in H^{2k}(B;\mathbb{K})$, then P is not a trivial bundle.

Example 2.5.11. For $G = \mathrm{GL}(n;\mathbb{C})$ or $G = \mathrm{U}(n)$ and $\lambda : \mathfrak{g} \to \mathbb{K}$, $\lambda(A) := \frac{1}{2\pi i}\mathrm{tr}(A)$, the characteristic class $c_\lambda(P) =: c_1(P)$ is called the ***1. Chern class*** of P. (The field $\mathbb{K}$ can be taken $\mathbb{K} = \mathbb{C}$ for $G = \mathrm{GL}(n;\mathbb{C})$ or $\mathbb{K} = \mathbb{R}$ for $G = \mathrm{U}(n)$.)

Example 2.5.12. Let $P \to B$ be a $\mathrm{U}(1)$-principal bundle over a closed surface B (i.e. B is a compact two dimensional manifold with no boundary). Then $c_1(P) = \left[\frac{1}{2\pi i}\bar{\Omega}\right] \in H^2_{dR}(B;\mathbb{R})$. If $c_1(P) = 0$, there exists a form $\eta \in \Omega^1(B; i\mathbb{R})$ such that $\bar{\Omega} = d\eta$. Integrating over the base and using Stokes theorem, we obtain:

$$\int_B \bar{\Omega} = \int_B d\eta = \int_{\partial B} \eta = 0\,.$$

Hence if $\int_B \bar{\Omega} \neq 0$, then the bundle cannot be trivial.

Example 2.5.13. For $G = \mathrm{GL}(n;\mathbb{C})$ or $G = \mathrm{U}(n)$ and $\lambda : \mathfrak{g} \to \mathbb{K}$,

$$\lambda(A_1, \dots, A_n) := \frac{1}{(2\pi i)^n} A_1 \wedge \dots \wedge A_n \in \mathrm{End}(\Lambda^n \mathbb{C}^n) \cong \mathbb{C},$$

the characteristic class $c_\lambda(P) =: c_n(P)$ is called the ***n^{th} Chern class*** of P. (As above, the field $\mathbb{K}$ can be taken $\mathbb{K} = \mathbb{C}$ for $G = \mathrm{GL}(n;\mathbb{C})$ or $\mathbb{K} = \mathbb{R}$ for $G = \mathrm{U}(n)$.)

Example 2.5.14. For $G = \mathrm{SO}(2m)$, we have $\mathfrak{g} = \mathfrak{so}(2m) \cong \Lambda^2\mathbb{R}^{2m}$. Let $\lambda : \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_{m} \to \mathbb{R}$ be defined as

$$\lambda(\sigma_1, \dots, \sigma_m) := \sigma_1 \wedge \dots \wedge \sigma_m \in \Lambda^{2m}\mathbb{R}^{2m}.$$

(The map $\mathrm{Pf} : \sigma \mapsto \lambda(\sigma, \dots, \sigma)$ is called the ***Pfaffian***.) The characteristic class

$$\left[\frac{\mathrm{Pf}(\bar{\Omega})}{(2\pi)^m \cdot m!}\right] =: e(P) \in H^{2m}(B;\mathbb{R})$$

is called the ***Euler class*** of P.

Example 2.5.15. For $G = \mathrm{SO}(2) = \mathrm{U}(1)$, we have $\mathfrak{so}(2) \cong \mathbb{R}$ and $\mathfrak{u}(1) \cong i\mathbb{R}$. Since $\mathrm{Pf} : \mathfrak{so}(2) \to \mathbb{R}$, $\sigma \mapsto \sigma$ and $\mathrm{tr} : \mathfrak{u}(1) \to i\mathbb{R}$, $A \mapsto A$, we have:

$$e(P) = \left[\frac{\mathrm{Pf}(\bar{\Omega})}{(2\pi)^1 \cdot 1!}\right] = \left[\frac{\mathrm{tr}(\bar{\Omega})}{2\pi i}\right] = c_1(P).$$

2.6 Parallel transport

In this section, let $P \to B$ be a G-principal bundle with a fixed connection 1-form ω.

Lemma 2.6.1
For any (piecewise) smooth curve $c : I \to B$, $t_0 \in I$ and any point $p \in P_{c(t_0)}$ there exists a unique (piecewise) smooth curve $\tilde{c} : I \to P$ with the following three properties:

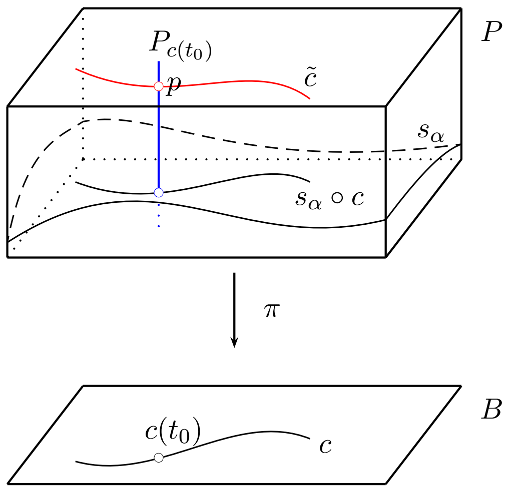
(i) $c = \tilde{c} \circ \pi$, *i.e. $\tilde{c}$ is a* **lift** *of c.*

(ii) $\forall\, t \in I$: $\dot{\tilde{c}}(t) \in H_{\tilde{c}(t)}$, *i.e. $\dot{\tilde{c}}$ is horizontal.*

(iii) $\tilde{c}(t_0) = p$.

The curve $\tilde{c}$ is called the **horizontal lift** *of c with initial condition (iii).*

Proof. W.l.o.g. we assume c to be smooth and $c(I) \subset U_\alpha$ with a local section $s_\alpha : U_\alpha \to P|_{U_\alpha}$. A curve $\tilde{c}$ satisfies (i) iff for any $t \in I$, we have $\tilde{c}(t) = s_\alpha(c(t)) \cdot h_\alpha(t)$ for some function $h_\alpha : I \to G$.



Then $h_\alpha(t_0)$ is uniquely determined by condition (iii). We express condition (ii) in terms of the function h_α. At $t = t_1$, we have:

$$
\begin{aligned}
0 &= \omega\left(\dot{\tilde{c}}(t_1)\right)\\
&= \omega\left(\left.\frac{d}{dt}\right|_{t=t_1} s_\alpha(c(t)) \cdot h_\alpha(t)\right)\\
&= \omega\left(dL_{s_\alpha(c(t_1))\cdot h_\alpha(t_1)}\left(\left.\frac{d}{dt}\right|_{t=t_1}\left(h_\alpha(t_1)^{-1} \cdot h_\alpha(t)\right)\right) + dR_{h_\alpha(t_1)}\left(\underbrace{\left.\frac{d}{dt}\right|_{t=t_1} s_\alpha\left(c\left(t_1\right)\right)}_{=:\dot{s}_\alpha(t_1)}\right)\right)\\
&= \omega\left(\overline{\underbrace{\left.\frac{d}{dt}\right|_{t=t_1}\left(h_\alpha(t_1)^{-1} \cdot h_\alpha(t)\right)}_{\in T_eG \cong \mathfrak{g}}}\right) + \left(R^*_{h_\alpha(t_1)}\omega\right)\left(\dot{s}_\alpha(t_1)\right)\\
&= \left.\frac{d}{dt}\right|_{t=t_1}\left(h_\alpha(t_1)^{-1} \cdot h_\alpha(t)\right) + \mathrm{Ad}_{h_\alpha(t_1)^{-1}} \circ \omega\left(\dot{s}_\alpha(t_1)\right)\\
&= dL_{h_\alpha(t_1)^{-1}}\left(\dot{h}_\alpha(t_1)\right) + \left(dL_{h_\alpha(t_1)^{-1}} \circ dR_{h_\alpha(t_1)}\right)\left(\omega\left(\dot{s}_\alpha(t_1)\right)\right)
\end{aligned}
$$

Applying $(dL_{h_\alpha(t_1)^{-1}})^{-1}$ to both sides of the last equation, we obtain the equivalent condition

$$\dot{h}_\alpha(t_1) = -dR_{h_\alpha(t_1)}\left(\omega\left(\dot{s}_\alpha(t_1)\right)\right) . \tag{2.9}$$

This is a first order ODE for the function h_α, which has, for a given initial condition, a unique solution defined on all of I. □

Remark 2.6.2. The fact that the solution to (2.9) exists on all of I is not apparent from the usual Picard-Lindelöf theorem on ODEs. In case that G is a matrix group, $G \subset \mathrm{GL}(n;\mathbb{K})$, then (2.9) reads

$$\dot{h}_\alpha = -dR_{h_\alpha}\left(\omega\left(\dot{s}_\alpha\right)\right) = -\omega\left(\dot{s}_\alpha\right)\cdot h_\alpha.$$

This is a *linear* ODE, hence the solution exists on the whole interval I.
In the general case, one can argue as follows: Suppose the maximal solution to (2.9) exists only up to t_1 where t_1 is smaller than the right border of I. Choose a horizontal lift $\hat{c}$ of c in a neighborhood of t_1. For some $\tau < t_1$ choose $g \in G$ such that $\tilde{c}(\tau) = \hat{c}(\tau)\cdot g$. Then $\bar{c} := \hat{c}\cdot g$ is another horizontal lift of c in a neighborhood of t_1 (compare Remark 2.6.5.5 below). It coincides with $\tilde{c}$ at τ, by uniqueness they coincide whereever they are both defined. Hence $\bar{c}$ extends $\tilde{c}$ beyond t_1 contradicting the maximality of t_1.

Remark 2.6.3. Let $\varrho : G \to \mathrm{Aut}(V)$ be a representation and $\mathcal{V} := P\times_\varrho V$ the associated vector bundle. For $c : I \to B$ let $\tilde{c}$ be a horizontal lift. Then for any fixed $v \in V$, the map

$$I \to \mathcal{V}\,, t \mapsto [\tilde{c}(t), v]\,,$$

is a parallel section of $\mathcal{V}$ along c. Indeed, covariant differention by t yields:

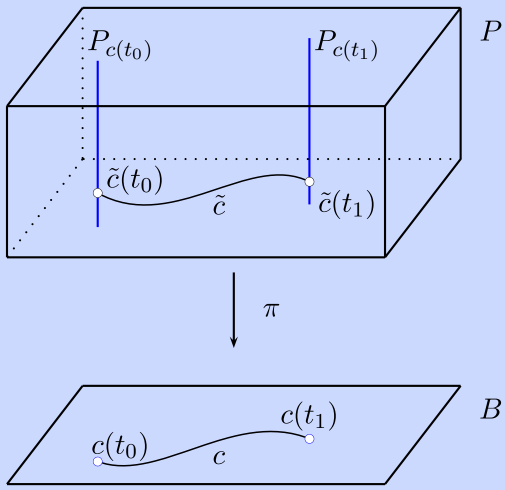
$$\frac{\nabla}{dt}\,[\tilde{c}(t), v] = \Bigg[\tilde{c}(t), \underbrace{\frac{d}{dt}v}_{=0} + \varrho_*\Big(\underbrace{\omega_{\tilde{c}(t)}\left(\dot{s}_\alpha(t)\right)}_{=0}\Big)\Bigg] = 0\,.$$

In case P is the frame bundle of a vector bundle E and ϱ is the standard representation, then $\tilde{c}(t) = (b_1(t),\dots,b_n(t))$ is a curve of basis vectors and $\tilde{c}$ is horizontal iff $b_1,\dots,b_n$ are parallel.

Definition 2.6.4
For a fixed curve $c : [t_0, t_1] \to B$, we get a map $\Gamma(c) : P_{c(t_0)} \to P_{c(t_1)}$ by setting

$$\Gamma(c)(p) := \tilde{c}(t_1),$$

where $\tilde{c}$ is the horizontal lift of c with initial condition $\tilde{c}(t_0) = p$.
$\Gamma(c)$ is called the ***parallel transport*** along c.



Remark 2.6.5. The parallel transport has the following properties:

1. If c is constant, then $\tilde{c}$ is also constant, hence $\Gamma(c) = \mathrm{id}$.
2. If $c' = c \circ \varphi$, where φ is an orientation preserving reparametrization, then $\tilde{c}' := \tilde{c} \circ \varphi$ is a horizontal lift for c' with the same initial condition as $\tilde{c}$. Hence $\Gamma(c) = \Gamma(c')$.
3. If $c' = c \circ \varphi$, where φ is an orientation reversing reparametrization, then $\tilde{c}' := \tilde{c} \circ \varphi$ is a horizontal lift for c' with the initial condition $\tilde{c}'(t_0) = \tilde{c}(t_1)$. Hence $\Gamma(c') = \Gamma(c)^{-1}$. In particular, $\Gamma(c)$ is always a diffeomorphism.
4. For the concatenation $c_2 * c_1$ of piecewise smooth curves, we have $\Gamma(c_2 * c_1) = \Gamma(c_2) \circ \Gamma(c_1)$.
5. If $\tilde{c}$ is the horizontal lift of c with initial condition $c(t_0) = p$, then for $g \in G$, $\tilde{c} \cdot g$ is the horizontal lift of c with initial condition $c(t_0) = p \cdot g$. Hence for any $g \in G$, we have $R_g \circ \Gamma(c) = \Gamma(c) \circ R_g$.

Remark 2.6.6. As seen above, $\Gamma(c)$ does not depend on a particular parametrization of the curve c. But in general, it *does* depend on c. For a closed curve c, we have $\Gamma(c) \neq \mathrm{id}$ in general. This is related to curvature, as we shall see soon.

As we have seen, for matrix groups $G \subset \mathrm{GL}(n; \mathbb{K})$, the horizontal lift is the solution of a linear first order ODE. So let us consider the following linear ODE of first order on $[0, t]$:

$$\begin{aligned} \dot{v}(t) &= -A(t) \cdot v(t) \\ v(0) &= v_0 . \end{aligned} \tag{2.10}$$

If all $A(t)$ commute, i.e. $t \mapsto A(t)$ takes values in an abelian subalgebra of $\mathrm{Mat}(n \times n; \mathbb{K})$, then the solution of (2.10) is given by $v(t) = \exp\left(-\int_0^t A(\tau) d\tau\right) \cdot v_0$. Indeed, differentiating by t, we get:

$$\begin{aligned} \dot{v}(t) &= \frac{d}{dt} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\int_0^t A(\tau) \right)^j \cdot v_0 \\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \Bigg\{ A(t) \cdot \int_0^t A(\tau) d\tau \ldots \int_0^t A(\tau) d\tau \\ &\qquad\qquad + \ldots + \int_0^t A(\tau) d\tau \ldots \int_0^t A(\tau) d\tau \cdot A(t) \Bigg\} \cdot v_0 \\ &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \cdot j \cdot A(t) \cdot \left(\int_0^t A(\tau) d\tau \right)^{j-1} \cdot v_0 \end{aligned}$$

$$
\begin{aligned}
&= -A(t) \cdot \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} \cdot \left(\int_0^t A(\tau) d\tau \right)^{j-1} \cdot v_0 \\
&= -A(t) \cdot \sum_{j=0}^{\infty} \frac{(-1)^{j}}{j!} \cdot \left(\int_0^t A(\tau) d\tau \right)^{j} \cdot v_0 \\
&= -A(t) \cdot v_0 \,.
\end{aligned}
$$

For the third equation we used the fact that all $A(\tau)$ commute to move $A(t)$ in front. In the general case, this is not possible and we have an ordering problem. This problem is fixed as follows:

Lemma 2.6.7
Let $I = [0, L] \subset \mathbb{R}$ be a fixed interval and let $A : I \to \mathrm{Mat}(n \times n; \mathbb{K})$ be a continuous curve. Then the solution of the ODE (2.10) *is given by:*

$$
v(t) = \sum_{j=0}^{\infty} (-1)^j \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 \, A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \tag{2.11}
$$

$$
= \lim_{N \to \infty} \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{N-1}{N} t \right) \right) \ldots \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{1}{N} t \right) \right) \left(\mathbb{1}_n - \frac{t}{N} A(0) \right) v_0 \tag{2.12}
$$

Proof.
a) The difference quotient $\frac{v(s+\epsilon)-v(s)}{\epsilon} = \dot{v}(s) + \mathrm{O}(\epsilon) = -A(s) \cdot v(s) + \mathrm{O}(\epsilon)$ yields

$$
v(s+\epsilon) = v(s) - \epsilon A(s) \cdot v(s) + \mathrm{O}\left(\epsilon^2\right) = (\mathbb{1}_n - \epsilon A(s)) \cdot v(s) + \mathrm{O}\left(\epsilon^2\right) .
$$

Setting $s = \frac{k}{N} t$ and $\epsilon = \frac{t}{N}$, we get:

$$
v\left(\frac{k+1}{N} t\right) = \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{k}{N} t \right) \right) \cdot v\left(\frac{k}{N} t \right) + \mathrm{O}\left(\frac{t^2}{N^2} \right) .
$$

Setting iteratively $k = N-1, N-2, \ldots, 0$, we get:

$$
\begin{aligned}
v(t) &= v\left(\frac{N}{N} t \right) \\
&= \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{N-1}{N} t \right) \right) \cdot v\left(\frac{N-1}{N} t \right) + \mathrm{O}\left(\frac{t^2}{N^2} \right) \\
&= \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{N-1}{N} t \right) \right) \cdot \left(\mathbb{1}_n - \frac{t}{N} A\left(\frac{N-2}{N} t \right) \right) \cdot v\left(\frac{N-2}{N} t \right) + 2\mathrm{O}\left(\frac{t^2}{N^2} \right)
\end{aligned}
$$

$$\vdots$$

$$= \left(\mathbb{1}_n - \frac{t}{N}A\left(\frac{N-1}{N}t\right)\right) \cdot \ldots \cdot \left(\mathbb{1}_n - \frac{t}{N}A(0)\right) \cdot v_0 + \underbrace{N\mathrm{O}\left(\frac{t^2}{N^2}\right)}_{\mathrm{O}(\frac{1}{N})}.$$

This proves (2.12).

b) By a simple induction on j, we show that

$$\int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 = \frac{t^j}{j!}.$$

Indeed, for $j = 1$, the claim is obviously true. For the induction step from $j-1$ to j, we have:

$$\int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 = \int_0^t d\tau_j \frac{\tau_j^{j-1}}{(j-1)!} = \frac{1}{j} \cdot \frac{\tau^j}{(j-1)!} = \frac{\tau^j}{j!}.$$

c) Let $\|\cdot\|$ be the operator norm on $\mathrm{Mat}(n \times n; \mathbb{K})$. Then we have:

$$\begin{aligned}
&\left\| \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \right\| \\
&\qquad \leq \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 \underbrace{\|A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1)\|}_{\leq \|A\|^j_{C^0(I)}} \cdot v_0 \\
&\qquad \leq \frac{t^j}{j!} \|A\|^j_{C^0(I)},
\end{aligned}$$

whence

$$\left\| t \mapsto \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \right\|_{C^0(I)} \leq \frac{L^j}{j!} \cdot \|A\|^j_{C^0(I)}.$$

This tells us that the series in (2.11) converges absolutely in the Banach space $C^0(I; \mathrm{Mat}(n \times n; \mathbb{K}))$. We further need to control the C^1-norm of the series:

$$\begin{aligned}
&\left\| \frac{d}{dt} \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \ldots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \right\| \\
&\quad = \left\| \int_0^t d\tau_{j-1} \int_0^{\tau_{j-1}} d\tau_{j-2} \ldots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(t) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \right\| \\
&\quad \leq \frac{t^{j-1}}{(j-1)!} \cdot \|A\|^j_{C^0(I)},
\end{aligned}$$

whence

$$\left\| \frac{d}{dt}\Big(t \mapsto \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \Big) \right\|_{C^0(I)} \\ \leq \frac{L^{j-1}}{(j-1)!} \cdot \|A\|^j_{C^0(I)} .$$

Together, we have the required estimate of the C^1-norm:

$$\left\| t \mapsto \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \right\|_{C^0(I)} \\ \leq \left(\frac{L^j}{j!} + \frac{L^{j-1}}{(j-1)!} \right) \cdot \|A\|_{C^0(I)} .$$

Hence the series in (2.11) converges absolutely in the Banach space $C^1(I; \mathrm{Mat}(n \times n; \mathbb{K}))$. This implies that the series defines a C^1-function and we may differentiate termwise.

d) Doing so, we obtain:

$$\begin{aligned} &\frac{d}{dt} \sum_{j=0}^{\infty} (-1)^j \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \\ = \; &\sum_{j=1}^{\infty} (-1)^j \frac{d}{dt} \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_j) \cdot A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \\ = \; &\sum_{j=1}^{\infty} (-1)^j A(t) \int_0^t d\tau_{j-1} \int_0^{\tau_{j-1}} d\tau_{j-2} \dots \int_0^{\tau_2} d\tau_1 A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \\ = \; &-A(t) \cdot \sum_{j=1}^{\infty} (-1)^{j-1} \int_0^t d\tau_{j-1} \int_0^{\tau_{j-1}} d\tau_{j-2} \dots \int_0^{\tau_2} d\tau_1 A(\tau_{j-1}) \cdot \ldots \cdot A(\tau_1) \cdot v_0 \\ = \; &-A(t) \cdot v(t) . \end{aligned}$$

□

Definition 2.6.8. The solution operator to the ODE (2.10)

$$\begin{aligned}
&\mathrm{Pexp}\left(-\int_0^t A(\tau)d\tau\right) \\
&:= \sum_{j=0}^{\infty}(-1)^j \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j-1} \dots \int_0^{\tau_2} d\tau_1 A(\tau_j)\cdot A(\tau_{j-1})\cdot \ldots \cdot A(\tau_1) \qquad (2.13)\\
&= \lim_{N\to\infty}\left(\mathbb{1}_n - \frac{t}{N}A\left(\frac{N-1}{N}t\right)\right)\cdot \ldots \cdot \left(\mathbb{1}_n - \frac{t}{N}A\left(\frac{1}{N}t\right)\right)\cdot\left(\mathbb{1}_n - \frac{t}{N}A(0)\right) \qquad (2.14)
\end{aligned}$$

is called the ***path-ordered exponential*** of A.

Lemma 2.6.7 says that the solution to (2.10) is given by

$$v(t) = \mathrm{Pexp}\left(-\int_0^t A(\tau)d\tau\right)\cdot v_0.$$

Remark 2.6.9. Let G be abelian, let $c : I \to B$ be a closed curve contained in a U_α, $c(I) \subset U_\alpha$, on which a section s_α is defined. Assume that the curve bounds a surface S, also contained in U_α. Using Stokes's theorem and $\Omega_\alpha = d\omega_\alpha$ we have

$$\Gamma(c) = \exp\left(-\int_I \omega_\alpha\left(\dot{c}(t)\right)\,dt\right) = \exp\left(-\int_c \omega_\alpha\right) = \exp\left(-\int_S d\omega_\alpha\right) = \exp\left(-\int_S \Omega_\alpha\right).$$

This shows that in general, $\Gamma(c) \neq \mathrm{id}$, if $\Omega \neq 0$.

Now let $G \subset \mathrm{GL}(n;\mathbb{K})$ be a (not necessarily abelian) matrix group and let $P \to B$ be a G-principal bundle with connection 1-form ω. For any $b_0 \in B$, let $c_L : [0,1] \to B$ be a 1-parameter family of closed curves satisfying $c_L(0) = c_L(1) = b_0$ and $\mathrm{length}(c_L) = \mathrm{O}(L)$. W.l.o.g. assume that every c_L is parametrized proportionally to arc-length, i.e. $\|\dot{c}_L\| = \mathrm{const} = \mathrm{O}(L)$. Let c_L bound a surface $S_L \subset B$ such that S_L is contained in the ball of radius $C \cdot L$ about b_0 (w.r.t. some metric) where C is a fixed constant and $\mathrm{area}(S_L) = \mathrm{O}(L^2)$. Then, for sufficiently small L, we have $S_L \subset U_\alpha$, where $U_\alpha \subset B$ is an open neighborhood of b_0 with a section $s_\alpha : U_\alpha \to P|_{U_\alpha}$.

Now for an arbitrary fixed L, denoting c_L by c and S_L by S, we have:

$$\begin{aligned}\Gamma(c) &= \operatorname{Pexp}\left(-\int_c \omega_\alpha\right)\\ &= \mathbb{1}_n - \int_0^1 d\tau\, \omega_\alpha\big(c(\tau)\big)\big(\dot c(\tau)\big)\\ &\quad + \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1\, \underbrace{\Big(\omega_\alpha\big(c(\tau_2)\big)\big(\dot c(\tau_2)\big)\Big)}_{\in\mathfrak{g}} \cdot \underbrace{\Big(\omega_\alpha\big(c(\tau_1)\big)\big(\dot c(\tau_1)\big)\Big)}_{\in\mathfrak{g}} + \mathrm{O}\left(L^3\right) . \end{aligned} \tag{2.15}$$

Using Stokes' theorem, we find for the first integral:

$$\int_0^1 d\tau\, \omega_\alpha\big(c(\tau)\big)\big(\dot c(\tau)\big) = \int_c \omega_\alpha = \int_S d\omega_\alpha = \mathrm{O}\left(L^2\right) .$$

To estimate the second integral, we set

$$\int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \left(\Big(\omega_\alpha\big(c(\tau_2)\big)\big(\dot c(\tau_2)\big)\Big) \cdot \Big(\omega_\alpha\big(c(\tau_1)\big)\big(\dot c(\tau_1)\big)\Big)\right) = I_s + I_a\,,$$

where $I_{s/a} := \frac{1}{2} \cdot \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \left(\omega_\alpha(\dot c(\tau_2)) \cdot \omega_\alpha(\dot c(\tau_1)) \mp \omega_\alpha(\dot c(\tau_1)) \cdot \omega_\alpha(\dot c(\tau_2))\right)$. Then we have:

$$\begin{aligned} 2\cdot I_s &= \iint\limits_{0\le\tau_1\le\tau_2\le 1} d\tau_2\, d\tau_1\, \omega_\alpha\big(\dot c(\tau_2)\big)\cdot\omega_\alpha\big(\dot c(\tau_1)\big) + \iint\limits_{0\le\tau_2\le\tau_1\le 1} d\tau_2\, d\tau_1\, \omega_\alpha\big(\dot c(\tau_2)\big)\cdot\omega_\alpha\big(\dot c(\tau_1)\big)\\ &= \int_0^1\int_0^1 d\tau_2\, d\tau_1\, \omega_\alpha\big(\dot c(\tau_2)\big)\cdot\omega_\alpha\big(\dot c(\tau_1)\big)\\ &= \left(\int_0^1 d\tau\, \omega_\alpha\big(\dot c(\tau)\big)\right)^2\\ &= \mathrm{O}\left(L^4\right) . \end{aligned}$$

Hence I_s is swallowed by the error term $\mathrm{O}(L^3)$ in (2.15). To determine I_a, we introduce local coordinates $x^1, \ldots x^n$ around b_0 such that b_0 has the coordinates $(0, \ldots, 0)$ and we write $\omega_\alpha = \sum_{j=1}^n \omega_{\alpha,j} dx^j =: \omega_j dx^j$. We compute:

$$
\begin{aligned}
&\int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \omega_\alpha\big(\dot c(\tau_2)\big) \cdot \omega_\alpha\big(\,\cdot\, c(\tau_1)\big) \\
&\quad = \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \omega_j\big(c(\tau_2)\big)\dot c^j(\tau_2) \cdot \omega_k\big(c(\tau_1)\big)\dot c^k(\tau_1) \\
&\quad = \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \omega_j(0)\dot c^j(\tau_2) \cdot \omega_k\big(c(\tau_1)\big)\dot c^k(\tau_1) \\
&\qquad + \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \underbrace{\Big(\omega_j\big(c(\tau_2)\big) - \omega_j(0)\Big)}_{\mathrm{O}(L)} \cdot \underbrace{\dot c^j(\tau_2)}_{\mathrm{O}(L)} \cdot \underbrace{\omega_k\big(c(\tau_1)\big)}_{\mathrm{O}(1)} \cdot \underbrace{\dot c^k(\tau_1)}_{\mathrm{O}(L)} \\
&\quad = \omega_j(0) \cdot \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \dot c^j(\tau_2) \cdot \omega_k\big(c(\tau_1)\big) \cdot \dot c^k(\tau_1) + \mathrm{O}\left(L^3\right) \\
&\quad = \omega_j(0) \cdot \omega_k(0) \cdot \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \dot c^j(\tau_2) \cdot \dot c^k(\tau_1) + \mathrm{O}\left(L^3\right) .
\end{aligned}
$$

For the term I_a, we thus get:

$$
\begin{aligned}
2 \cdot I_a &= \omega_j(0) \cdot \omega_k(0) \cdot \int_0^1 \int_0^{\tau_2} d\tau_2\, d\tau_1\, \Big(\dot c^j(\tau_2) \cdot \dot c^k(\tau_1) - \dot c^j(\tau_1) \cdot \dot c^k(\tau_2)\Big) + \mathrm{O}\left(L^3\right) \\
&= \omega_j(0) \cdot \omega_k(0) \cdot \int_0^1 d\tau_2\, \Big(\dot c^j(\tau_2) c^k(\tau_2) - c^j(\tau_2)\dot c^k(\tau_1)\Big) + \mathrm{O}\left(L^3\right) \\
&= \omega_j(0) \cdot \omega_k(0) \cdot \int_c \Big(x^k dx^j - x^j dx^k\Big) + \mathrm{O}\left(L^3\right) \\
&\overset{\text{Stokes}}{=} \omega_j(0) \cdot \omega_k(0) \cdot \int_S \Big(dx^k \wedge dx^j - dx^j \wedge dx^k\Big) + \mathrm{O}\left(L^3\right) \\
&= -\left[\omega_j(0), \omega_k(0)\right] \cdot \int_S dx^j \wedge dx^k + \mathrm{O}(L^3) \\
&= -\int_S \left[\omega_j, \omega_k\right] dx^j \wedge dx^k + \int_S \underbrace{\left(\left[\omega_j, \omega_k\right] - \left[\omega_j, \omega_k\right](0)\right)}_{\mathrm{O}(L)} dx^j \wedge dx^k + \mathrm{O}\left(L^3\right) \\
&= -\int_S \left[\omega_j, \omega_k\right] dx^j \wedge dx^k + \mathrm{O}\left(L^3\right)
\end{aligned}
$$

$$= \quad -\int\limits_S [\omega_\alpha, \omega_\alpha] + \mathrm{O}\left(L^3\right) .$$

We thus have:

$$\begin{aligned} \Gamma(c) &= \mathbb{1}_n - \int\limits_S d\omega_\alpha - \frac{1}{2}\int\limits_S [\omega_\alpha, \omega_\alpha] + \mathrm{O}\left(L^3\right) \\ &= \mathbb{1}_n - \int\limits_S \Omega_\alpha + \mathrm{O}\left(L^3\right) \qquad (L \searrow 0) . \end{aligned}$$

2.7 Gauge transformations

Definition 2.7.1. Let $P \xrightarrow{\pi} B$ be a G-principal bundle. A diffeomorphism $f : P \to P$ is called an ***automorphism*** of P iff

$$\forall\, g \in G, \, , \forall\, p \in P : \; f(p \cdot g) = f(p) \cdot g .$$

$\mathrm{Aut}(P) := \{\text{automorphisms of } P\}$ is called the ***automorphism group*** of P.

Remark 2.7.2

1. $\mathrm{Aut}(P) \subset \mathrm{Diff}(P)$ is a subgroup.

2. Any $f \in P$ takes the fibers of P to fibers of P. Indeed, if $p, p' \in P$ are in the same fiber, then there exists a (unique) $g \in G$ such that $p' = p \cdot g$. Applying f, we find $f(p') = f(p \cdot g) = f(p) \cdot g$, thus $f(p), f(p')$ are in the same fiber again. This implies that there is a (unique) smooth map $\bar{f} : B \to B$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"] \\ B \arrow[r, "\bar{f}"'] & B \end{tikzcd}$$

3. $\mathrm{Aut}(P)$ acts from the right on $\mathcal{C}(P) := \{\text{connection 1-forms on } P\}$ by pull-back: We first check that for any $\omega \in \mathcal{C}(P)$ and any $f \in \mathrm{Aut}(P)$, the pull-back $f^*\omega$ is again a connection 1-form, i.e. $f^*\omega \in \mathcal{C}(P)$. Indeed, for any $g \in G$, we have:

$$\begin{aligned} R_g^* \left(f^*\omega\right) &= \left(f \circ R_g\right)^* \omega = \left(R_g \circ f\right)^* \omega = f^* \left(R_g^*\omega\right) = f^* \left(\mathrm{Ad}_{g^{-1}} \circ \omega\right) \\ &= \mathrm{Ad}_{g^{-1}} \circ \left(f^*\omega\right) . \end{aligned}$$

For any $X \in \mathfrak{g}$, we find:

$$(f^*\omega)\left(\bar{X}(p)\right) = \omega\left(df\left(\bar{X}(p)\right)\right) = \omega\left(df \circ dL_p(X)\right) = \omega\left(dL_{f(p)}(X)\right) = \omega\left(\bar{X}(f(p))\right) \\ = X\,.$$

Here we have used $(f \circ L_p)(g) = f(p \cdot g) = f(p) \cdot g = L_{f(p)}(g)$.

Finally, the pull-back is indeed a right action, since for any $f, g \in \mathrm{Aut}(P)$, we have $(f \circ g)^*\omega = g^*(f^*\omega)$.

Definition 2.7.3. An automorphism $f \in \mathrm{Aut}(P)$ with $\bar{f} = \mathrm{id}_B$ is called a ***gauge transformation*** of P. The group

$$\mathcal{G}(P) := \{\text{gauge transformations on } P\}$$

of gauge transformations on P is called the ***gauge group*** of P.

Remark 2.7.4. The map $\mathrm{Aut}(P) \to \mathrm{Diff}(B)$, $f \mapsto \bar{f}$, is a group homomorphism. Hence $\mathcal{G}(P) = \ker(f \mapsto \bar{f}) \subset \mathrm{Aut}(P)$ is a subgroup.

Example 2.7.5. If G is abelian, then each smooth map $g : B \to G$ gives rise to a gauge transformation by $f(p) := p \cdot g(\pi(p))$. Indeed, we have:

$$f\left(p \cdot g'\right) = p \cdot g' \cdot g\big(\pi(p)\big) = p \cdot g\big(\pi(p)\big) \cdot g' = f(p) \cdot g'\,.$$

In the next to last equality, we used that G is abelian. In fact, in the abelian case, all gauge transformations are of this form (see the general case below).

Remark 2.7.6. If G is non-abelian, this construction does not give a gauge transformation unless $g : B \to Z(G)$.

Remark 2.7.7. Let $P \to B$ be a G-principal bundle. Let us consider the associated bundle $P \times_\alpha G := P \times G/\sim$, where α denotes the conjugation action of G on itself, so that $[p, g] \sim [p \cdot h, h^{-1} \cdot g \cdot h]$. The fibers of $P \times_\alpha G$ carry a group structure making them isomorphic to G. Indeed, the multiplication $[p, g] \cdot [p, g'] := [p, g \cdot g']$ is well-defined, because

$$\left[ph, h^{-1}gh\right] \cdot \left[ph, h^{-1}g'h\right] = \left[ph, h^{-1}gh \cdot h^{-1}g'h\right] = \left[ph, h^{-1}gg'h\right] = \left[p, gg'\right]\,.$$

If $b \mapsto [p(b), g(b)]$ is a smooth section of $P \times_\alpha G$, then there is a unique $f \in \mathcal{G}(P)$ such that $f(p(b)) = p(b) \cdot g(b)$: For any $p' \in P$, we find $p' = p(b) \cdot h$, where $b = \pi(p)$ and $h \in G$ is uniquely determined. We then have:

$$f\left(p'\right) = f\big(p(b) \cdot h\big) = f\big(p(b)\big) \cdot h = p(b) \cdot g(b) \cdot h = p' \cdot h^{-1} \cdot g(b) \cdot h\,.$$

This shows that f is uniquely determined by the section $b \mapsto [p(b), g(b)]$. As to existence, $f(p') = f(p(b) \cdot h) := p' \cdot h^{-1} \cdot g(b) \cdot h$ defines a gauge transformation.
Conversely, given a gauge transformation $f \in \mathcal{G}(P)$, then for any $p \in P$ there exists a unique $g(p) \in G$ such that $f(p) = p \cdot g(p)$. For $p' = ph$ we find on the one hand

$$f\left(p'\right) = p' \cdot g\left(p'\right) = p \cdot h \cdot g\left(p'\right)$$

and on the other hand

$$f\left(p'\right) = f(ph) = f(p)h = p \cdot g(p) \cdot h.$$

Thus $g(p') = h^{-1} \cdot g(p) \cdot h$. Therefore $\pi(p) \mapsto [p, g(p)]$ is a well-defined smooth section of $P \times_\alpha G$ giving rise to the gauge transformation f.
This yields an isomorphism of groups:

$$\{\mathcal{C}^\infty\text{-sections of } P \times_\alpha G\} \cong \mathcal{G}(P)\,.$$

Note that $P \times_\alpha G$ is a group bundle (with typical fiber the Lie group G) but not a G-principal bundle. In general, this group bundle is not trivial, but it always has smooth global sections, e.g. the map $\pi(p) \mapsto [p, e]$ which corresponds to $\mathrm{id} \in \mathcal{G}(P)$.

Definition 2.7.8. Let $b \in B$ be an arbitrary point in the basis of a G-principal bundle $P \to B$. The kernel of the group homomorphism

$$\mathcal{G}(P) \to \mathrm{Diff}\,(P_b)\ , f \mapsto f|_{P_b}\,,$$

given by

$$\mathcal{G}_b(P) := \{f \in \mathcal{G}(P) \,|\, f|_{P_b} = \mathrm{id}_{P_b}\}\ .$$

is called the ***reduced gauge group***.

The reduced gauge group fits into the following table of groups and homomorphisms (where the horizontal maps are the natural inclusions):

$$\begin{tikzcd}
\mathcal{G}_b(P) \arrow[r] & \mathcal{G}(P) \arrow[r] \arrow[d, "|_{P_b}"'] & \mathrm{Aut}(P) \arrow[r] \arrow[d] & \mathrm{Diff}(P) \\
 & \mathrm{Diff}\,(P_b) & \mathrm{Diff}(B) &
\end{tikzcd}$$

At least one reason to define the reduced gauge is the following nice property:

Proposition 2.7.9
If B is connected, then the action of the reduced gauge group $\mathcal{G}_b(P)$ on the space of connections $\mathcal{C}(P)$ is free.

Proof. Let $f \in \mathcal{G}_b(P)$ and $\omega \in \mathcal{C}(P)$ be a connection 1-form such that $f^*\omega = \omega$. Then we need to show that $f = \mathrm{id}_P$. To this end, fix a point $p \in P$ and choose a curve $c : [0,1] \to B$ such that $c(0) = b$ and $c(1) = \pi(p)$. Let $\tilde{c} : [0,1] \to P$ be the ω-horizontal lift of c with initial condition $\tilde{c}(1) = p$. Now we put $\hat{c} := f \circ \tilde{c}$. Obviously, $\hat{c}$ is a lift of c. We show that $\hat{c}$ is ω-horizontal:
For any $X \in H_q^\omega$, we have

$$\omega\big(df(X)\big) = f^*\omega(X) = \omega(X) = 0\,.$$

Thus $df(X) \in H^\omega_{f(q)}$. Hence df preserves the horizontal spaces: we have $df(H_q^\omega) \subset H^\omega_{f(q)}$, and since f is a diffeomorphism, we have also $\dim df(H_q^\omega) = \dim H_q^\omega = \dim B$, which yields $df(H_q^\omega) = H^\omega_{f(q)}$. This implies that $\hat{c}$ is horizontal, because $\tilde{c}$ is.
Now we have:

$$\hat{c}(0) = f\big(\underbrace{\tilde{c}(0)}_{\in P_b}\big) \overset{f\in\mathcal{G}_b(P)}{=} \tilde{c}(0)\,,$$

which says that $\hat{c}$ is a horizontal lift of c with the same initial condition as $\tilde{c}$ and hence coincides with $\tilde{c}$. We conclude that

$$f(p) = f\left(\tilde{c}(1)\right) = \hat{c}(1) = \tilde{c}(1) = p\,,$$

whence $f = \mathrm{id}_P$. $\square$

3 Applications to Physics

3.1 The Hodge-star operator

Let V be an n-dimensional $\mathbb{R}$-vector space, equipped with a (not necessarily definite) non-degenerate inner product $\langle \cdot, \cdot \rangle$. Let $e_1, \ldots, e_n$ be a generalized orthonormal basis of V, i.e.

$$\langle e_i, e_j \rangle = \begin{cases} 0 & : \ i \neq j \\ \epsilon_j = \pm 1 & : \ i = j \end{cases}$$

Then there is an inner product on $\Lambda^k V^*$, naturally induced by the one on V, denoted by the same symbol $\langle \cdot, \cdot \rangle$ and defined by:

$$\langle \omega, \eta \rangle := \sum_{i_1 < \ldots < i_k} \epsilon_{i_1} \cdot \ldots \cdot \epsilon_{i_k} \cdot \omega \left(e_{i_1}, \ldots, e_{i_k} \right) \cdot \eta \left(e_{i_1}, \ldots, e_{i_k} \right) .$$

Lemma 3.1.1
The definition of the inner product on $\Lambda^k V^*$ above does not depend on the choice of generalized orthonormal basis $e_1, \ldots, e_n$.

Proof. Let $f_1, \ldots, f_n$ be another generalized orthonormal basis of V, i.e. $\langle f_i, f_j \rangle = \epsilon'_j \cdot \delta_{ij}$, $\epsilon'_j = \pm 1$. We write the unique $A \in \mathrm{Aut}(V)$ such that $Ae_i = f_i$ in matrix coefficients with respect to $e_1, \ldots, e_n$ as $f_i = Ae_i = \sum_{i=1}^n A_i^j e_j$. We then have:

$$\begin{aligned} \delta_{ij} \cdot \epsilon'_j &= \langle f_i, f_j \rangle \\ &= \langle Ae_i, Ae_j \rangle \\ &= \sum_{k,l=1}^n \left\langle A_i^k e_k, A_j^l e_l \right\rangle \\ &= \sum_{k,l=1}^n A_i^k \cdot A_j^l \cdot \underbrace{\langle e_k, e_l \rangle}_{=\delta_{kl} \cdot \epsilon_k} \\ &= \sum_{k=1}^n A_i^k \cdot A_j^k \cdot \epsilon_k \, . \end{aligned}$$

Putting

$$\epsilon := \begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon'_2 n \end{pmatrix} \quad \text{and} \quad \epsilon' := \begin{pmatrix} \epsilon'_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon'_n \end{pmatrix},$$

we have $\epsilon' = A \cdot \epsilon \cdot A^*$, hence $A^* = \epsilon \cdot A^{-1} \cdot \epsilon'$, and thus

$$A^* \epsilon' A = \epsilon \cdot A^{-1} \cdot \underbrace{\epsilon' \cdot \epsilon'}_{=\mathbb{1}_n} \cdot A = \epsilon \,.$$

We thus find $\delta_{ij}\epsilon_i = \sum_{l=1}^n A^i_l \cdot A^j_l \cdot \epsilon'_l$. Inserting this into the definition of the inner product on $\Lambda^k V^*$, we find:

$$\begin{aligned}
&\sum_{i_1<\ldots<i_k} \epsilon'_{i_1} \cdot \ldots \cdot \epsilon'_{i_k} \cdot \omega\left(f_{i_1}, \ldots, f_{i_k}\right) \cdot \eta\left(f_{i_1}, \ldots, f_{i_k}\right) \\
&= \frac{1}{k!} \sum_{i_1,\ldots,i_k} \epsilon'_{i_1} \cdot \ldots \cdot \epsilon'_{i_k} \cdot \omega\left(f_{i_1}, \ldots, f_{i_k}\right) \cdot \eta\left(f_{i_1}, \ldots, f_{i_k}\right) \\
&= \frac{1}{k!} \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_k,l_1,\ldots,l_k}} \epsilon'_{i_1} \cdot \ldots \cdot \epsilon'_{i_k} \cdot A^{j_1}_{i_1} \cdot \ldots \cdot A^{j_k}_{i_k} \cdot A^{l_1}_{i_1} \cdot \ldots \cdot A^{l_k}_{i_k} \cdot \omega\left(e_{j_1}, \ldots, e_{j_k}\right) \cdot \eta\left(e_{l_1}, \ldots, e_{l_k}\right) \\
&= \frac{1}{k!} \sum_{j_1,\ldots,j_k} \epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_k} \cdot \omega\left(e_{j_1}, \ldots, e_{j_k}\right) \cdot \eta\left(e_{j_1}, \ldots, e_{j_k}\right) \\
&= \sum_{j_1<\ldots<j_k} \epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_k} \cdot \omega\left(e_{j_1}, \ldots, e_{j_k}\right) \cdot \eta\left(e_{j_1}, \ldots, e_{j_k}\right) \\
&= \langle \omega, \eta \rangle \,.
\end{aligned}$$

□

Remark 3.1.2. If $e_1, \ldots, e_n$ is a generalized orthonormal basis of V and $e_1^*, \ldots, e_n^*$ is the dual basis, i.e. $e_i^*(e_j) = \delta_{ij}$, then $\{e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}\}_{i_1<\ldots<i_k}$ is a generalized orthonormal basis of $\Lambda^k V^*$ with

$$\left\langle e^*_{i_1} \wedge \ldots \wedge e^*_{i_k}, e^*_{i_1} \wedge \ldots \wedge e^*_{i_k} \right\rangle = \epsilon_{i_1} \cdot \ldots \cdot \epsilon_{i_k} \,.$$

Definition 3.1.3. Let V be an n-dimensional $\mathbb{R}$-vector space with a fixed orientation. Let $e_1, \ldots, e_n$ be a positively oriented generalized orthonormal basis of V. Then $\mathrm{vol} := e_1^* \wedge \ldots \wedge e_n^* \in \Lambda^n V^*$ is called the ***volume form*** on V associated with the given orientation.

Remark 3.1.4. The volume form defined above does not depend on the choice of generalized orthonormal basis, hence it is well-defined as an element of $\Lambda^n V^*$:
Since $\Lambda^n V^*$ is 1-dimensional, and

$$\langle e_1^* \wedge \ldots \wedge e_n^*, e_1^* \wedge \ldots \wedge e_n^* \rangle = \epsilon_1 \cdot \ldots \cdot \epsilon_n = (-1)^{\mathrm{ind}\langle \cdot , \cdot \rangle} \,,$$

the element $e_1^* \wedge \ldots \wedge e_n^*$ is determined up to sign independently of the choice of generalized orthonormal basis $e_1, \ldots, e_n$. The orientation determines the sign.

Lemma 3.1.5
There exists a unique linear map $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$ *such that* $\forall\, \omega \in \Lambda^k V^*$, $\forall\, \eta \in \Lambda^{n-k} V^*$, *we have:*

$$\omega \wedge \eta = \langle * \,\omega, \eta \rangle \cdot \mathrm{vol} \,. \tag{3.1}$$

Proof. For any $\sigma \in \Lambda^n V^*$, there is a unique $a_\sigma \in \mathbb{R}$ such that $\sigma = a_\sigma \cdot \mathrm{vol}$. We may thus formally write $a_\sigma = \frac{\sigma}{\mathrm{vol}}$. Now for any fixed $\omega \in \Lambda^k V^*$, the map

$$\Lambda^{n-k} V^* \to \mathbb{R} \,, \eta \mapsto \frac{\omega \wedge \eta}{\mathrm{vol}} \,,$$

is linear. Since $\langle \cdot , \cdot \rangle$ is non-degenerate there is a unique element $*\omega \in \Lambda^{n-k} V^*$ satisfying $\frac{\omega \wedge \eta}{\mathrm{vol}} = \langle *\omega, \eta \rangle$, whence $\omega \wedge \eta = \langle *\omega, \eta \rangle \cdot \mathrm{vol}$ for all $\eta \in \Lambda^{n-k} V^*$. The map $\Lambda^k V^* \to \Lambda^{n-k} V^*$, $\omega \mapsto *\omega$, is linear, since $\omega \mapsto \frac{\omega \wedge \eta}{\mathrm{vol}}$ is linear. □

Definition 3.1.6. The map $* : \Lambda^k V^* \to \Lambda^{n-k} V^*$ is called the ***Hodge-star operator*** associated with the inner product $\langle \cdot , \cdot \rangle$.

Remark 3.1.7. The Hodge-star operator $*$ depends on the inner product.

Proposition 3.1.8
Let V be an oriented n-dimensional $\mathbb{R}$-vector space, equipped with a non-degenerate inner product $\langle \cdot, \cdot \rangle$ of index p. Then the Hodge-star operator $*$ associated with $\langle \cdot, \cdot \rangle$ has the following properties:

1. *For a positively oriented generalized orthonormal basis $e_1, \ldots, e_n$ of V and the dual basis $e_1^*, \ldots, e_n^*$ of V^*, we have:*
$$* \left(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \right) = \epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_{n-k}} \cdot \operatorname{sign}(IJ) \cdot e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^* \,, \tag{3.2}$$
where $(IJ) = (i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is a permutation of $(1, \ldots, n)$.
2. *$\forall\, \omega \in \Lambda^k V^*$, we have:*
$$* * \omega = (-1)^{k(n-k)+p} \cdot \omega \,. \tag{3.3}$$
3. *$\forall\, \omega, \eta \in \Lambda^k V^*$, we have:*
$$\langle * \omega, * \eta \rangle = (-1)^p \cdot \langle \omega, \eta \rangle \,. \tag{3.4}$$
4. *$\forall\, \omega, \eta \in \Lambda^k V^*$, we have:*
$$\omega \wedge * \eta = \eta \wedge * \omega = (-1)^p \cdot \langle \omega, \eta \rangle \cdot \mathrm{vol} \,. \tag{3.5}$$
5. *$\forall\, \omega \in \Lambda^k V^*$, $\forall\, \eta \in \Lambda^{n-k} V^*$, we have:*
$$\omega \wedge \eta = (-1)^{k(n-k)} \cdot \langle \omega, * \eta \rangle \cdot \mathrm{vol} \,. \tag{3.6}$$

Proof.

1. If $\{i_1, \ldots, i_k, j_1, \ldots, j_{n-k}\} \neq \{1, \ldots, n\}$, then $\{e_{i_1}, \ldots, e_{i_k}\} \cap \{e_{j_1}, \ldots, e_{j_{n-k}}\} \neq \emptyset$, so that $e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \wedge e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^* = 0$ and thus $\langle *(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*), e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^* \rangle = 0$. Hence $*(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*) = c \cdot e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^*$, where $\{j_1, \ldots, j_{n-k}\}$ is complementary to $\{i_1, \ldots, i_k\}$ in $\{1, \ldots, n\}$ (in other words, $IJ = (i_1, \ldots, i_k, j_1 \ldots, j_{n-k})$ is a permutation of $(1, \ldots, n)$). To determine the constant c, we compute:
$$\begin{aligned} \operatorname{sign}(IJ) \cdot \mathrm{vol} &= e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \wedge e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^* \\ &= \left\langle * \left(e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \right), e_{j_1}^* \wedge \ldots \wedge e_{j_{n-k}}^* \right\rangle \cdot \mathrm{vol} \\ &= c \cdot \epsilon_{j_1} \cdot \ldots \cdot \epsilon_{j_{n-k}} \cdot \mathrm{vol} \,. \end{aligned}$$
2. exercise
3. exercise

4. exercise

5. We compute, using (3.3) and (3.4):

$$\begin{aligned}
\omega \wedge \eta &= \langle *\omega, \eta \rangle \cdot \mathrm{vol} \\
&\overset{(3.4)}{=} (-1)^p \cdot \langle * * \omega, *\eta \rangle \cdot \mathrm{vol} \\
&\overset{(3.3)}{=} (-1)^{k(n-k)+2p} \langle \omega, *\eta \rangle \cdot \mathrm{vol} \\
&= (-1)^{k(n-k)} \cdot \langle \omega, *\eta \rangle \cdot \mathrm{vol}\,.
\end{aligned}$$

□

Remark 3.1.9. Let us consider the important special case of the Hodge-star operator $* : \Lambda^2 V^* \to \Lambda^2 V^*$ on 2-forms on a 4-dimensional euclidean vector space V, i.e. $n = 4$, $k = 2$, $p = 0$. By (3.3), we have $* \circ * = (-1)^{2(4-2)} = 1$. By (3.4), $*$ is an isometry. Hence $*$ has eigenvalues ± 1, and we have the eigenspace decomposition:

$$\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*\,,$$

where $\Lambda^2_\pm V^* = \{\omega \in \Lambda^2 \mid * \omega = \pm\omega\}$ is the space of ***self-dual*** resp. ***anti-self-dual*** 2-forms. Choosing an orthonormal basis $e_1, \ldots, e_4$ of V, we have:

$$\begin{aligned}
*(e_1^* \wedge e_2^*) &= e_3^* \wedge e_4^* \\
*(e_1^* \wedge e_3^*) &= -e_2^* \wedge e_4^* \\
*(e_1^* \wedge e_4^*) &= e_2^* \wedge e_3^*\,.
\end{aligned}$$

Hence

$$\begin{aligned}
e_1^* \wedge e_2^* \pm e_3^* \wedge e_4^* &\in \Lambda^2_\pm V^*\,, \\
e_1^* \wedge e_3^* \mp e_2^* \wedge e_4^* &\in \Lambda^2_\pm V^*\,, \\
e_1^* \wedge e_4^* \pm e_2^* \wedge e_3^* &\in \Lambda^2_\pm V^*\,.
\end{aligned}$$

These elements are easily seen to be linearly independent: indeed, they are pairwise orthogonal. We thus have $\dim \Lambda^2_\pm V^* \geq 3$. In fact, $\dim \Lambda^2 V^* = \binom{4}{2} = 6$, so that $\dim \Lambda^2_\pm V^* = 3$, and the elements given above form bases of $\Lambda^2_\pm V^*$.

Remark 3.1.10. A reversal of orientation turns the volume form into its negative. By (3.1), the same holds for the Hodge-star operator. In the special situation above, the subspaces $\Lambda^2_\pm V^*$ are interchanged upon reversal of orientation.

Remark 3.1.11. If W is another $\mathbb{R}$-vector space with inner product $\langle\cdot,\cdot\rangle$, then $V\otimes W$ carries a natural inner product characterized by

$$\left\langle v\otimes w, v'\otimes w'\right\rangle = \left\langle v,v'\right\rangle\cdot\left\langle w,w'\right\rangle\ .$$

This induces a natural Hodge-star operator on W-valued forms by

$$*:\Lambda^k V^*\otimes W\to\Lambda^{n-k}V^*\otimes W\,,\quad *(\omega\otimes w):=(*\omega)\otimes w\,.$$

3.2 Electrodynamics

Throughout this section, let M be an oriented Lorentzian 4-manifold. This manifold is the mathematical model for spacetime in general relativity. For example, the spacetime of special relativity is Minkowski space.
Furthermore, let $P\to M$ be a U(1)-principal bundle. For any $\omega\in\mathcal{C}(P)$ let Ω be its curvature form. The 2-form $s^*\Omega$ for some local section s does not depend on the choice of s. This yields a well-defined 2-form $\bar\Omega\in\Omega^2(M;i\mathbb{R})$. We write $\Omega=iF$, $F\in\Omega^2(M;\mathbb{R})$. The Bianchi identity (2.6) tells us that $dF=0$.
Now we introduce local coordinates (t,x,y,z) on M such that $\langle\frac{\partial}{\partial t},\frac{\partial}{\partial t}\rangle<0$ and $\langle\frac{\partial}{\partial x},\frac{\partial}{\partial x}\rangle$, $\langle\frac{\partial}{\partial y},\frac{\partial}{\partial y}\rangle$, $\langle\frac{\partial}{\partial z},\frac{\partial}{\partial z}\rangle>0$. With respect to these coordinates, we write

$$F=E_x\,dx\wedge dt+E_y\,dy\wedge dt+E_z\,dz\wedge dt+B_x\,dy\wedge dz+B_y\,dz\wedge dx+B_z\,dx\wedge dy\,.$$

Now we compute dF in these coordinates:

$$\begin{aligned}
dF &= \frac{\partial E_x}{\partial y}\,dy\wedge dx\wedge dt+\frac{\partial E_x}{\partial z}\,dz\wedge dx\wedge dt+\frac{\partial E_y}{\partial x}\,dx\wedge dy\wedge dt+\frac{\partial E_y}{\partial z}\,dz\wedge dy\wedge dt\\
&\quad+\frac{\partial E_z}{\partial x}\,dx\wedge dz\wedge dt+\frac{\partial E_z}{\partial y}\,dy\wedge dz\wedge dt+\frac{\partial B_x}{\partial t}\,dt\wedge dy\wedge dz+\frac{\partial B_x}{\partial x}\,dx\wedge dy\wedge dz\\
&\quad+\frac{\partial B_y}{\partial t}\,dt\wedge dz\wedge dx+\frac{\partial B_y}{\partial y}\,dy\wedge dz\wedge dx+\frac{\partial B_z}{\partial t}\,dt\wedge dx\wedge dy+\frac{\partial B_z}{\partial z}\,dz\wedge dx\wedge dy\\
&=\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)\cdot dx\wedge dy\wedge dz+\left(-\frac{\partial E_z}{\partial y}+\frac{\partial E_z}{\partial y}+\frac{\partial B_x}{\partial t}\right)\cdot dt\wedge dy\wedge dz\\
&\quad+\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{\partial B_y}{\partial t}\right)\cdot dt\wedge dz\wedge dy+\left(-\frac{\partial E_x}{\partial y}+\frac{\partial E_y}{\partial x}+\frac{\partial B_z}{\partial t}\right)\cdot dt\wedge dx\wedge dy\,.
\end{aligned}$$

To abbreviate this, we introduce the time dependent vector fields $\vec{B}:=(B_x,B_y,B_z)$ and $\vec{E}:=(E_x,E_y,E_z)$. In terms of classical electrodynamics, $\vec{E}$ is the ***electric field*** and $\vec{B}$ is the ***magnetic field***. Note that the definition of these vector fields depends on the choice of coordinate system. Then we have:

$$dF=0\Leftrightarrow\qquad \operatorname{div}\vec{B}=0\qquad (\textit{\textbf{Gauß' law}})\tag{3.7}$$

$$\text{and}\qquad \frac{\partial\vec{B}}{\partial t}+\operatorname{rot}\vec{E}=0\qquad (\textit{\textbf{Faraday's law}})\,.\tag{3.8}$$

Hence the first two of the classical Maxwell equations for the electric field $\vec{E}$ and the magnetic field $\vec{B}$ have shown up as special instances of the Bianchi identity for a connection on a U(1)-principal bundle. To derive the two remaining (non-homogeneous) Maxwell equations, we need to introduce an appropriate action functional for the connection ω.
Let $J \in \Omega^3(M;\mathbb{R})$ and pick a "back-ground" connection (or reference connection) $\omega_0 \in \mathcal{C}(P)$. Then for any connection $\omega \in \mathcal{C}(P)$, we have that $s^*(\omega - \omega_0)$ for some local section s is independent of the choice of s and yields a well-defined 1-form $iA = iA(\omega,\omega_0) \in \Omega^1(M;i\mathbb{R})$. We then have $dA = F - F_0$. With these data, we introduce the ***Lagrangian***

$$\mathcal{L} : \mathcal{C}(P) \to \Omega^4(M;i\mathbb{R}), \quad \mathcal{L}(\omega) := \frac{1}{2} F \wedge *F + A \wedge J\,.$$

We say that ω is ***critical*** for $\mathcal{L}$ iff

$$\forall\,\text{open } U \Subset M, \forall\,\eta \in \Omega^1(M;\mathbb{R}), \operatorname{supp}(\eta) \subset U : \qquad \frac{d}{dt}\Big|_{t=0} \int\limits_{\bar{U}} \mathcal{L}\left(\omega_{t,\eta}\right) = 0\,,$$

where $\omega_{t,\eta} \in \mathcal{C}(P)$ is such that $A(\omega_{t,\eta},\omega_0) = A(\omega,\omega_0) + t\eta$. For the corresponding curvature, we obtain:

$$F\left(\omega_{t,\eta}\right) - F(\omega) = d\left(A(\omega_{t,\eta},\omega_0) - A\left(\omega,\omega_0\right)\right) = t \cdot d\eta\,,$$

hence $F(\omega_{t,\eta}) = F(\omega) + t \cdot d\eta$. To compute the Euler-Lagrange equations for the Lagrangian $\mathcal{L}$, we write $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ with $\mathcal{L}_1(\omega) := \frac{1}{2} F \wedge *F$ and $\mathcal{L}_2(\omega) = A \wedge J$. Now we compute:

$$\begin{aligned}
\frac{\partial}{\partial t}\Big|_{t=0} \mathcal{L}_1\left(\omega_{t,\eta}\right) &= \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} (F + t \cdot d\eta) \wedge *(F + t \cdot d\eta) \\
&= \frac{1}{2} \cdot (d\eta \wedge *F + F \wedge *d\eta) \\
&\overset{(3.5)}{=} d\eta \wedge *F\,,
\end{aligned}$$

hence

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int\limits_{\bar{U}} \mathcal{L}_1(\omega_{t,\eta}) &= \int\limits_{\bar{U}} d\eta \wedge *F \\
&= \int\limits_{\bar{U}} d\left(\eta \wedge *F\right) + \eta \wedge d\left(*F\right) \\
&\overset{\text{Stokes}}{=} \int\limits_{\bar{U}} \eta \wedge d\left(*F\right) \\
&\overset{\operatorname{supp}(\eta)\subset U}{=} \int\limits_{M} \eta \wedge d\left(*F\right)\,.
\end{aligned}$$

Similarly, for $\mathcal{L}_2$, we compute:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_{\bar{U}} \mathcal{L}_2\left(\omega_{t,\eta}\right) &= \frac{d}{dt}\Big|_{t=0} \int_{\bar{U}} (A + t\cdot\eta) \wedge J \\
&= \int_{\bar{U}} \eta \wedge J \\
&\overset{\mathrm{supp}(\eta)\subset U}{=} \int_{M} \eta \wedge J\,.
\end{aligned}$$

Hence

$$\begin{aligned}
& \omega \in \mathcal{C}(P) \text{ is critical for } \mathcal{L} \\
\Leftrightarrow \quad & \forall\, \eta \in \Omega^1(M;\mathbb{R})\,, \mathrm{supp}(\eta) \Subset M : \quad \int_M \eta \wedge (d * F + J) = 0 \\
\Leftrightarrow \quad & d * F + J = 0\,.
\end{aligned}$$

We observe that the Lagrangian $\mathcal{L}$ depends on the choice of the background connection ω_0 but the Euler-Lagrange equation $d * F + J = 0$ does not. This is because replacing ω_0 by some other background connection $\tilde{\omega}_0$ yields for the Lagrangians

$$\mathcal{L}(\omega) - \tilde{\mathcal{L}}(\omega) = A\left(\tilde{\omega}_0, \omega_0\right) \wedge J.$$

Hence, after integration, the Lagrangians differ only by a constant.

In the local coordinates as above, we write:

$$J = \varrho \cdot dx \wedge dy \wedge dz - j_x \cdot dt \wedge dy \wedge dz - j_y \cdot dt \wedge dz \wedge dx - j_z \cdot dt \wedge dx \wedge dy\,.$$

As for $\vec{E}$ and $\vec{B}$ above, we write the coefficients as a time dependent vector field $\vec{j} := (j_x, j_y, j_z)$. In terms of classical electrodynamics, ϱ is the ***electric charge density*** and $\vec{j}$ is the ***electric current density***.

Now for Minkowski space with the standard coordinates (t, x, y, z), we compute:

$$\begin{aligned}
*dt \wedge dx &= dy \wedge dz & *dy \wedge dz &= -dt \wedge dx \\
*dt \wedge dy &= dz \wedge dx & *dz \wedge dx &= -dt \wedge dy \\
*dt \wedge dz &= dx \wedge dy & *dx \wedge dy &= -dt \wedge dz\,.
\end{aligned}$$

Hence for $*F$, we obtain:

$$\begin{aligned}
*F \;=\; & -E_x\, dy \wedge dz - E_y\, dz \wedge dx - E_z\, dx \wedge dy \\
& + B_x\, dx \wedge dt + B_y\, dy \wedge dt - B_z\, dz \wedge dt\,,
\end{aligned}$$

and for $d * F$, we obtain by a computation similar to the one for dF above:

$$\begin{aligned} d(*F) &= \left(-\mathrm{div}\vec{E}\right) \cdot dx \wedge dy \wedge dz + \left(\mathrm{rot}\vec{B} - \frac{\partial \vec{E}}{\partial t}\right)_x \cdot dt \wedge dy \wedge dz \\ &\quad + \left(\mathrm{rot}\vec{B} - \frac{\partial \vec{E}}{\partial t}\right)_y \cdot dt \wedge dz \wedge dx + \left(\mathrm{rot}\vec{B} - \frac{\partial \vec{E}}{\partial t}\right)_z \cdot dt \wedge dx \wedge dy \,. \end{aligned}$$

Hence on Minkowski space with the standard coordinates, we have:

$$\begin{aligned} \omega \text{ critical for } \mathcal{L} \Leftrightarrow \quad & d(*F) + J = 0 \\ \Leftrightarrow \quad & \mathrm{div}\vec{E} = \varrho \qquad (\textbf{\textit{Coulomb's law}}) \qquad (3.9) \\ & \text{and } \mathrm{rot}\vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \qquad (\textbf{\textit{Ampère's law}})\,. \qquad (3.10) \end{aligned}$$

From the equation $d(*F) + J = 0$, we deduce $0 = d(d(*F) + J) = dJ$. In standard coordinates on Minkowski space, we thus have

$$\begin{aligned} 0 = dJ &= \left(\frac{\partial \varrho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\right) \cdot dt \wedge dx \wedge dy \wedge dz \\ &\Leftrightarrow \frac{\partial \varrho}{\partial t} + \mathrm{div}\vec{j} = 0\,. \qquad (\textbf{\textit{continuity equation}}) \qquad (3.11) \end{aligned}$$

To illustrate this equation, let $B \subset \mathbb{R}^3$ be compact with smooth boundary. Then we have:

$$\begin{aligned} 0 &= \int_{[t_0,t_1]\times B} dJ \\ &\overset{\text{Stokes}}{=} \int_{\partial([t_0,t_1]\times B)} J \\ &= \underbrace{\int_B \varrho(t_1)\, dx\, dy\, dz}_{\text{charge of } B \text{ at time } t_1} - \underbrace{\int_B \varrho(t_0)\, dx\, dy\, dz}_{\text{charge of } B \text{ at time } t_0} + \underbrace{\int_{t_0}^{t_1}\int_{\partial B} \left\langle \vec{j}, \nu \right\rangle \mathrm{dvol}_{\partial B}\, dt}_{\text{flux through } \partial B}\,. \end{aligned}$$

(Here ν denotes the exterior normal of ∂B). Hence the continuity equation yields the conservation of charge.

How do we feel the electromagnetic field F? A test particle (of mass 1 and charge 1) is described by its worldline, meaning a timelike smooth curve $c : I \to M$, $\langle c', c' \rangle < 0$. In standard coordinates on Minkowski space we write

$$c(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau)) = (t(\tau), \vec{c}(\tau)) \quad \text{and} \quad c'(\tau) = \left(t'(\tau), \vec{c}\,'(\tau)\right)\,.$$

The condition for c to be timelike reads $\langle c', c' \rangle = -(t')^2 + |\vec{c}'|^2 < 0$. For the observed velocity

$$\vec{v} = \frac{d\vec{c}}{dt} = \frac{d\vec{c}}{d\tau} \cdot \frac{d\tau}{dt} = \frac{\vec{c}'}{t'}\,,$$

we thus have the condition $|\vec{v}|^2 < 1$, i.e. the observed velocity of the test particle is less than the speed of light. The force imposed on the test particle in the electromagnetic field is given by the curvature F, so by Newton's law, we have the following equation of motion:[1]

$$\frac{\nabla}{d\tau} c' + F\left(c', \cdot\right)^\sharp = 0\,. \tag{3.12}$$

Remark 3.2.1. For any timelike smooth curve c satisfying (3.12), we have:

$$\frac{d}{dt}\left\langle c', c' \right\rangle = 2\left\langle \frac{\nabla}{d\tau} c', c' \right\rangle = 2\left\langle F\left(c', \cdot\right)^\sharp, c' \right\rangle = 2F\left(c', c'\right) = 0\,,$$

i.e. c' is parametrized proportionally to eigentime.

Remark 3.2.2. Note that since (3.12) is a linear ODE of second order, for any $p \in M$ and any $X \in T_pM$, there exists a unique maximal solution c to (3.12) satisfying $c(t_0) = p$ and $c'(t_0) = X$.

W.l.o.g. we will henceforth assume c to be parametrized by eigentime, so that $\langle c', c' \rangle = -(t')^2 + |\vec{c}'|^2 = -1$. This can of course always be achieved by an appropriate rescaling. We will henceforth also assume that $t' > 0$, which can be achieved by replacing τ by $-\tau$.

Recall that the mass m of a test particle with rest mass m_0 varies with the velocity of the particle as

$$m = \frac{m_0}{\sqrt{1 - |\vec{v}|^2}} = \frac{m_0}{\sqrt{1 - \left|\frac{\vec{c}'}{t'}\right|^2}} = \frac{m_0 \cdot t'}{\sqrt{(t')^2 - |\vec{c}'|^2}} = m_0 \cdot t'\,.$$

Now let us compute the equation of motion in the standard coordinates on Minkowski space. For the left hand side of (3.12), we have:

$$\frac{d}{dt}\, m\,\vec{v} = \frac{d}{dt}\, m_0 \cdot t' \cdot \frac{\vec{c}'}{t'} = m_0 \cdot \frac{d}{dt}\,\vec{c}' = m_0 \cdot \frac{d\tau}{dt} \cdot \vec{c}'' = m_0 \cdot \frac{\vec{c}''}{t'}\,.$$

[1] For any $\eta \in T_x^*M$, the vector $\eta^\sharp \in T_xM$ is defined as the dual to η with respect to the (non-degenerate) inner product, i.e. for any $Y \in T_xM$, we have: $\eta(Y) = \langle \eta^\sharp, Y \rangle$.

For the right hand side we have:

$$\begin{aligned}
F\left(c',\cdot\right)^\sharp &= F\left(t'\cdot\frac{\partial}{\partial t}+x'\cdot\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+z'\cdot\frac{\partial}{\partial z},\cdot\right)^\sharp \\
&= \Big(-t'\cdot\big(E_x\,dx+E_y\,dy+E_z\,dz\big)+x'\cdot\big(E_x\,dt-B_y\,dz+B_z\,dy\big) \\
&\qquad +y'\cdot\big(E_y\,dt+B_x\,dz-B_z\,dx\big)+z'\cdot\big(E_z\,dt-B_x\,dy+B_y\,dx\big)\Big)^\sharp \\
&= \big(x'\cdot E_x+y'\cdot E_y+z'\cdot E_z\big)\cdot dt^\sharp+\big(-t'\cdot E_x-y'\cdot B_z+z'\cdot B_y\big)\cdot dx^\sharp \\
&\quad +\big(-t'\cdot E_y+x'\cdot B_z-z'\cdot B_x\big)\cdot dy^\sharp+\big(-t'\cdot E_z-x'\cdot B_y+y'\cdot B_x\big)\cdot dz^\sharp \\
&= -\left\langle\vec{c}\,',\vec{E}\right\rangle\cdot\frac{\partial}{\partial t}+\left(-t'\cdot\vec{E}+\vec{B}\times\vec{c}\,'\right).
\end{aligned}$$

Hence (3.12) is equivalent to the equations:

$$\begin{aligned}
t''+\langle\vec{c}\,',\vec{E}\rangle &= 0 \\
\text{and}\qquad \vec{v}\,''-t'\cdot\vec{E}+\vec{B}\times\vec{c}\,' &= 0\,.
\end{aligned}$$

Now note that $0=\frac{d}{dt}\langle c',c'\rangle=-2t't''+2\langle\vec{c}\,',\vec{c}\,''\rangle$ yields $t'=\langle\vec{v},\vec{c}\,''\rangle$. Hence the first equation follows from the second by scalar multiplication with $\vec{v}$. We thus found:

$$\begin{aligned}
\frac{\nabla}{d\tau}c'+F\left(c',\cdot\right)^\sharp=0 \quad&\Leftrightarrow\quad \frac{\vec{c}\,''}{t'}-\vec{E}+\vec{B}\times\vec{v}=0 \\
&\Leftrightarrow\quad \frac{d}{dt}\left(m\cdot\vec{v}\right)=\vec{E}+\vec{v}\times\vec{B}\,. \qquad (\textbf{\textit{Lorentz force law}}) \quad (3.13)
\end{aligned}$$

Let $\mathcal{L}$ be the Lagrangian for classical electrodynamics as defined above. So far, we only considered variations of $\mathcal{L}$ with respect to the connection ω on the U(1)-principal bundle $P\to M$. We derived the two inhomogeneous Maxwell equations as the Euler-Lagrange equations for this variation. But the Lagrangian $\mathcal{L}:=\frac{1}{2}F\wedge *F+A\wedge J$ also depends on the chosen Riemannian metric, since the Hodge-star operator does.

In general relativity, the metric is to be considered as a dynamical variable, so we should also study variations of the metric in $\mathcal{L}$ and compute the Euler-Lagrange equations thereof. So we take another look at the Lagrangian $\mathcal{L}$, varying the metric this time (and fixing the connection ω). First of all, we have:

$$\mathcal{L}_1(\omega,g)=\frac{1}{2}F\wedge *_g F=\frac{1}{2}\langle F,F\rangle_g\cdot\mathrm{vol}_g\,,$$

so we should first compute the derivatives of the two factors separately. To this end, let $g(t)$ be a smooth 1-parameter family of Riemannian metrics on M such that $g(0)=g$

and $\frac{d}{dt}\Big|_{t=0} g(t) = h \in \odot^2 T^*M$. Computing in local coordinates, we find:[2]

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathrm{vol}_{g(t)} &= \frac{d}{dt}\Big|_{t=0} \sqrt{-\det g_{ij}(t)} \cdot dx^0 \wedge \ldots \wedge dx^3 \\
&= -\frac{1}{2} \left(-\det g_{ij}\right)^{-1/2} \cdot \det\left(g_{ij}\right) \cdot \mathrm{tr}\Big(\left(g^{ij} \cdot h_{jk}\right)^i_k\Big) \cdot dx^0 \wedge \ldots \wedge dx^3 \\
&= \frac{1}{2} \left(-\det g_{ij}\right)^{1/2} \cdot g^{ij} \cdot h_{ji} \cdot dx^0 \wedge \ldots \wedge dx^3 \\
&= \frac{1}{2} \mathrm{tr}_g(h) \cdot \mathrm{vol}_g .
\end{aligned}$$

For the curvature term, we find:[3]

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \langle F, F\rangle_{g(t)} &= \frac{d}{dt}\Big|_{t=0} \Big\langle \sum_{\alpha<\beta} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \sum_{\gamma<\delta} F_{\gamma\delta} dx^\gamma \wedge dx^\delta \Big\rangle_{g(t)} \\
&= \frac{1}{4} \cdot \frac{d}{dt}\Big|_{t=0} F_{\alpha\beta} \cdot F_{\gamma\delta} \left\langle dx^\alpha \wedge dx^\beta, dx^\gamma \wedge dx^\delta \right\rangle_{g(t)} \\
&= \frac{1}{4} \cdot F_{\alpha\beta} \cdot F_{\gamma\delta} \frac{d}{dt}\Big|_{t=0} \Big(g^{\alpha\gamma}(t) \cdot g^{\beta\delta}(t) - g^{\alpha\delta}(t) \cdot g^{\beta\gamma}(t) \Big) \\
&= \frac{1}{4} \cdot F_{\alpha\beta} F_{\gamma\delta} \cdot \Big(- g^{\alpha i}\, h_{ij}\, g^{j\gamma}\, g^{\beta\delta} - g^{\alpha\gamma}\, g^{\beta i}\, h_{ij}\, g^{j\delta} \\
&\quad + g^{\alpha i}\, h_{ij}\, g^{j\delta}\, g^{\beta\gamma} + g^{\alpha\delta}\, g^{\beta i}\, h_{ij}\, g^{j\gamma} \Big) \\
&= \frac{1}{4} \cdot \Big(- F^{i\delta} F^j{}_\delta - F^{\gamma i} F_\gamma{}^i + F^{i\gamma} F_\gamma{}^j + F^{\delta i} F^j{}_\delta \Big) \cdot h_{ij} \\
&= F^{\delta i} F^j{}_\delta\, h_{ij} .
\end{aligned}$$

Collecting the terms, we find:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathcal{L}_1(\omega, g(t)) &= \frac{1}{2} \cdot \Big(F^{\delta i} F^j{}_\delta\, h_{ij} + \langle F, F\rangle_g \cdot \frac{1}{2} \cdot \mathrm{tr}_g(h) \Big) \\
&= -\frac{1}{2} \cdot T^{ij} h_{ij} \cdot \mathrm{vol}_g ,
\end{aligned}$$

where $T^{ij} := -F^{\delta i} F^j{}_\delta - \frac{1}{2} \cdot \langle F, F\rangle_g \cdot g^{ij}$. The $(2,0)$-tensor field $T := T^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ is called the ***energy momentum tensor (field)*** of ω (or of F).

To see at which point the energy momentum tensor becomes important, we briefly recall that general relativity deals with yet another action functional of the metric, given by the geometric Lagrangian or the ***Einstein-Hilbert action***:

$$\mathcal{L}_{\mathrm{geom}}(g) := -\frac{1}{2} \cdot \mathrm{scal}_g \cdot \mathrm{vol}_g .$$

One computes that for a smooth curve $g(t)$ as above with $g'(0) = h$, we have: $\frac{d}{dt}\Big|_{t=0} \mathrm{scal}_{g(t)} = -\mathrm{ric}_g^{ij} h_{ij} + \mathrm{div}(X)$ for some vector field X (which is of no further interest

[2] Here we use the formula: $(\det A(t))^{-1} \cdot \frac{d}{dt} \det A(t)) = \mathrm{tr}(A(t)^{-1} \cdot \frac{d}{dt} A(t))$.

[3] Here we use: $0 = \frac{d}{dt} g^{ik}(t) \cdot g_{kj}(t) = \dot{g}^{ik} \cdot g_{kj} + g^{ik} \cdot \dot{g}_{kj}$, which implies $\dot{g}^{il} = -g^{ij} \cdot \dot{g}_{jk} \cdot g^{kl}$.

for us, since the divergence term vanishes upon integration anyway). If $h \in \odot^2 T^*M$ has compact support, then we get for the corresponding action functional:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int\limits_M \mathcal{L}_{\mathrm{geom}}(g(t)) &= -\frac{1}{2}\cdot \int\limits_M \left(-\mathrm{ric}_g^{ij}\, h_{ij} + \frac{1}{2}\cdot g^{ij}\, h_{ij}\cdot \mathrm{scal}_g\right)\cdot \mathrm{vol}_g \\
&= \frac{1}{2}\cdot \int\limits_M \left(\mathrm{ric}_g^{ij} - \frac{1}{2}\cdot g^{ij}\cdot \mathrm{scal}_g\right)\cdot h_{ij}\cdot \mathrm{vol}_g\,.
\end{aligned}$$

Putting the two action principles together, we find:

$$g \text{ is critical for } \mathcal{L}_{\mathrm{geom}}(g) + \mathcal{L}_1(\omega, g) + \mathcal{L}_2(\omega) \tag{3.14}$$

$$\Leftrightarrow \quad \forall\, h \in \odot^2 T^*M,\ \mathrm{supp}(h) \Subset M : \quad \int\limits_M \left(\mathrm{ric}_g^{ij} - \frac{1}{2}\cdot g^{ij}\cdot \mathrm{scal}_g - T^{ij}\right)\cdot h_{ij}\cdot \mathrm{vol}_g$$

$$\Leftrightarrow \quad \mathrm{ric}_g - \frac{1}{2}\mathrm{scal}_g\cdot g = T \qquad (\textbf{\textit{Einstein field equations}}) \tag{3.15}$$

(Here $T := T_{ij}dx^i \otimes dx^j$ is the $(0,2)$-tensor field associated to the $(2,0)$-tensor field defined above.)
Next we want to express the energy momentum tensor T in terms of the electric field $\vec{E}$ and the magnetic field $\vec{B}$. On Minkowski space with the standard coordinates we find:

$$\begin{aligned}
\langle F, F\rangle &= -\sum_{k=1}^{3} F_{0k}\, F_{0k} + \sum_{1\le i<k\le 3} F_{ik}\, F_{ik} \\
&= -\left\langle \vec{E}, \vec{E}\right\rangle + \left\langle \vec{B}, \vec{B}\right\rangle \\
&= -|\vec{E}|^2 + |\vec{B}|^2 \\
T^{00} &= F^{0k}F^{0}{}_{k} - \frac{1}{2}\cdot\left(|\vec{B}|^2 - |\vec{E}|^2\right)\cdot g^{00} \\
&= |\vec{E}|^2 + \frac{1}{2}\cdot\left(|\vec{B}|^2 - |\vec{E}|^2\right) \\
&= \frac{1}{2}\cdot\left(|\vec{E}|^2 + |\vec{B}|^2\right) \qquad\qquad (\textbf{\textit{energy density}}) \\
T^{01} &= F^{0k}F^{1}{}_{k} - \frac{1}{2}\cdot\left(|\vec{B}|^2 - |\vec{E}|^2\right)\cdot g^{01} \\
&= E_2B_3 - E_3B_2 \\
&= \left(\vec{E}\times\vec{B}\right)_1 \\
\text{and similarly}\quad T^{02} &= \left(\vec{E}\times\vec{B}\right)_2 \\
T^{03} &= \left(\vec{E}\times\vec{B}\right)_3 .
\end{aligned}$$

The vector field $(T^{01}, T^{02}, T^{03}) = \vec{E}\times\vec{B} =: \vec{S}$ is called the ***Poynting vector***. Finally, for $1 \leq i, j \leq 3$, we find:

$$
\begin{aligned}
T^{ij} &= F^{i0}F^{j}{}_{0} + \sum_{k=1}^{3} F^{ik}F^{j}{}_{k} - \frac{1}{2}\cdot\left(\left|\vec{B}\right|^2 - \left|\vec{E}\right|^2\right)\cdot g^{ij} \\
&= \begin{cases} -E_iE_j - B_jB_i & : \quad i \neq j \\ -E_i^2 + \left|\vec{B}\right|^2 - B_i^2 - \frac{1}{2}\cdot\left(\left|\vec{B}\right|^2 - \left|\vec{E}\right|^2\right) & : \quad i = j \end{cases} \\
&= -E_iE_j - B_iB_j + \frac{1}{2}\cdot\left(\left|\vec{B}\right|^2 + \left|\vec{E}\right|^2\right)\cdot g^{ij} \\
&=: -\sigma^{ij}\,.
\end{aligned}
$$

The $(2,0)$-tensor field

$$\sigma := \vec{E}\otimes\vec{E} + \vec{B}\otimes\vec{B} - \frac{1}{2}\cdot\left(\left|\vec{B}\right|^2 + \left|\vec{E}\right|^2\right)\cdot g$$

is called the ***Maxwell stress tensor***. (Note that the g on the right is the inverse metric.)
Now we exploit some geometrical properties of the electromagnetic Lagrangian (resp. the associated action functional) to derive further physical properties of the energy momentum tensor.

Conformal invariance

Let V be an oriented n-dimensional $\mathbb{R}$-vector space with inner product $g = \langle\cdot,\cdot\rangle$. We discuss how the Hodge-star operator $*$ changes, if we rescale the metric g to $g' = \lambda^2 \cdot g$ by a positive factor $\lambda > 0$:
If $e_1, \ldots, e_n$ is a generalized orthonormal basis of V for g, then $e_1' := \frac{1}{\lambda}\cdot e_1, \ldots, e_n' := \frac{1}{\lambda}\cdot e_n$ is a generalized orthonormal basis of V for g'. If $e_1^*, \ldots, e_n^*$ is the basis of V^* dual to $e_1, \ldots, e_n$, then $(e_1^*)' := \lambda\cdot e_1^*, \ldots, (e_n^*)' := \lambda\cdot e_n^*$ is the basis of V^* dual to $e_1', \ldots, e_n'$. Correspondingly, $(e_{i_1}^*)' \wedge \ldots \wedge (e_{i_k}^*)' = \lambda^k e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*$, $1 \leq i_1 < \ldots < i_k \leq n$, is a generalized orthonormal basis for the inner product on $\Lambda^k V^*$ induced by g'. Hence this product itself is given as $\langle\cdot,\cdot\rangle' = \lambda^{-2k}\cdot\langle\cdot,\cdot\rangle$. In particular, we have for the volume forms induced from the two inner products:

$$\mathrm{vol}_{g'} = \lambda^n \cdot \mathrm{vol}_g\,.$$

For any $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^{n-k}V^*$, we then have:

$$
\begin{aligned}
\langle\omega, *\eta\rangle\cdot\mathrm{vol}_g &= \omega\wedge\eta \\
&= \left\langle\omega, *'\eta\right\rangle'\cdot\mathrm{vol}_{g'} \\
&= \lambda^{-2k}\cdot\left\langle\omega, *'\eta\right\rangle\cdot\lambda^n\cdot\mathrm{vol}_g \\
&= \lambda^{n-2k}\cdot\left\langle\omega, *'\eta\right\rangle\cdot\mathrm{vol}_g\,,
\end{aligned}
$$

whence $*' = \lambda^{2k-n}\cdot *$. In particular, if $2k = n$, then $*' = *$.

In electrodynamics, we have $n = 4$, $k = 2$, hence the electrodynamical Lagrangian $\mathcal{L}$ is ***conformally invariant***, meaning that for $g' = \lambda \cdot g$, $g \in \mathcal{C}^\infty(M)$, $\lambda > 0$, we have $\mathcal{L}(\omega, g') = \mathcal{L}(\omega, g)$. So let us compute the effect of the conformal invariance on the energy momentum tensor. To this end, we take the family $g(t) := (1+t) \cdot g$ of conformally equivalent metrics, so that $g(0) = g$, $\dot{g}(0) = g$. We then have:

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \mathcal{L}_1(\omega, g) \\ &= \frac{d}{dt}\Big|_{t=0} \mathcal{L}_1(\omega, g(t)) \\ &= -\frac{1}{2} \cdot T \cdot \dot{g}(0) \cdot \mathrm{vol}_g \\ &= -\frac{1}{2} \cdot T \cdot g \cdot \mathrm{vol}_g \\ &= -\frac{1}{2} \mathrm{tr}_g(T) \cdot \mathrm{vol}_g \,, \end{aligned}$$

which yields $\mathrm{tr}_g(T) = 0$. Hence by conformal invariance of the electrodynamic Lagrangian, the energy momentum tensor is trace free.

Diffeomorphism invariance

For any diffeomorphism $\varphi \in \mathrm{Diff}(M)$ with $\mathrm{supp}(\varphi) \subset U \Subset M$, we have the pull-back diagram:

$$\begin{tikzcd} \varphi^* P \arrow[r, "\Phi"] \arrow[d] & P \arrow[d] \\ M \arrow[r, "\varphi"] & M \end{tikzcd}$$

So let us compute the effect of a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ and its induced bundle isomorphism $\Phi : \varphi^* P \to P$ on the action functional for the electrodynamics Lagrangian:

$$\begin{aligned} \int_U \mathcal{L}_1\left(\Phi^*\omega, \varphi^* g\right) &= \frac{1}{2} \cdot \int_U \varphi^* F \wedge *_{\varphi^* g}\, \varphi^* F \\ &= \frac{1}{2} \cdot \int_U \langle \varphi^* F, \varphi^* F \rangle_{\varphi^* g} \cdot \varphi^* \mathrm{vol}_g \\ &= \frac{1}{2} \cdot \int_U \langle F, F \rangle_g \circ \varphi \cdot \varphi^* \mathrm{vol}_g \\ &= \frac{1}{2} \cdot \int_U \langle F, F \rangle_g \cdot \mathrm{vol}_g \\ &= \frac{1}{2} \cdot \int_U \mathcal{L}_1(\omega, g) \,. \end{aligned}$$

Hence the action functional given by the electrodynamic Lagrangian $\mathcal{L}$ is invariant under (compactly supported) diffeomorphisms of the basis M. Note that in contrast to the pointwise conformal invariance, this diffeomorphism invariance is not a pointwise invariance of the Lagrangian density itself, but only of the corresponding action functional.
Now let us compute the effect of diffeomorphism invariance on the energy momentum tensor. To this end, let $X \in \mathfrak{X}(M)$ be a smooth vector field with compact support and let φ_t be its flow. Then we study the family $g_t := \varphi_t^* g$. We first claim that $h = \frac{d}{dt}\Big|_{t=0} \varphi_t^* g = \mathcal{L}_X g = 2\nabla^{\mathrm{sym}} X$, where $\nabla^{\mathrm{sym}} X$ is the symmetrization of the covariant derivative, to be defined in the following justification of the claim:

$$
\begin{aligned}
(\mathcal{L}_X g)(Y,Z) &= \mathcal{L}_X(g(Y,Z)) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z) \\
&= \partial_X g(Y,Z) - g([X,Y],Z) - g(Y,[X,Z]) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X) \\
&= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \\
&=: 2(\nabla^{\mathrm{sym}} X)(Y,Z)\,.
\end{aligned}
$$

Putting the above family g_t of metrics into the action functional, we may now compute the effect of diffeomorphism invariance. (Note that to compute the derivative with respect to t of $\Phi_t^*\omega$, we need to identify the forms $\Phi_t^*\omega$ for different t, which by construction live on the different bundles $\varphi_t^* P$. This is most easily done via the local sections $s_{\alpha,t}$ of $\varphi_t^* P$ given by $s_{\alpha,t} := \Phi_t^{-1} \circ s_\alpha \circ \varphi_t$, where s_α is any local section of P.) For the variation of the electrodynamic Lagrangian $\mathcal{L}_1$ along the family $(\Phi_t^*\omega, \varphi_t^* g)$, we thus find:

$$
\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} \int_U \mathcal{L}_1(\omega, g) \\
&= \frac{d}{dt}\Big|_{t=0} \int_U \mathcal{L}_1\left(\Phi_t^*\omega, \varphi_t^* g\right) \\
&= \frac{d}{dt}\Big|_{t=0} \int_U \mathcal{L}_1\left(\Phi_t^*\omega, g\right) + \frac{d}{dt}\Big|_{t=0} \int_U \mathcal{L}_1\left(\omega, \varphi_t^* g\right) \\
&= \int_U \eta \wedge d * F - \frac{1}{2} \cdot \int_U T \cdot h \cdot \mathrm{vol}_g\,,
\end{aligned}
$$

where η is the following 1-form:[4]

$$
\begin{aligned}
i\eta &:= \frac{d}{dt}\Big|_{t=0} s_{\alpha,t}^*\, \Phi_t^*\omega \\
&= \frac{d}{dt}\Big|_{t=0} \varphi_t^*\, s_\alpha^*\omega \\
&= \mathcal{L}_X\left(s_\alpha^*\omega\right)
\end{aligned}
$$

[4]Here we use the so called Cartan's magic formula: for any $\alpha \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$, we have $\mathcal{L}_X\alpha = d(\iota_X\alpha) + \iota_X(d\alpha)$. Here ι_X denotes the insertion of X in the first slot of a form, i.e. $\iota_X\beta = \beta(X, \ldots)$.

$$\begin{aligned}
&\overset{\text{Cartan}}{=} \quad \iota_X d\,(s_\alpha^*\omega) + d\big(\underbrace{s_\alpha^*\omega(X)}_{=:f}\big) \\
&= \quad i \cdot (\iota_X F + df)\ .
\end{aligned}$$

We thus have as a consequence of the diffeomorphism invariance:

$$\begin{aligned}
\int_M T \cdot \nabla^{\text{sym}} X \cdot \text{vol}_g \quad &= \quad \int_M (\iota_X F + df) \wedge d * F \\
&= \quad \int_M \iota_X F \wedge d * F + d(f \cdot d * F) \\
&\overset{\text{Stokes}}{=} \quad \int_M \langle \iota_X F, *d * F\rangle \cdot \text{vol}_g\ .
\end{aligned}$$

To see what this equations means for the energy momentum tensor T, we define the divergence of T (and similarly of any $(2,0)$-tensor) as:

$$\text{div}(T) := \sum_{i=1}^{n} (\nabla_{e_i} T)\,(e_i^*, \cdot) \in \mathfrak{X}(M)\,,$$

where $e_1, \ldots, e_n$ is a generalized orthonormal basis of TM. The vector field $\text{div}(T)$ does not depend on the choice of orthonormal basis.
With this definition, we find $\int_M T \cdot \nabla^{\text{sym}} X \cdot \text{vol}_g = -\int_M \langle \text{div}(T), X\rangle \cdot \text{vol}_g$ by Stokes theorem, and hence:

$$\begin{aligned}
&\forall\, X \in \mathfrak{X}(M), \text{supp}(X) \Subset M : \quad -\int_M \langle \text{div}(T), X\rangle \cdot \text{vol}_g = \int_M \langle \iota_X F, *d * F\rangle \cdot \text{vol}_g \\
\Leftrightarrow \quad &\forall\, X \in TM : \quad -\langle \text{div}(T), X\rangle = \langle \iota_X F, *d * F\rangle\ . \qquad (3.16)
\end{aligned}$$

If ω is critical for $\mathcal{L}$, thus $d * F = J$, then $-\langle \text{div}(T), X\rangle = \langle \iota_X F, *J\rangle$. So in Minkowski space with standard coordinates, we find for $X = \frac{\partial}{\partial t}$ in the left hand side of (3.16):

$$\begin{aligned}
-\left\langle \text{div}(T), \frac{\partial}{\partial t}\right\rangle \quad &= \quad (-\text{div}(T))^0 \\
&= \quad -\partial_i T^{i0} \\
&= \quad -\frac{\partial}{\partial t} T^{00} - \sum_{i=1}^{3} \frac{\partial}{\partial x^i} T^{i0} \\
&= \quad -\frac{1}{2} \cdot \frac{\partial}{\partial t}\left(|\vec{E}|^2 + |\vec{B}|^2\right) - \text{div}\big(\vec{S}\big)\ .
\end{aligned}$$

For $X = \frac{\partial}{\partial t}$ we find in the right hand side of (3.16):

$$\left\langle \iota_{\frac{\partial}{\partial t}} F, *J\right\rangle \quad = \quad \left\langle \sum_{i=1}^{3} -E_{x^i} dx^i, \varrho dt - \sum_{i=1}^{3} j_{x^i} dx^i \right\rangle \quad = \quad \left\langle \vec{E}, \vec{j}\right\rangle\ .$$

Finally, the diffeomorphism invariance of the action functional associated with the electromagnetic Lagrangian $\mathcal{L}_1$ results in the equation:

$$\frac{1}{2} \cdot \frac{\partial}{\partial t}\left(\left|\vec{E}\right|^2 + \left|\vec{B}\right|^2\right) + \operatorname{div}\left(\vec{S}\right) \;=\; -\left\langle \vec{E}, \vec{j} \right\rangle . \qquad (\textbf{\textit{Poynting's theorem}}) \tag{3.17}$$

In case $\vec{j} = 0$, we can (similarly to what we did for the continuity equation) use Stokes theorem to get an interpretation of the Poynting vector $\vec{S}$. So let $B \subset \mathbb{R}^3$ be compact with smooth boundary. Then we have:

$$\begin{aligned} 0 \quad &= \int\limits_{[t_0,t_1]\times B} \frac{1}{2} \cdot \frac{\partial}{\partial t}\left(\left|\vec{E}\right|^2 + \left|\vec{B}\right|^2\right) + \operatorname{div}\left(\vec{S}\right) \\ &\overset{\text{Stokes}}{=} \underbrace{\frac{1}{2} \cdot \int\limits_{B} \left(\left|\vec{E}\right|^2 + \left|\vec{B}\right|^2\right)(t_1)}_{\text{energy at time } t_1} - \underbrace{\frac{1}{2} \cdot \int\limits_{B} \left(\left|\vec{E}\right|^2 + \left|\vec{B}\right|^2\right)(t_0)}_{\text{energy at time } t_0} \\ &\quad - \underbrace{\int\limits_{t_0}^{t_1} \int\limits_{\partial B} \left\langle \vec{S}, \vec{\nu} \right\rangle (t)\, d\mathrm{vol}_{\partial B}\, dt}_{\text{energy flux through } \partial B} . \end{aligned}$$

(Here ν denotes the exterior normal of ∂B). This yields an interpretation of the Poynting vector $\vec{S}$ as the current density of the energy of the electromagnetic field F.

Gauge invariance

For $\omega \in \mathcal{C}(P)$ and $\varphi \in \mathcal{G}(P)$, we have: $\mathcal{L}_1(\varphi^*\omega, g) = \mathcal{L}_1(\omega, g)$. Indeed, since $G = \mathrm{U}(1)$ is abelian, for $\omega' := \varphi^*\omega$ we have $\bar{\Omega}' = \bar{\Omega}$, hence $F' = F$ and

$$\mathcal{L}_1\left(\omega', g\right) = \frac{1}{2} F' \wedge *F' = \frac{1}{2} F \wedge *F = \mathcal{L}_1(\omega, g) .$$

3.3 Yang-Mills fields

Let M be a smooth manifold and let $E \to M$ be a smooth $\mathbb{K}$-vector bundle ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) with covariant derivative ∇. Recall that differential forms $\eta \in \Omega^k(M; E)$ with values in E are just smooth sections of the vector bundle $\Lambda^k T^* M \otimes E$.

Definition 3.3.1. The exterior derivative $d^\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$ associated with ∇ is defined by:

$$\begin{aligned} d^\nabla \eta\,(X_0, \ldots, X_k) \quad &:= \quad \sum_{i=0}^{k} (-1)^i \nabla_{X_i} \left(\eta \left(X_0, \ldots, \widehat{X_i}, \ldots, X_k \right) \right) \\ &\quad + \sum_{i<j} (-1)^{i+j}\, \eta \left([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k \right) . \end{aligned}$$

Remark 3.3.2. This exterior derivative does not satisfy $d^\nabla \circ d^\nabla = 0$ in general! Indeed, on $\Omega^0(M; E) = \Gamma(E)$, we find:

$$\begin{aligned} (d^\nabla \circ d^\nabla \sigma)(X, Y) \quad &= \quad \nabla_X (d^\nabla \sigma(Y)) - \nabla_Y (d^\nabla \sigma(X)) - d^\nabla \sigma([X, Y]) \\ &= \quad \nabla_X (\nabla_Y \sigma) - \nabla_Y (\nabla_X \sigma) - \nabla_{[X,Y]} \sigma \\ &= \quad R(X, Y) \cdot \sigma\,, \end{aligned}$$

where R is the curvature tensor of the covariant derivative ∇. Indeed, $d^\nabla \circ d^\nabla \equiv 0$ iff the curvature $R \equiv 0$.

Remark 3.3.3. Let E carry a Riemannian resp. Hermitean metric $\langle \cdot, \cdot \rangle$. For $\eta \in \Omega^k(M; E)$ and $\mu \in \Omega^l(M; E)$, we can build $\omega \wedge \eta \in \Omega^{k+l}(M; E \otimes E)$ by writing in local coordinates:

$$\begin{aligned} \eta \quad &= \quad \sum_{i_1 < \ldots < i_k} \eta_{i_1, \ldots, i_k} \otimes dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ \mu \quad &= \quad \sum_{j_1 < \ldots < j_l} \eta_{j_1, \ldots, j_l} \otimes dx^{j_1} \wedge \ldots \wedge dx^{j_l} \\ \eta \wedge \mu \quad &:= \quad \sum_{\substack{i_1 < \ldots < i_k \\ j_1 < \ldots < j_l}} \eta_{i_1, \ldots, i_k} \otimes \mu_{j_1, \ldots, j_l} \otimes dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} . \end{aligned}$$

Using the metric in E, we can also build a real resp. complex valued $k+l$ form out of η and μ by setting:

$$\langle \eta \wedge \mu \rangle \quad := \quad \sum_{\substack{i_1<\ldots<i_k \\ j_1<\ldots<j_l}} \langle \eta_{i_1,\ldots,i_k}, \mu_{j_1,\ldots,j_l} \rangle \, dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} \, .$$

If ∇ is a metric connection with respect to $\langle \cdot, \cdot \rangle$, then we have:

$$d\langle \eta \wedge \mu \rangle = \langle d^\nabla \eta \wedge \mu \rangle + (-1)^k \langle \eta \wedge d^\nabla \mu \rangle \, .$$

Now let M be a Riemannian 4-manifold and let $P \to M$ be an $\mathrm{SU}(N)$-principal bundle, $N \geq 2$. On the Lie algebra $\mathfrak{su}(N)$, we have an Ad-invariant positiv definite symmetric bilinear form defined by $(A,B) \mapsto -\mathrm{tr}(A \cdot B)$. Bilinearity and symmetry are fairly obvious. Let us check that the expression is real valued, i.e. for any $A, B \in \mathfrak{su}(N)$, we have $-\mathrm{tr}(A \cdot B) \in \mathbb{R}$:

$$\begin{aligned}
\overline{-\mathrm{tr}(A \cdot B)} &= -\mathrm{tr}(\overline{A \cdot B}) \\
&= -\mathrm{tr}(\bar{A} \cdot \bar{B}) \\
&= -\mathrm{tr}((\bar{A} \cdot \bar{B})^t) \\
&= -\mathrm{tr}(\bar{B}^t \cdot \bar{A}^t) \\
&= -\mathrm{tr}(B^* \cdot A^*) \\
&= -\mathrm{tr}((-B) \cdot (-A)) \\
&= -\mathrm{tr}(B \cdot A) \\
&= -\mathrm{tr}(A \cdot B) \, .
\end{aligned}$$

To see that $(A,B) \mapsto -\mathrm{tr}(A \cdot B)$ is positive definite, we compute:

$$-\mathrm{tr}(A \cdot A) = -\sum_{i,j=1}^{N} A^i_j \cdot A^j_i = \sum_{i,j=1}^{N} A^i_j \bar{A}^i_j = \sum_{i,j=1}^{N} |A^i_j|^2 \geq 0$$

and obviously, $-\mathrm{tr}(A \cdot A) = 0$ iff $A = 0$.
Finally, $\lambda : (A,B) \mapsto -\mathrm{tr}(A \cdot B)$ is Ad-invariant, since for matrix groups, the adjoint representation Ad is given by conjugation, and we have:

$$\begin{aligned}
\lambda(\mathrm{Ad}_g A, \mathrm{Ad}_g B) &= -\mathrm{tr}(g \cdot A \cdot g^{-1} \cdot g \cdot B \cdot g^{-1}) \\
&= -\mathrm{tr}(g \cdot A \cdot B \cdot g^{-1}) \\
&= -\mathrm{tr}(A \cdot B \cdot g^{-1} \cdot g) \\
&= -\mathrm{tr}(A \cdot B) \\
&= \lambda(A,B) \, .
\end{aligned}$$

By the Ad-invariance, the inner product λ on $\mathfrak{g} = \mathfrak{su}(N)$ gives a well-defined Riemannian metric λ on the adjoint bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ by:

$$\lambda([p,A],[p,B]) := \lambda(A,B) = -\mathrm{tr}(A \cdot B) \, .$$

As explained in Remark 2.3.9, any connection 1-form $\omega \in \mathcal{C}(P)$ yields a covariant derivative ∇^ω on the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ by:

$$\nabla^\omega_X [s, A] := [s, \partial_X s + \mathrm{ad}(s^*\omega(X)) \cdot A] .$$

In fact, ∇^ω is a metric connection with respect to the Riemannian metric λ. On the one hand, we have:

$$\partial_X \lambda([s, A], [s, B]) = -\partial_X \mathrm{tr}(A \cdot B) = -\mathrm{tr}((\partial_X A) \cdot B + A \cdot \partial_X(B)) .$$

On the other hand, we find:

$$\begin{aligned}
&\lambda(\nabla^\omega_X [s, A], [s, B]) + \lambda([s, A], \nabla^\omega_X [s, B]) \\
&\quad= \lambda([s, \partial_X A + \mathrm{ad}(s^*\omega(X)) \cdot A], [s, B]) + \lambda([s, A], [s, \partial_X B + \mathrm{ad}(s^*\omega(Y)) \cdot B]) \\
&\quad= -\mathrm{tr}((\partial_X A) \cdot B + [s^*\omega(X), A] \cdot B) - \mathrm{tr}(A \cdot (\partial_X B) + A \cdot [s^*\omega(Y), B]) \\
&\quad= \partial_X(-\mathrm{tr}(A \cdot B)) - \underbrace{\mathrm{tr}([s^*\omega(X), A] \cdot B + A \cdot [s^*\omega, B])}_{=\mathrm{tr}([s^*\omega(X), A \cdot B])=0} \\
&\quad= \partial_X \lambda([s, A], [s, B]) .
\end{aligned}$$

In the abelian case, the Bianchi identity implies that for any connection ω on P, the curvature 2-form $\bar{\Omega}$ on the base M is closed. What does the Bianchi identity tell us in the nonabelian case? To answer this question, let $\omega \in \mathcal{C}(P)$ be a connection 1-form and let $\bar{\Omega} \in \Omega(M; P \times_{\mathrm{Ad}} \mathfrak{g})$ be its curvature 2-form on M. For any $x \in M$, let $X, Y, Z \in T_x M$ be tangent vectors, and extend them to vector fields around $x \in M$ such that $[X, Y]_x = [X, Z]_x = [Y, Z]_x = 0$. Further let $s : U \to P|_U$ be a local section around x with $ds_x(T_x M) = H_{s(x)}$. Then we have:

$$\begin{aligned}
\left(d^\omega \bar{\Omega}\right)_x (X, Y, Z) &:= \left(d^{\nabla^\omega} \bar{\Omega}\right)(X, Y, Z) \\
&= \nabla^\omega_X \bar{\Omega}(Y, Z) - \nabla^\omega_Y \bar{\Omega}(X, Z) + \nabla^\omega_Z \bar{\Omega}(X, Y) \\
&= \nabla^\omega_X [s, (s^*\Omega)(Y, Z)] - \nabla^\omega_Y [s, (s^*\Omega)(X, Z)] + \nabla^\omega_Z [s, (s^*\Omega)(X, Y)] \\
&= [s, \partial_X (s^*\omega)(Y, Z) + \mathrm{ad}(s^*\omega(X)) s^*\Omega(Y, Z)] \\
&\quad -[s, \partial_Y (s^*\omega)(X, Z) + \mathrm{ad}(s^*\omega(Y)) s^*\Omega(X, Z)] \\
&\quad +[s, \partial_Z (s^*\omega)(X, Y) + \mathrm{ad}(s^*\omega(Z)) s^*\Omega(X, Y)] \\
&= [s, ds^*\Omega(X, Y, Z)] + [s, \mathrm{ad}(\underbrace{\omega(ds(X))}_{=0}) \cdot \Omega(ds(Y), ds(Z))] \\
&\quad -[s, \mathrm{ad}(\underbrace{\omega(ds(Y))}_{=0}) \cdot \Omega(ds(X), ds(Z))] \\
&\quad +[s, \mathrm{ad}(\underbrace{\omega(ds(Z))}_{=0}) \cdot \Omega(ds(X), ds(Y))] \\
&= [s, d\Omega(ds(X), ds(Y), ds(Z))] \\
&\overset{2.4.5}{=} 0 .
\end{aligned}$$

Hence the Bianchi identity 2.4.5 is equivalent to the statement $d^\omega \bar{\Omega} = 0$. Note that this is a nonlinear equation in ω! With respect to local sections $s_\alpha : U_\alpha \to P|_{U_\alpha}$, this equation reads $d\Omega_\alpha + [\omega_\alpha, \Omega_\alpha] = 0$.

Definition 3.3.4. The ***Yang-Mills Lagrangian*** $\mathcal{L}_{\mathrm{YM}}$ is given as:

$$\mathcal{L}_{\mathrm{YM}} : \mathcal{C}(P) \to \Omega^4(M;\mathbb{R}), \quad \omega \mapsto \frac{1}{2} \cdot \lambda(\bar{\Omega} \wedge *\bar{\Omega}).$$

Remark 3.3.5. By Remark 3.3.3, the real valued 4-form $\lambda(\bar{\Omega}\wedge *\bar{\Omega})$ is build from a scalar product on the adjoint bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$. By the definition of the Hodge-star operator, we can likewise write this form as $\lambda(\bar{\Omega} \wedge *\bar{\Omega}) = \langle \bar{\Omega}, \bar{\Omega} \rangle \cdot \mathrm{vol}$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\Lambda^2 T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})$ induced from the Riemannian metric on M and the metric λ on $P \times_{\mathrm{Ad}} \mathfrak{g}$. Stated this way, the action functional for the Yang-Mills Lagrangian is given by the L^2-norm square of the curvature 2-form, hence $\int_M \mathcal{L}_{\mathrm{YM}}(\omega) = \frac{1}{2} \cdot \|\bar{\Omega}\|^2_{L^2} \geq 0$ for any $\omega \in \mathcal{C}(P)$.

Remark 3.3.6. The Yang-Mills Lagrangian is gauge invariant, i.e. for any $\omega \in \mathcal{C}(P)$, $\varphi \in \mathcal{G}(P)$, we have $\mathcal{L}_{\mathrm{YM}}(\varphi^*\omega) = \mathcal{L}_{\mathrm{YM}}(\omega)$. Indeed, for $\omega' := \varphi^*\omega$ and $X, Y \in TM$, we find:

$$\begin{aligned} \bar{\Omega}'(X,Y) &= [s, \Omega'(ds(X), ds(Y))] \\ &= [s, (\varphi^*\Omega)(ds(X), ds(Y))] \\ &= [\varphi^{-1} \circ s', (s')^*\Omega(X,Y)] \qquad \text{where } s' := \varphi \circ s \\ &= [s', \mathrm{Ad}_{g^{-1}\circ s'}(s')^*\Omega(X,Y)]\,. \end{aligned}$$

Here $g : P \to G$ is the section of the group bundle $P \times_\alpha \mathrm{SU}(N)$ associated with φ as explained in Remark 2.7.7, i.e. $\varphi(p) = p \cdot g(p)$. We thus have $\bar{\Omega}' = \mathrm{Ad}_{g^{-1}}\bar{\Omega}$, hence by the Ad-invariance of the inner product: $\lambda(\bar{\Omega}' \wedge *\bar{\Omega}') = \lambda(\bar{\Omega} \wedge *\Omega)$.

Definition 3.3.7. A connection 1-form $\omega \in \mathcal{C}(P)$ is called ***Yang-Mills connection*** iff ω is critical for the action functional associated with the Lagrangian $\mathcal{L}_{\mathrm{YM}}$.

To understand what it means for a connection 1-form to be critical in this sense, let us compute the Euler-Lagrange equations for this action functional. To this end, let $\omega_t = \omega + t\eta$ be a variation of connection 1-forms, i.e. $\omega, \omega_t \in \mathcal{C}(P)$, hence $\eta \in \Omega^1_{\mathrm{Ad}}(P; \mathfrak{su}(N))$ is an Ad-invariant 1-form on P. Then we find for the curvatures:[5]

$$\begin{aligned}
\Omega_t &= d\omega_t + \frac{1}{2}[\omega_t, \omega_t] \\
&= d\omega + td\eta + \frac{1}{2}[\omega, \omega] + \frac{1}{2}t([\omega, \eta] + [\eta, \omega]) + \mathrm{O}(t^2) \\
&= \Omega + t(d\eta + [\omega, \eta]) + \mathrm{O}(t^2)\,,
\end{aligned}$$

whence $\bar{\Omega}_t = \bar{\Omega} + t \cdot d^\omega \bar{\eta} + \mathrm{O}(t^2)$. For any $\bar{\eta} \in \Omega^1(M; \mathfrak{su}(N))$, $\mathrm{supp}(\eta) \Subset M$, we thus have:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_{\bar{U}} \mathcal{L}_{\mathrm{YM}}(\omega_t) &= -\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \int_{\bar{U}} \mathrm{tr}(\bar{\Omega}_t \wedge *\bar{\Omega}_t) \\
&= -\frac{1}{2} \int_{\bar{U}} \mathrm{tr}\big(\bar{\Omega} \wedge *d^\omega\bar{\eta} + d^\omega\bar{\eta} \wedge *\bar{\Omega}\big) \\
&= -\int_{\bar{U}} \mathrm{tr}(d^\omega\bar{\eta} \wedge \bar{\Omega}) \\
&= -\int_{\bar{U}} d(\mathrm{tr}(\bar{\eta} \wedge *\bar{\Omega})) + \mathrm{tr}(\bar{\eta} \wedge d^\omega * \bar{\Omega}) \qquad (\nabla^\omega \text{ metric}) \\
&\overset{\text{Stokes}}{=} -\int_{\bar{U}} \mathrm{tr}(\bar{\eta} \wedge d^\omega * \bar{\Omega})\,.
\end{aligned}$$

We thus found:

$$\begin{aligned}
&\omega \text{ is critical for } \mathcal{L}_{\mathrm{YM}} \\
&\quad \Leftrightarrow \quad \forall\, \bar{\eta} \in \Omega^1(M; P \times_{\mathrm{Ad}} \mathfrak{g}), \mathrm{supp}(\eta) \Subset M: \quad \int_M \mathrm{tr}(\bar{\eta} \wedge d^\omega * \bar{\Omega}) = 0 \\
&\quad \Leftrightarrow \quad d^\omega * \bar{\Omega} = 0\,.
\end{aligned}$$

[5] Here we use that for 1-forms ω, η, the bracket $[\omega, \eta]$ is symmetric in ω, η. Indeed, by definition, we have for any X, Y:

$$[\omega, \eta](X, Y) := [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)] = -[\eta(Y), \omega(X)] + [\eta(X), \omega(Y)] = [\eta, \omega](X, Y)\,.$$

Corollary 3.3.8
If the curvature form $\bar{\Omega}$ of a connection $\omega \in \mathcal{C}(P)$ is (anti-)selfdual, then ω is a Yang-Mills connection.

Proof. In this case $d^\omega * \bar{\Omega} = \pm d^\omega \bar{\Omega} = 0$ by the Bianchi identity. □

Definition 3.3.9. A connection 1-form ω with self-dual curvature form $\bar{\Omega} \in \Omega^2(M; P \times_{\mathrm{Ad}} \mathfrak{g})$ is called ***instanton***.

Definition 3.3.10. Let $P \to M$ be a $\mathrm{GL}(n;\mathbb{R})$-principal bundle. The ***1. Pontrjagin class*** $p_1(P)$ of P is the de Rham cohomology class

$$p_1(P) := \left[\frac{1}{8\pi^2} \cdot \Big(\mathrm{tr}\big(\bar{\Omega}\big) \wedge \mathrm{tr}\big(\bar{\Omega}\big) - \mathrm{tr}\big(\bar{\Omega} \wedge \bar{\Omega}\big)\Big)\right] \in H^4_{\mathrm{dR}}(M)\,,$$

where $\bar{\Omega} \in \Omega^2(M; P \times_{\mathrm{Ad}} \mathfrak{g})$ is the curvature 2-form on M of any connection $\omega \in \mathcal{C}(P)$. The first Pontrjagin class $p_1(E)$ of a real vector bundle $E \to M$ is the de Rham cohomology class $p_1(P)$, where P is the frame bundle of P.

Here, the term $\mathrm{tr}(\bar{\Omega} \wedge \bar{\Omega})$ is to be understood in the sense of Remark 3.3.3.

Remark 3.3.11. The first Pontrjagin class of a real vector bundle has the following properties, similar to those of the first Chern class for complex vector bundles:

1. $p_1(E)$ is independent of the choice of connection $\omega \in \mathcal{C}(P)$ on the frame bundle of E.
2. If the vector bundle E is trivial, then $p_1(E) = 0$.
3. For a smooth map $\varphi : N \to M$ and a real vector bundle $E \to M$, we have $p_1(\varphi^* E) = \varphi^* p_1(E)$.

Remark 3.3.12. On an oriented connected compact 4-manifold M, the integration of differential forms yields an isomorphism

$$H^4_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathbb{R}\,, \quad [\omega] \mapsto \int_M \omega\,.$$

Using this isomorphism, one often identifies cohomology classes in $H^4_{\mathrm{dR}}(M)$ with their evaluation by integration over M. So we will not distinguish in notation between the first Pontrjagin class $p_1(E)$ of a real vector bundle $E \to M$ over a compact, oriented 4-manifold and the real number given by integrating over M a form $\eta \in \Omega^4(M)$ representing $p_1(E)$.

By 3.3.5, we know that the action functional for the Yang-Mills Lagrangian is nonnegative. The following Theorem gives a sharp lower bound in terms of the first Pontrjagin class of the adjoint bundle:

Theorem 3.3.13
Let $P \to M$ be an $\mathrm{SU}(N)$-principal bundle on a compact, oriented 4-manifold M. Then we have for any $\omega \in \mathcal{C}(P)$:

$$\int_M \mathcal{L}_{\mathrm{YM}}(\omega) \quad \geq \quad \frac{2\pi^2}{N} \cdot \left| p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) \right| . \tag{3.18}$$

Furthermore, we have:

1. *If $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) < 0$, then P has no selfdual connections. For any $\omega \in \mathcal{C}(P)$, we have $\int_M \mathcal{L}_{\mathrm{YM}}(\omega) \geq -\frac{2\pi^2}{N} \cdot p_1(P \times_{\mathrm{Ad}} \mathfrak{g})$ with equality iff ω is anti-selfdual.*
2. *If $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) = 0$, then $\omega \in \mathcal{C}(P)$ is (anti-)selfdual iff $\bar{\Omega} \equiv 0$, i.e. iff ω is flat.*
3. *If $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) > 0$, then P has no anti-selfdual connections. For any $\omega \in \mathcal{C}(P)$, we have $\int_M \mathcal{L}_{\mathrm{YM}}(\omega) \geq \frac{2\pi^2}{N} \cdot p_1(P \times_{\mathrm{Ad}} \mathfrak{g})$ with equality iff ω is selfdual.*

Proof. By definition, $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) = \frac{1}{8\pi^2} \int_M \big(\mathrm{tr}(\bar{\Phi}) \wedge \mathrm{tr}(\bar{\Phi}) - \mathrm{tr}(\bar{\Phi} \wedge \bar{\Phi})\big)$, where $\bar{\Phi}$ is the curvature of any connection φ on the frame bundle of $P \times_{\mathrm{Ad}} \mathfrak{g}$. Given a connection $\omega \in \mathcal{C}(P)$, we obtain a connection φ on the frame bundle of $P \times_{\mathrm{Ad}} \mathfrak{g}$ via the covariant derivative on $P \times_{\mathrm{Ad}} \mathfrak{g}$ induced from ω (see Example 2.3.3 and Remark 2.3.9). The curvature $\bar{\Phi}$ of this particular connection φ is related to the curvature $\bar{\Omega}$ of ω as $\bar{\Phi} = \mathrm{ad} \circ \bar{\Omega}$.
Now recall that the scalar product $\lambda(A, B) := -\mathrm{tr}(A \cdot B)$ on $\mathfrak{g} = \mathfrak{su}(N)$ is Ad-invariant,

hence for any $g \in \mathrm{SU}(N)$, we have:

$$\lambda\Big(\mathrm{Ad}_g(A), \mathrm{Ad}_g(B)\Big) = \lambda(A, B)\,.$$

Inserting a curve $t \mapsto g(t) := \exp(tX)$, $X \in \mathfrak{g} = \mathfrak{su}(N)$, and differentiating with respect to t, we obtain:

$$\lambda\Big(\mathrm{ad}(X)(A), B\Big) + \lambda\Big(A, \mathrm{ad}(X)(B)\Big) = 0\,.$$

Hence $\mathrm{ad}(X)$ is skew symmetric with respect to λ and hence trace free as endomorphism on $\mathfrak{g}$. Applying this to the first Pontrjagin class, we observe:

$$\mathrm{tr}\big(\bar{\Phi}\big) = \mathrm{tr}\big(\mathrm{ad} \circ \bar{\Omega}\big) = 0\,.$$

Next we claim that the map $\lambda' : (A, B) \mapsto -\mathrm{tr}(\mathrm{ad}(A) \circ \mathrm{ad}(B))$ defines another positive definite, Ad-invariant scalar product on the Lie algebra $\mathfrak{g} = \mathfrak{su}(N)$. (The bilinear form $(A, B) \mapsto \mathrm{tr}(\mathrm{ad}(A) \circ \mathrm{ad}(B))$ is the so called ***Killing form*** of $\mathfrak{g}$. It can be defined for any Lie group G, and it is negative definite iff G is semisimple.) It follows from an elementary fact in representation theory, that the two bilinear forms λ, λ' are related by a constant. For $\mathfrak{g} = \mathfrak{su}(N)$, we have: $2N \cdot \lambda = \lambda'$.
With these observations, we obtain for the first Pontrjagin class of the adjoint bundle:

$$\begin{aligned}
p_1\big(P \times_{\mathrm{Ad}} \mathfrak{g}\big) &= \frac{1}{8\pi^2} \cdot \int_M \Big(\mathrm{tr}\big(\bar{\Phi}\big) \wedge \mathrm{tr}\big(\bar{\Phi}\big) - \mathrm{tr}\big(\bar{\Phi} \wedge \bar{\Phi}\big)\Big) \\
&= \frac{1}{8\pi^2} \cdot \int_M \Big(\mathrm{tr}\big(\mathrm{ad} \circ \bar{\Omega}\big) \wedge \mathrm{tr}\big(\mathrm{ad} \circ \bar{\Omega}\big) - \mathrm{tr}\big(\mathrm{ad} \circ \bar{\Omega} \wedge \mathrm{ad} \circ \bar{\Omega}\big)\Big) \\
&= -\frac{1}{8\pi^2} \cdot \int_M \mathrm{tr}\Big(\mathrm{ad} \circ \bar{\Omega} \wedge \mathrm{ad} \circ \bar{\Omega}\Big) \\
&= -\frac{2N}{8\pi^2} \cdot \int_M \mathrm{tr}\big(\bar{\Omega} \wedge \bar{\Omega}\big) \\
&= \frac{2N}{8\pi^2} \cdot \int_M \lambda\big(\bar{\Omega}, *\bar{\Omega}\big) \cdot \mathrm{vol}\,.
\end{aligned}$$

Now we use the fact, that the L^2-norm square (with respect to the scalar product on $\Lambda^2 T^*M \otimes (P \times_{\mathrm{Ad}} \mathfrak{g})$ induced from the Riemannian metric on M and the metric λ on $P \times_{\mathrm{Ad}} \mathfrak{g}$) of $\bar{\Omega} \mp *\bar{\Omega}$ is nonnegative, to obtain:

$$\begin{aligned}
0 &\le \big(\bar{\Omega} \mp *\bar{\Omega}, \bar{\Omega} \mp *\bar{\Omega}\big)_{L^2} \\
b &= \|\bar{\Omega}\|^2_{L^2} + \| * \bar{\Omega}\|^2_{L^2} \mp 2\big(\bar{\Omega}, *\bar{\Omega}\big)_{L^2} \\
&= 2\Big(\|\bar{\Omega}\| \mp \big(\bar{\Omega}, *\bar{\Omega}\big)_{L^2}\Big) \qquad (* \text{ is an isometry}) \\
&= 2\left(2 \cdot \int_M \mathcal{L}_{\mathrm{YM}}(\omega) \mp \frac{8\pi^2}{2N} \cdot p_1\big(P \times_{\mathrm{Ad}} \mathfrak{g}\big)\right)\,.
\end{aligned}$$

This yields the estimate (3.18). We also see from this computation, that equality holds iff $\bar{\Omega} = \pm * \bar{\Omega}$.
Now, if $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) < 0$, then $\int_M \mathcal{L}_{\mathrm{YM}}(\omega) \geq -\frac{2\pi^2}{N} \cdot p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) > 0$. But if ω was a connection with $\bar{\Omega} = *\bar{\Omega}$, then we would have $\int_M \mathcal{L}_{\mathrm{YM}}(\omega) = \frac{2\pi^2}{N} \cdot p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) < 0$. Hence such a connection cannot exist. With a similar argument for the case $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) > 0$, we have proved the assertions 1. and 3.
As to assertion 2., let ω be a connection with $*\bar{\Omega} = \pm\bar{\Omega}$. We then have:

$$2 \int\limits_M \mathcal{L}_{\mathrm{YM}}(\omega) = \big\|\bar{\Omega}\big\|^2_{L^2} = \pm\big(\bar{\Omega}, *\bar{\Omega}\big)_{L^2} = \pm\frac{8\pi^2}{2N} \cdot p_1\big(P \times_{\mathrm{Ad}} \mathfrak{g}\big)\,.$$

Hence if $p_1(P \times_{\mathrm{Ad}} \mathfrak{g}) = 0$, then $*\bar{\Omega} = \pm\bar{\Omega}$ iff $\bar{\Omega} \equiv 0$. $\square$

4 Algebraic Topology

4.1 Homotopy theory

Definition 4.1.1. Let X, Y be topological spaces, and let

$$\mathcal{C}(X,Y) := \{f : X \to Y \text{ continuous}\}.$$

Then f_0, $f_1 \in \mathcal{C}(X,Y)$ are called ***homotopic***, if there exists an $f \in \mathcal{C}(X \times I, Y)$, $I = [0,1]$, satisfying $f(\cdot, 0) = f_0$ and $f(\cdot, 1) = f_1$. In this case, we write $f_0 \simeq f_1$. The map f is called a ***homotopy*** from f_0 to f_1.

Example 4.1.2. Let $X = Y = \mathbb{R}^n$ and take $\forall\, x \in X$: $f_0(x) := x$, $f_1(x) := 0$. Then $f_0 \simeq f_1$ by $f(x,t) := t \cdot x$.

Remark 4.1.3. The relation $\simeq$ is an equivalence relation on $\mathcal{C}(X,Y)$:

* For any $f_0 \in \mathcal{C}(X,Y)$, we have $f_0 \simeq f_0$ by the constant homotopy $f(x,t) := f_0(x)$. This shows reflexivity.
* As to symmetry, let f be a homotopy from f_0 to f_1. Then $\tilde{f}(x,t) := f(x, 1-t)$ is a homotopy from f_1 to f_0.
* As to transitivity, let $\tilde{f}$ be a homotopy from f_0 to f_1, and let $\hat{f}$ be a homotopy from f_1 to f_2. Then

$$f(x,t) := \begin{cases} \tilde{f}(x, 2t) & : \quad t \in [0, \frac{1}{2}] \\ \hat{f}(x, 2t-1) & : \quad t \in [\frac{1}{2}, 1] \end{cases}$$

  defines a homotopy from f_0 to f_2.

Example 4.1.4. As in Example 4.1.2 above, take $X = Y = \mathbb{R}^n$. Then any two maps $f, g \in \mathcal{C}(X,Y)$ are homotopic: as in Example 4.1.2 one sees $f \simeq 0$, $g \simeq 0$, where 0 is the constant map $x \mapsto 0$. By symmetry and transitivity, this implies $f \simeq g$.

Definition 4.1.5. Topological spaces X, Y are called ***homotopy equivalent*** iff there exist $f \in \mathcal{C}(X,Y)$, $g \in \mathcal{C}(Y,X)$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_Y$. In this case, f and g are called ***homotopy equivalences*** and f, g are ***homotopy iverses*** of each other. We denote homotopy equivalence by $X \simeq Y$.

Remark 4.1.6. Homotopy equivalence is an equivalence relation on the class of all topological spaces.

Definition 4.1.7. A topological space X with $X \simeq \{*\}$ is called ***contractible***.

Example 4.1.8. $\mathbb{R}^n$ is contractible: Take $f : \{0\} \to \mathbb{R}^n$, $0 \mapsto 0$ and $g : \mathbb{R}^n \to \{0\}$, $x \mapsto 0$. Then $f \circ g = 0 \simeq \mathrm{id}_{\mathbb{R}^n}$ and $g \circ f = \mathrm{id}_{\{0\}}$. Hence f and g are homotopy inverses of each other.

Remark 4.1.9. A homeomorphism $f : X \to Y$ is a homotopy equivalence, but homotopic spaces X, Y are in general not homeomorphic, as the previous example has shown.

Example 4.1.10. Let $X = S^n$, $Y = \mathbb{R}^{n+1} - \{0\}$, and let $f : S^n \to \mathbb{R}^{n+1} - \{0\}$, $x \mapsto x$, and $g : \mathbb{R}^{n+1} - \{0\} \to S^n$, $y \mapsto \frac{y}{\|y\|}$. Then $g \circ f = \mathrm{id}_{S^n}$ and $f \circ g = g \simeq \mathrm{id}_{\mathbb{R}^{n+1}-\{0\}}$ by $G(y,t) := (1 - t + \frac{t}{\|y\|}) \cdot y$.

Definition 4.1.11. Let f_0, $f_1 \in \mathcal{C}(X,Y)$ and $A \subset X$. Then f_0, f_1 are called ***homotopic relative*** A iff there is a homotopy $f \in \mathcal{C}(X \times I, Y)$ from f_0 to f_1 satisfying:

$$\forall\, a \in A, \forall\, t \in I : f(a,t) = f_0(a).$$

In this case we write $f_0 \simeq f_1$ rel. A.

Remark 4.1.12. As above one easily sees that the relation $\simeq$ rel. A is an equivalence relation on $\mathcal{C}(X,Y)$.

Definition 4.1.13. Let X be a topological space and $x \in X$, $n \in \mathbb{N}_0$.

$$\pi_n(X,x) := \{f \in \mathcal{C}(S^n, X) \mid f(NP) = x\} \,/ \simeq \text{rel. } \{NP\}$$

is called the ***n*th *homotopy group*** of (X,x). Here $NP \in S^n$ is a fixed point (which we call north pole).

Remark 4.1.14. For $n = 0$, we have $S^0 = \{NP, SP\}$ so that

$$\big\{f \in \mathcal{C}(S^0, X) \mid f(NP) = x\big\} \cong X$$

by $x' \mapsto (f : NP \mapsto x, SP \mapsto x')$. Further, we have $f \simeq f'$ rel. NP iff there exists a continuous curve $g \in \mathcal{C}(I,X)$ satisfying $g(0) = f(SP)$, $g(1) = f'(SP)$. Hence we have $\pi_0(X,x) \overset{1:1}{\longleftrightarrow} \{\text{path components of } X\}$.

Remark 4.1.15. $\pi_0(X,x)$ carries no canonical group structure.
In contrast, for $n \geq 1$, $\pi_n(X,x)$ is a group. To define the group structure we first introduce a different model for the homotopy groups $\pi_n(X,x)$. Namely, let

$$I^n := \underbrace{I \times \cdots \times I}_{n}$$

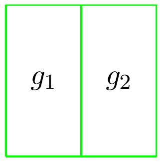
be the n-dimensional standard cube, and let $\psi : I^n \to S^n$ be a fixed continuous map such that $\psi|_{\mathring{I}^n} : \mathring{I}^n \to S^n - \{NP\}$ is a homeomorphism and $\psi(\partial I^n) = \{NP\}$.
Then for any $f \in \mathcal{C}(S^n, X)$ with $f(NP) = x$, we have $f \circ \psi \in \mathcal{C}(I^n, X)$ satisfying $(f \circ \psi)(\partial I^n) = \{x\}$. Conversely, for any $g \in \mathcal{C}(I^n, X)$ satisfying $g(\partial I^n) = \{x\}$, we find $f \in \mathcal{C}(S^n, X)$ with $f(NP) = x$ such that $g = f \circ \psi$. This gives $1:1$-correspondences:

$$\begin{array}{rrcl} & \{f \in \mathcal{C}\left(S^n, X\right) \mid f(NP) = x\} & \overset{1:1}{\longleftrightarrow} & \{g \in \mathcal{C}(I^n, X) \mid g\left(\partial I^n\right) = \{x\}\} \\ & \simeq \text{rel. } \{NP\} & \overset{1:1}{\longleftrightarrow} & \simeq \text{rel. } \partial I^n \\ \text{hence} & \pi_n(X,x) & \overset{1:1}{\longleftrightarrow} & \{g \in \mathcal{C}\left(I^n, X\right) \mid g\left(\partial I^n\right) = \{x\}\} \,/ \simeq \text{rel. } \partial I^n\,. \end{array}$$

With this new model for the homotopy groups, we can easily define the multiplication in $\pi_n(X,x)$, $n \geq 1$, by the concatenation of maps $g_1, g_2 : I^n \to X$, $g_1(\partial I^n) = g_2(\partial I^n) = \{x\}$.

We write schematically, the coloured lines indicating the parts mapped to $\{x\}$:

$$g_1 * g_2$$



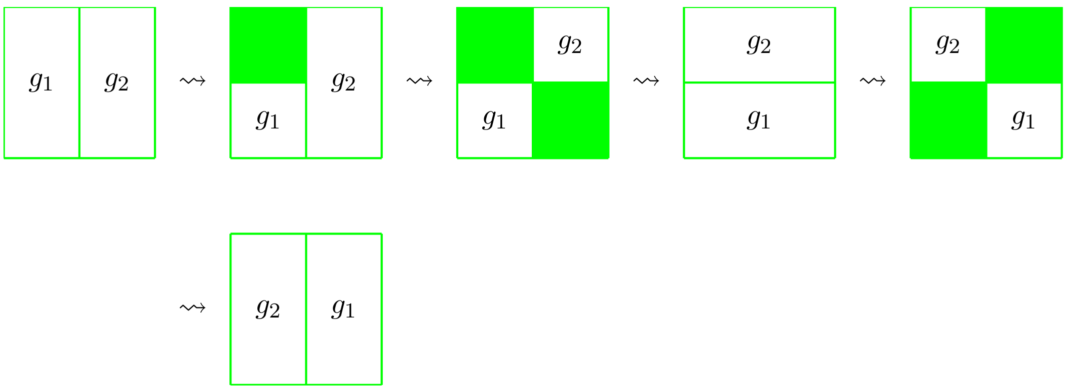
More explicitly, we have:

$$(g_1 * g_2)(t_1, \dots, t_n) := \begin{cases} g_1(2t_1, t_2, \dots, t_n) & : \quad t_1 \in [0, \frac{1}{2}] \\ g_2(2t_1 - 1, t_2, \dots, t_n) & : \quad t_1 \in [\frac{1}{2}, 1] \end{cases}$$

This induces a map $\pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x)$ by $([g_1],[g_2]) \mapsto [g_1]*[g_2] := [g_1 * g_2]$. This map induces a group structure on the homotopy groups $\pi_n(X,x)$. The neutral element is represented by the constant map $I^n \to \{x\} \subset X$.

Proposition 4.1.16
For $n \geq 2$, the homotopy groups $\pi_n(X,x)$ are abelian.

Proof. We give the proof by schematically performing the following chain of homotopies from $g_1 * g_2$ to $g_2 * g_1$ (here again, the coloured parts are those which are mapped to the base point $\{x\}$):



$\square$

Remark 4.1.17. In general, $\pi_1(X,x)$ is not abelian.

Definition 4.1.18. $\pi_1(X, x)$ is also called the ***fundamental group*** of (X, x).
If $\pi_0(X, x) = \{x\}$, i.e. X is path connected, then X is called ***simply connected*** iff $\pi_1(X, x) = \{e\}$ for any and hence all $x \in X$. This means, that any continuous loop in X starting and ending at x can be deformed continuously to the constant map in x.

Lemma 4.1.19
Let $f_0, f_1 \in \mathcal{C}(X, Y)$, let g_0, $g_1 \in \mathcal{C}(Y, Z)$, and let $A \subset X$, $B \subset Y$ with $f_i(A) \subset B$, $i = 0, 1$. If $f_0 \simeq f_1$ rel. A and $g_0 \simeq g_1$ rel. B, then $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel. A.

Proof. exercise. □

Corollary 4.1.20
If $[f_0] = [f_1] \in \pi_n(X, x)$, and $g \in \mathcal{C}(X, Y)$, then for $y = g(x)$, we have

$$[g \circ f_0] = [g \circ f_1] \in \pi_n(Y, y).$$

Thus, any continuous map $g : X \to Y$ induces a group homomorphism

$$\begin{array}{rcl} g_\sharp : \pi_n(X, x) & \to & \pi_n(Y, g(x)) \\ {[f]} & \mapsto & [g \circ f]. \end{array}$$

Corollary 4.1.21
Let $g_0, g_1 \in \mathcal{C}(X, Y)$ with $g_0 \simeq g_1$ rel. $\{x\}$. Then we have

$$(g_0)_\sharp = (g_1)_\sharp : \pi_n(X, x) \to \pi_n(Y, g(x)).$$

Remark 4.1.22. Let X, Y, Z be topological spaces, $x \in X$, $y \in Y$, $z \in Z$ and $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ with $f(x) = y$, $g(y) = z$. Then it follows directly from the definition that $(g \circ f)_\sharp = g_\sharp \circ f_\sharp$. It is also clear that $(\mathrm{id}_X)_\sharp = \mathrm{id}_{\pi_n(X,x)}$.

Remark 4.1.23. Let $(X, x) \simeq (Y, y)$, i.e. there exist $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, X)$ with $f(x) = y$ and $g(y) = x$ such that $f \circ g \simeq \mathrm{id}_Y$ rel.$\{y\}$ and $g \circ f \simeq \mathrm{id}_X$ rel.$\{x\}$. For the induced homomorphisms on homotopy groups, we find:

$$f_\sharp \circ g_\sharp = (f \circ g)_\sharp = (\mathrm{id}_Y)_\sharp = \mathrm{id}_{\pi_n(Y,y)}$$

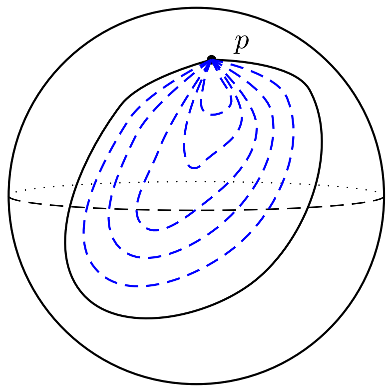
and similarly $g_\sharp \circ f_\sharp = \mathrm{id}_{\pi_n(X,x)}$. Hence $f_\sharp$ and $g_\sharp$ are both group isomorphisms, inverse to each other.
In particular, homotopy equivalent spaces have isomorphic homotopy groups. Contractible spaces thus have trivial homotopy groups.

Example 4.1.24. For $X = S^1$, one can show that the map

$$\mathbb{Z} \to \pi_1\left(S^1, 1\right), \; k \mapsto \left[z \mapsto z^k\right],$$

is an isomorphism. Hence $\pi_1(S^1, 1) \cong \mathbb{Z}$, in particular, S^1 is not simply connected. By Example 4.1.10, we then also have that $\mathbb{R}^2 - \{0\} \simeq S^1$ is not contractible.

Example 4.1.25. For $n \geq 2$, we have that S^n is simply connected.



Remark 4.1.26. In general, we have: $\pi_i(S^n) = \{e\}$ if $i < n$ and $\pi_n(S^n) = \mathbb{Z}$. The higher homotopy groups $\pi_m(S^n)$, $m > n$ are not known in general.

Definition 4.1.27. A fiber bundle with discrete fiber is called a ***covering***.

Example 4.1.28. The map $\exp : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$, is a covering; actually, it is a $\mathbb{Z}$-principal bundle, where the action is defined as $(t, k) \mapsto t + k$.

Similarly the map $\exp : \mathbb{R}^n \to T^n = \overbrace{S^1 \times \ldots \times S^1}^{n}$, given by

$$(t_1, \ldots, t_n) \mapsto \left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right),$$

is a $\mathbb{Z}^n$-principal bundle.

Example 4.1.29. If X is a connected differentiable manifold, then there exists a covering $\bar{X} \to X$ such that $\bar{X}$ is simply connected. The covering $\bar{X} \to X$ is unique up to isomorphism, and is called the ***universal covering***.

Example 4.1.30. The map $S^1 \to S^1$, $z \mapsto z^k$, is a (k-fold) covering; actually, it is a $\mathbb{Z}_k$-principal bundle.

Lemma 4.1.31 (Lifting Lemma)
Let $p : \tilde{Y} \to Y$ be a covering, $\tilde{y} \in \tilde{Y}$, $y = p(\tilde{y}) \in Y$. Let X be a path connected topological space, $x \in X$, and let $f : X \to Y$ be a continuous map with $f(x) = y$.

A **lift** *of f through $\tilde{y}$ is a continuous map $\tilde{f} : X \to \tilde{Y}$ with $f(x) = \tilde{y}$ satisfying $p \circ \tilde{f} = f$.*

$$\begin{tikzcd} & \tilde{Y} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Such a lift exists iff $f_\sharp(\pi_1(X, x)) \subset p_\sharp(\pi_1(\tilde{Y}, \tilde{y}))$.

Proof. If $\tilde{f}$ is such a lift, then we have for any $[c] \in \pi_1(X, x)$:

$$f_\sharp([c]) = [f \circ c] = \left[p \circ \tilde{f} \circ c\right] = p_\sharp\Big(\underbrace{[\tilde{f} \circ c]}_{\in \pi_1(\tilde{Y}, \tilde{y})} \Big) \in p_\sharp\left(\pi_1\left(\tilde{Y}, \tilde{y}\right)\right),$$

whence $f_\sharp(\pi_1(X, x)) \subset p_\sharp(\pi_1(\tilde{Y}, \tilde{y}))$.
The other direction is slightly more involved. $\square$

Corollary 4.1.32
If X is simply connected, then any $f \in \mathcal{C}(X,Y)$ can be lifted to any covering $\tilde{Y} \to Y$.

Example 4.1.33. Using the covering $\exp : \mathbb{R} \to S^1$, we can easily determine the higher homotopy groups of S^1: Namely, since for $n \geq 2$, we have $\pi_n(S^1, 1) = \{1\}$, any $u \in \mathcal{C}(S^n, S^1)$, $u(NP) = 1$, can be lifted to $\tilde{u} : S^n \to \mathbb{R}$, $\tilde{u}(NP) = 0$. We then have:

$$[u] = [\exp \circ \tilde{u}] = \exp_\sharp([\tilde{u}]) = 0 \in \pi_n(\mathbb{R}, 0) = \{0\},$$

since $\mathbb{R}$ is contractible. Thus the higher homotopy groups of S^1 are all trivial.

Definition 4.1.34. A sequence of groups and homomorphisms

$$\cdots \to G_{i+1} \overset{f_{i+1}}{\to} G_i \overset{f_i}{\to} G_{i-1} \overset{f_{i-1}}{\to} G_{i-2} \to \cdots$$

is called ***exact*** iff $\forall\, i$: $\ker(f_i) = \operatorname{im}(f_{i+1})$.

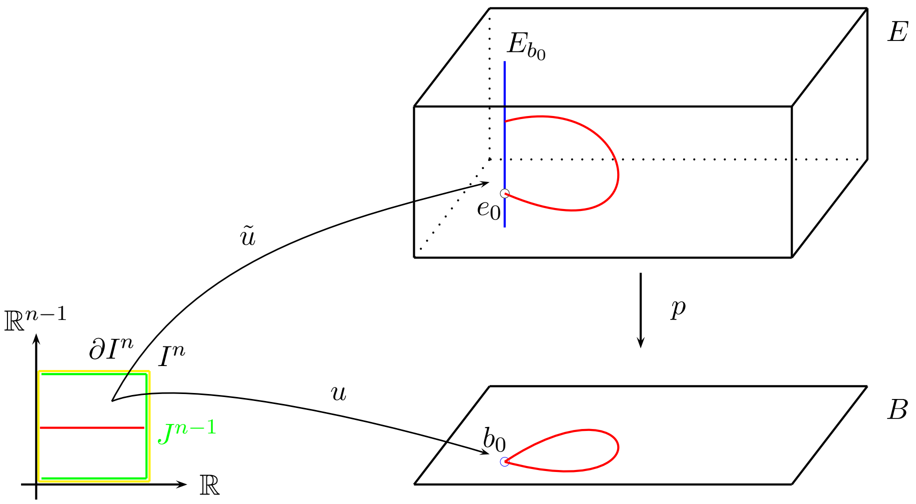
Let $p : E \to B$ be a fiber bundle, let $e_0 \in E$, $b_0 = p(e_0)$. Let $F := E_{b_0} = p^{-1}(b_0)$ be the fiber of p containing e_0. Denote the inclusion of that fiber into the total space E by $\iota : F \hookrightarrow E$. Then $p \circ \iota$ is the constant map to b_0, whence

$$0 = (p \circ \iota)_\sharp = p_\sharp \circ \iota_\sharp : \pi_n(F, e_0) \to \pi_n(B, b_0).$$

In particular, we have $\operatorname{im}(\iota_\sharp) \subset \ker p_\sharp$.

We now construct the so called ***boundary homomorphism*** (or connecting homomorphism) $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$:

We start with a map $u \in \mathcal{C}(I^n, B)$ representing a homotopy class $[u] \in \pi_n(B, b_0)$, i.e. $u(\partial I^n) = \{b_0\}$. Any red line in I^n as in the picture gets mapped under u to a closed curve in B, starting and ending in b_0 (we parametrize these lines starting at the right endpoint in J^{n-1}). Now we lift these closed curves in (B, b_0) to curves in (E, e_0) (meaning that the lifted curve starts in $e_0 \in E_{b_0}$ and ends in some point in E_0). We may choose the family of lifted curves to depend continuously on the initial points in $I^{n-1} \times \{1\}$ of the red lines in I^n.



By this lifting procedure, we obtain a map $\tilde{u} \in \mathcal{C}(I^n, E)$ such that $p \circ \tilde{u} = u$ and $\tilde{u}(J^{n-1}) = \{e_0\}$. Now we put $\partial[u] := [\tilde{u}|_{I^{n-1}\times\{0\}}] \in \mathcal{C}(I^{n-1}, E)$. By construction, we have $\tilde{u}(\partial I^{n-1}) = \{e_0\}$, since $\partial I^{n-1} \subset J^{n-1}$.
This construction yields the following relation between the homotopy groups of the fiber, the total space and the base of a fiber bundle:

Theorem 4.1.35
The sequence

$$\cdots \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{\iota_\sharp} \pi_n(E, e_0) \xrightarrow{p_\sharp} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \xrightarrow{\iota_\sharp} \cdots \xrightarrow{p_\sharp} \pi_1(B, b_0) \tag{4.1}$$

is exact.

Example 4.1.36. Let us consider the trivial bundle $E = B \times F$. Then in addition to the bundle projection $p : E \to B$, we have another projection $\hat{p} : E \to F$, and $\hat{p} \circ \iota = \mathrm{id}_F$. We thus find for the induced maps on homotopy groups:

$$\hat{p}_\sharp \circ \iota_\sharp = (\hat{p} \circ \iota)_\sharp = (\mathrm{id}_F)_\sharp = \mathrm{id}_{\pi_n(F, e_0)} ,$$

which implies that $\iota_\sharp$ is injective and $\hat{p}_\sharp$ is surjective. Hence the connecting homomorphisms $\partial : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ need to be trivial, and the long exact sequence 4.1 degenerates to a series of short exact sequences: for any $n \geq 1$, we have:

$$0 \to \pi_n (F, e_0) \xrightarrow{\iota_\sharp} \pi_n (E, (b_0, e_0)) \xrightarrow{p_\sharp} \pi_n (B, b_0) \to 0 .$$

Now the map $p_\sharp \times \hat{p}_\sharp : \pi_n(E,(b_0,e_0)) \to \pi_n(B,b_0) \times \pi_n(F,e_0)$ is surjective, since $p_\sharp$ and $\hat{p}_\sharp$ are. It is also injective: take $x \in \pi_n(E,(b_0,e_0))$ with $p_\sharp(x) = 0$ and $\hat{p}_\sharp(x) = 0$. By exactness, $x = \iota_\sharp(y)$ for some $y \in \pi_n(F,e_0)$. But we also have

$$0 = \hat{p}_\sharp(x) = \hat{p}_\sharp(\iota_\sharp(y)) = (\hat{p} \circ \iota)_\sharp(y) = (\mathrm{id}_F)_\sharp(y) = \mathrm{id}_{\pi_n(F,e_0)}(y) = y\,,$$

so that $x = \iota_\sharp(y) = \iota_\sharp(0) = 0$ as well.
Consequently, $\pi_n(B \times F,(b_0,e_0)) \xrightarrow{\approx} \pi_n(B,b_0) \times \pi_n(F,e_0)$.

Example 4.1.37. For the Hopf bundle $H : S^3 \to S^2$, we find:

$$\underbrace{\pi_3\left(S^1\right)}_{=\{0\}} \xrightarrow{\iota_\sharp} \pi_3\left(S^3\right) \xrightarrow{p_\sharp} \pi_3\left(S^2\right) \xrightarrow{\partial} \underbrace{\pi_2\left(S^1\right)}_{=\{0\}}\,.$$

Hence $H_\sharp : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism. Since $\pi_3(S^3) \cong \mathbb{Z}$, we find $\pi_3(S^2) \cong \mathbb{Z}$. Furthermore, since $\pi_3(S^3)$ is generated by $[\mathrm{id}_{S^3}]$, we have that $\pi_3(S^2)$ is generated by $H_\sharp([\mathrm{id}_{S^3}]) = [H \circ \mathrm{id}_{S^3}] = [H]$. The Hopf map $H : S^3 \to S^2$ thus represents a generator of $\pi_3(S^2)$.

4.2 Homology theory

Definition 4.2.1. A sequence of homomorphisms of abelian groups

$$\cdots \to A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1} \xrightarrow{f_{k-1}} A_{k-2} \to \cdots$$

is called a ***complex (of abelian groups)*** iff $\forall\, k$: $\mathrm{im}(f_k) \subset \ker(f_{k-1})$.

Definition 4.2.2. The ***k*^th^ *homology*** of a complex $(A_\bullet, f_\bullet)$ is the abelian group defined by:

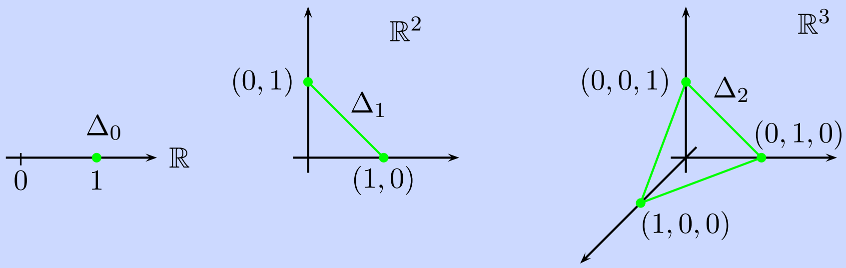
$$H_k\left(A_\bullet, f_\bullet\right) := \frac{\ker\left(f_k : A_k \to A_{k-1}\right)}{\mathrm{im}\left(f_{k+1} : A_{k+1} \to A_k\right)}$$

Remark 4.2.3. The $\mathrm{k^{th}}$ homology group of a complex $(A_\bullet, f_\bullet)$ measures the failure (at the $\mathrm{k^{th}}$ spot) of $(A_\bullet, f_\bullet)$ to be exact. In particular, $(A_\bullet, f_\bullet)$ is exact iff $\forall\, k$: $H^k(A_\bullet, f_\bullet) = 0$.

Definition 4.2.4. For any $n \in \mathbb{N}_0$, the set

$$\Delta_n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}$$

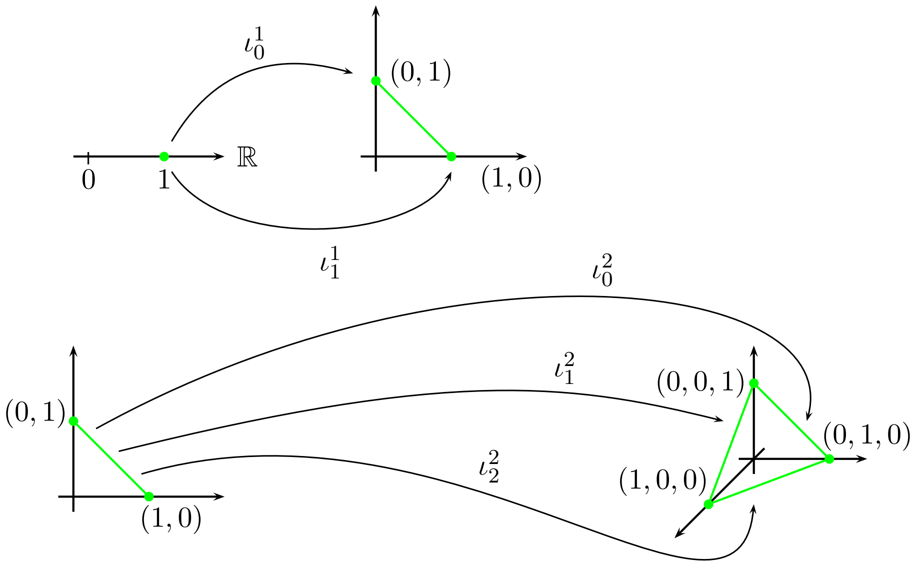
is called the ***n*^th^ *standard simplex***.



For a topological space X, a ***singular*** n***-simplex in*** X is a map $\sigma \in \mathcal{C}(\Delta_n, X)$. For any $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, n\}$, the $\boldsymbol{k}^{\text{th}}$ ***side*** of $\partial\Delta_n$ is the map

$$\iota_k^n : \Delta_{n-1} \to \Delta_n \,, (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{k-1}, 0, t_k, \ldots, t_{n-1}) \ .$$

Example 4.2.5.



Definition 4.2.6. Let R be a commutative ring with unit. Typical relevant examples are $R = \mathbb{Z}, \mathbb{Z}/k\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Then we set

$$\begin{aligned} C_n(X;R) &:= \{\text{formal (finite) linear combinations with coefficients in } R \\ &\qquad \text{of singular } n\text{-simplices in } X\} \\ &= \left\{ \sum_k \alpha_k \sigma_k \mid \alpha_k \in R, \sigma_k \in \mathcal{C}(\Delta_n, X) \right\} \\ &= \text{free } R\text{-module generated by } \mathcal{C}(\Delta_n, X)\ . \end{aligned}$$

Elements in $C_n(X;R)$ are called ***singular* n*-chains in* X**. We define a so called ***boundary operator*** $\partial_n : C_n(X;R) \to C_{n-1}(X;R)$ by setting

$$\partial_n(\sigma) := \sum_{k=0}^{n} (-1)^k \sigma \circ \iota_k^n\,,$$

on n-simplices and extending linearly to $C_n(X;R)$.

This operator ∂ satisfies $\partial \circ \partial \equiv 0 : C_n(X;R) \to C_{n-2}(X;R)$. Hence we obtain a complex of free R-modules

$$\cdots \leftarrow C_{n-1}(X;R) \overset{\partial}{\leftarrow} C_n(X;R) \overset{\partial}{\leftarrow} C_{n+1}(X;R) \overset{\partial}{\leftarrow} \cdots$$

Definition 4.2.7. Moreover we set

$$\begin{aligned} Z_n(X;R) &:= \ker\left(\partial : C_n(X;R) \to C_{n-1}(X;R)\right) \quad \text{and} \\ B_n(X;R) &:= \operatorname{im}\left(\partial : C_{n+1}(X;R) \to C_n(X;R)\right). \end{aligned}$$

Elements in $Z_n(X;R)$ are called ***(singular)* n*-cycles in*** X, elements of $B_n(X;R)$ are called ***(singular)* n*-boundaries in*** X.
The n^{th} homology of the singular chain complex $(C_\bullet(X;R), \partial)$ is called the $\mathbf{n}^{\text{th}}$ ***singular homology of* X *(with coefficients in* R*)***, and is denoted by:

$$\begin{aligned} H_n(X;R) &:= \frac{\ker\left(\partial : C_n(X;R) \to C_{n-1}(X;R)\right)}{\operatorname{im}\left(\partial : C_{n+1}(X;R) \to C_n(X;R)\right)} \\ &= \frac{Z_n(X;R)}{B_n(X;R)}\,. \end{aligned}$$

Let X, Y be topological spaces and $f \in \mathcal{C}(X,Y)$. For

$$f_*\left(\sum \alpha_j \sigma_j\right) := \sum \alpha_j \left(f \circ \sigma_j\right),$$

we find $\partial \circ f_* = f_* \circ \partial$. This implies that $f_*(Z_n(X;R)) \subset Z_n(Y;R)$ and $f_*(B_n(X;R)) \subset B_n(Y;R)$. Hence f_* descends to a map on homology, defined as $f_*([z]) := [f_*(z)]$.

Remark 4.2.8. Singular homology has the following properties:

1. *Functoriality*: For $f \in \mathcal{C}(X,Y)$ and $g \in \mathcal{C}(Y,Z)$, we have $(g \circ f)_* = g_* \circ f_*$, and $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X;R)}$.
2. *Homotopy invariance*: For $f, g \in \mathcal{C}(X,Y)$ with $f \simeq g$, we have $f_* = g_*$.
3. *Coefficients*: $H_n(\{p\};R) \cong \begin{cases} R & : \quad n = 0 \\ \{0\} & : \quad n \geq 1 \end{cases}$
4. *Mayer-Vietoris sequence*: For a topological space X and open subsets X_0, $X_1 \subset X$ with $X_0 \cup X_1 = X$, we have the inclusions $j^\nu : X_\nu \hookrightarrow X$ and $i^\nu : X_0 \cap X_1 \hookrightarrow X_\nu$, $\nu = 0, 1$. There exists a natural connecting homomorphism

$$\partial_n : H_n(X;R) \to H_{n-1}\left(X_0 \cap X_1;R\right)$$

such that the following sequence is exact (for simplicity, we drop the coefficient ring R in the notation):

$$\cdots \to H_n(X_0 \cap X_1) \xrightarrow{\begin{pmatrix} i^0_* \\ i^1_* \end{pmatrix}} H_n(X_0) \oplus H_n(X_1) \xrightarrow{(j^0_*, -j^1_*)} H_n(X) \xrightarrow{\partial} H_{n-1}(X_0 \cap X_1) \to \cdots$$

Proof. Assertion 1. follows directly from the definitions. To show assertion 3., we observe that for any $n \in \mathbb{N}$, there exists precisely one n-simplex in $X = \{p\}$, namely the constant map $\Delta_n \to \{p\}$. Hence $C_n(X;R) \cong R$ for any n and

$$\partial \sigma_n = \sum_{k=0}^{n} (-1)^k \underbrace{\sigma_n \circ \iota^n_k}_{=\sigma_{n-1}} = \underbrace{\left(\sum_{k=0}^{n} (-1)^k\right)}_{\substack{=0 \quad n \text{ odd} \\ =1 \quad n \text{ even}}} \sigma_{n-1}.$$

The singular chain complex thus reads:

$$\{0\} \leftarrow R \xleftarrow{0} R \xleftarrow{1} \cdots \xleftarrow{1} R \xleftarrow{0} R \xleftarrow{1} \cdots$$

Hence for $n \geq 1$, we have:

$$H_{2n}(X;R) = \frac{\ker(1:R \to R)}{\operatorname{im}(0:R \to R)} = \{0\} \quad \text{and} \quad H_{2n-1}(X;R) = \frac{\ker(0:R \to R)}{\operatorname{im}(1:R \to R)} = \{0\}.$$

Finally for $n = 0$, we have:

$$H_0(\{p\};R) = \frac{\ker(1:R \to \{0\})}{\operatorname{im}(0:R \to R)} = R.$$

To prove assertions 2. and 4. requires some more work (to be done in a lecture course on algebraic topology). □

Remark 4.2.9. Assertions 1. and 2. imply that if $X \simeq Y$, then for any $n \in \mathbb{N}$, we have: $H_n(X;R) \cong H_n(Y;R)$.

Remark 4.2.10. For $X = \emptyset$ and $n \in \mathbb{N}$, we have: $C_n(\emptyset;R) = \{0\}$ and hence $H_n(\emptyset;R) = \{0\}$.

Remark 4.2.11. In case $X_0 \cap X_1 = \emptyset$, i.e. $X = X_0 \sqcup X_1$, since $H_n(\emptyset;R) = \{0\}$, we obtain from the Mayer-Vietoris sequence the isomorphisms $H_n(X_0;R) \oplus H_n(X_1;R) \xrightarrow{\cong} H_n(X;R)$.

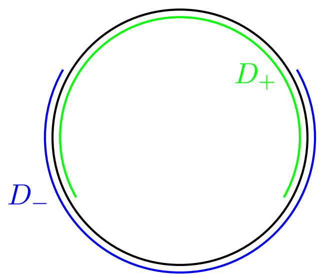
Remark 4.2.12. If X is path connected, then $H_0(X;R) \cong R$.

Example 4.2.13. Using the Mayer-Vietoris sequence, we inductively compute the singular homology of the spheres S^m

a) For $m = 0$, we have $S^0 = \{NP, SP\}$, so from Remark 4.2.11 and assertion 3. of Remark 4.2.8, we have:

$$H_n\left(S^0;R\right) = H_n\left(\{NP\};R\right) \oplus H_n\left(\{SP\};R\right) \cong \begin{cases} R^2 & : n = 0 \\ \{0\} & : n \geq 1 \end{cases}$$

b) For $m = 1$, we take $X_0 = D_-$, $X_1 = D_+$ as depicted alongside: Then clearly $D_\pm \simeq \{p\}$ and $D_+ \cap D_- \simeq \{p_1, p_2\} \cong S^0$.
Since S^1 is path connected, we have $H_0(S^1;R) \cong R$.
Since $D_\pm$ is contractible, we have $H_n(D_\pm;R) = \{0\}$ for $n \geq 1$ and $H_0(D_\pm;R) \cong R$.



For $n \geq 2$, the Mayer-Vietoris now reads:

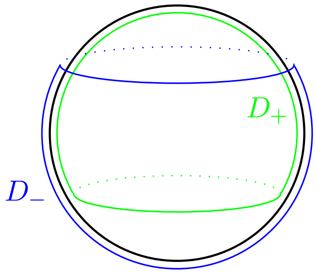
$$\cdots \to \underbrace{H_n\left(D_+;R\right)\oplus H_n\left(D_-;R\right)}_{\cong H_n(\{p_1\};R)\oplus H_n(\{p_1\};R)\cong\{0\}} \to H_n\left(S^1\right) \to \underbrace{H_{n-1}(D_+\cap D_-)}_{\cong H_{n-1}(S^0;R)\cong\{0\}} \to \cdots$$

So $H_n(S^1;R) = \{0\}$ for $n \geq 2$.

For $n = 1$, we have:

$$\underbrace{H_1(D_+;R)\oplus H_1(D_-;R)}_{\cong\{0\}} \to H_1(S^1) \to \underbrace{H_0(D_+\cap D_-;R)}_{\cong R^2} \xrightarrow{\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)} \underbrace{H_0(D_+;R)\oplus H_0(D_-;R)}_{\cong R^2}$$

Hence $H_1(S^1;R) = \ker\begin{pmatrix}1&1\\1&1\end{pmatrix} \cong \left\{\begin{pmatrix}x\\-x\end{pmatrix}\,\middle|\, x\in R\right\} \cong R.$



c) Similarly, for $m \geq 2$, we take $X_0 = D_+$, $X_1 = D_-$ to be sufficiently large balls around NP resp. $SP \in S^m$ such that $X_0 \cap X_1 \simeq S^{m-1}$.

For $k \geq 2$, the Mayer-Vietoris sequence

$$\begin{aligned}\underbrace{H_k\left(D_+;R\right)\oplus H_k\left(D_-;R\right)}_{\cong\{0\}} \quad &\to \quad H_k\left(S^m;R\right) \xrightarrow{\cong} \underbrace{H_{k-1}\left(D_+\cap D_-;R\right)}_{\cong H_{k-1}(S^{m-1};R)} \\ &\to \quad \underbrace{H_{n-1}\left(D_+;R\right)\oplus H_{n-1}\left(D_-;R\right)}_{\cong\{0\}}\end{aligned}$$

yields ismomorphisms $H_k(S^m;R) \cong H_{k-1}(S^{m-1};R)$. For $k = 0$, we have $H_0(S^m;R)\cong R$, since S^m is path connected. For $k = 1$, we have:

$$\underbrace{H_1\left(D_+;R\right)\oplus H_1\left(D_-;R\right)}_{\cong\{0\}} \to H_1\left(S^m;R\right) \to \underbrace{H_0\left(S^{m-1};R\right)}_{\cong R} \xrightarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} \underbrace{H_0\left(D_+;R\right)\oplus H_0\left(D_-;R\right)}_{\cong R^2}$$

whence $H_1(S^m;R)\cong\{0\}$.

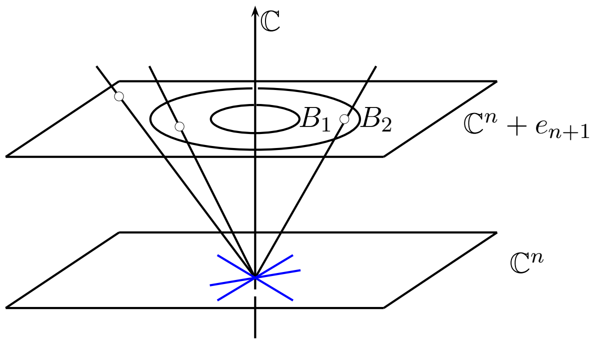
Thus, we have shown:

For $m \geq 1$: $\quad H_k(S^m;R) = \begin{cases} R & : k = 0, m \\ \{0\} & : \text{otherwise}\end{cases}$ $\quad$ and $\quad H_k(S^0;R) = \begin{cases} R^2 & : k = 0 \\ \{0\} & : \text{otherwise}\end{cases}$

Example 4.2.14. For the homology of the complex projective spaces, we find:

$$H_n(\mathbb{C}P^n; R) = \begin{cases} R & : \ k = 0, 2, 4, \dots 2n \\ \{0\} & : \ \text{otherwise} \end{cases} \tag{4.2}$$

For $n = 1$, we have $\mathbb{C}P^1 = S^2$, and $H_0(S^2; R) = H_2(S^2; R) = R$ whereas $H_1(S^2; R) = \{0\}$.
We proceed by the induction step from $n-1$ to n. To use the Mayer-Vietoris sequence, we first define an appropriate cover of $\mathbb{C}P^n$ as follows. By definition, $\mathbb{C}P^n$ is the space of all complex lines in $\mathbb{C}^{n+1}$. We depict $\mathbb{C}P^n$ as in the model from Remark 1.5.15:



Now we take two concentric balls $B_1 \subset B_2 \subset \mathbb{C}^n + e_{n+1}$, and we set:

$$\begin{aligned} X_0 &:= \{\ell \in \mathbb{C}P^n \mid \ell \cap B_2 \neq \emptyset\} \cong B_2 \simeq \{p\} \\ X_1 &:= \left\{\ell \in \mathbb{C}P^n \mid \ell \cap \bar{B}_1 = \emptyset\right\} \simeq \mathbb{C}P^{n-1}. \end{aligned}$$

Then we have $X_0 \cap X_1 \cong B_2 \backslash \bar{B}_1 \simeq S^{2n-1}$. The Mayer-Vietoris sequence now reads:

$$\underbrace{H_k(X_0 \cap X_1; R)}_{\cong H_k(S^{2n-1})} \to \underbrace{H_k(X_0; R) \oplus H_k(X_1; R)}_{\cong H_k(\{p\}; R) \oplus H_k(\mathbb{C}P^{n-1}; R)} \to H_k(\mathbb{C}P^n; R) \to \underbrace{H_{k-1}(X_0 \cap X_1; R)}_{\cong H_{k-1}(S^{2n-1})}$$

For $k \neq 0, 1, 2n-1, 2n$, we thus obtain isomorphisms $H_k(\mathbb{C}P^n; R) \cong H_k(\mathbb{C}P^{n-1}; R)$.

For $k = 0$, we have $H_0(\mathbb{C}P^n; R) = R$, since $\mathbb{C}P^n$ is path connected.

For $k = 1$, we have (using the induction hypothesis that (4.2) holds for $\mathbb{C}P^{n-1}$):

$$\underbrace{H_1\left(\mathbb{C}P^{n-1}; R\right)}_{\cong \{0\}} \to H_1(\mathbb{C}P^n; R) \to \underbrace{H_0\left(S^1; R\right)}_{\cong R} \xrightarrow{\binom{1}{1}} \underbrace{H_0(\{p\}; R) \oplus H_0\left(\mathbb{C}P^{n-1}; R\right)}_{\cong R^2}$$

Since the map on the right is injective, we find $H_1(\mathbb{C}P^n; R) = \{0\}$.

For $k = 2n-1$, the Mayer-Vietoris sequence reads:

$$\underbrace{H_{2n-1}(\{p\}; R) \oplus H_{2n-1}(\mathbb{C}P^{n-1}; R)}_{=\{0\}} \to H_{2n-1}(\mathbb{C}P^n; R) \to \underbrace{H_{2n-2}(S^{2n-1}; R)}_{=\{0\}}$$

whence $H_{2n-1}(\mathbb{C}P^n; R) = \{0\}$.

Finally, for $k = 2n$, the Mayer-Vietoris sequence reads:

$$\begin{aligned}\underbrace{H_{2n}\left(\{p\}; R\right) \oplus H_{2n}\left(\mathbb{C}P^{n-1}; R\right)}_{=\{0\}} \quad &\to \quad H_{2n}\left(\mathbb{C}P^n; R\right) \to \underbrace{H_{2n-1}\left(S^{2n-1}; R\right)}_{\cong R} \\ &\to \quad \underbrace{H_{2n-1}\left(\{p\}\right) \oplus H_{2n-1}\left(\mathbb{C}P^{n-1}; R\right)}_{=\{0\}}\end{aligned}$$

whence $H_{2n}(\mathbb{C}P^n; R) \cong R$.

Example 4.2.15. Connected sums provide yet another example of the usefulness of the Mayer-Vietoris sequence: Let M be a (topological) manifold, $x \in M$, and put $\dot{M} := M \backslash \{x\}$. Let X_0 be any ball containing x and define $X_1 := \dot{M}$. Then $X_0 \simeq \{x\}$ and $X_0 \cap X_1 \simeq S^{n-1}$. The Mayer-Vietoris sequence reads:

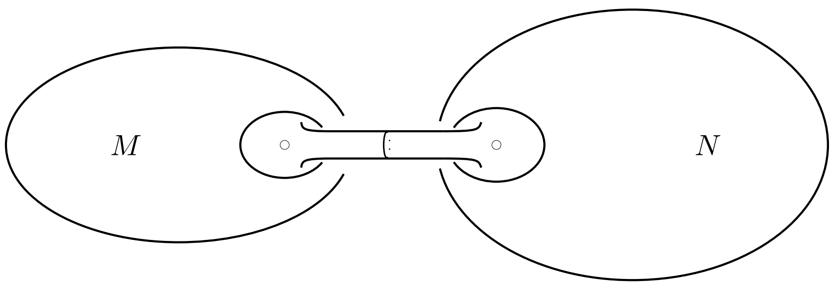
$$H_k\left(S^{n-1}; R\right) \to H_k\left(\{x\}; R\right) \oplus H_k\left(\dot{M}; R\right) \to H_k(M; R) \to H_{k-1}\left(S^{n-1}; R\right)$$

For $k \notin \{0, 1, n-1, n\}$, we thus have:

$$\{0\} \to H_k\left(\dot{M}; R\right) \to H_k(M; R) \to \{0\}$$

Hence, in these cases, the inclusion $\dot{M} \hookrightarrow M$ induces isomorphisms $H_k(\dot{M}; R) \cong H_k(M; R)$.

Now let N be another (topological) manifold. Then we can build the ***connected sum*** $M \sharp N$ by removing a small ball in M and N respectively and glueing the remaining parts of M and N along the boundaries of those balls (by filling in a small neck):



We take $X_0 := \dot{M}$, $X_1 := \dot{N}$, so that $X_0 \cap X_1 \cong S^{n-1} \times (0,1) \simeq S^{n-1}$. Then the Mayer-Vieroris sequence reads:

$$H_k\left(S^{n-1};R\right) \to H_k\left(\dot{M};R\right) \oplus H_k\left(\dot{N};R\right) \to H_k\left(M\sharp N;R\right) \to H_{k-1}\left(S^{n-1};R\right)$$

Again, for $k \notin \{0,1,n-1,n\}$, we obtain isomorphisms $H_k(M) \oplus H_k(N) \cong H_k(M\sharp N;R)$. This holds especially for $n=4$, $k=2$.

Example 4.2.16. Here is an example how homology groups are used to solve geometrical problems. We prove the following statement:
There exists no continuous map

$$f : \bar{B}^{n+1} = \left\{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\right\} \to S^n$$

satisfying $f|_{S^n} = \mathrm{id}_{S^n}$.

Namely, if there was such an f, then composing f with the inclusion $\iota : S^n \hookrightarrow \bar{B}^{n+1}$, we would have $f \circ \iota = \mathrm{id}_{S^n}$, hence on homomology groups, $(f \circ \iota)* = \mathrm{id}_{H_*(S^n;\mathbb{Z})}$. But since $\bar{B}^{n+1}$ is contractible, we have in the n^{th} homology:

$$\underbrace{H_n\left(S^n;\mathbb{Z}\right)}_{\cong\mathbb{Z}} \xrightarrow{\iota_*} \underbrace{H_n\left(\bar{B}^{n+1};\mathbb{Z}\right)}_{\cong\{0\}} \xrightarrow{f_*} \underbrace{H_n\left(S^n;\mathbb{Z}\right)}_{\cong\mathbb{Z}}$$

The identity of $\mathbb{Z}$ would thus factorize through $\{0\}$, which is impossible.

Now let us briefly discuss the relation between the homotopy and homology groups.

Definition 4.2.17. Fix a generator $[c] \in H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Let X be a path connected topological space, and let $x \in X$. Assume $[f] \in \pi_n(X,x)$, represented by $f \in \mathcal{C}(S^n, X)$. Then the ***Hurewicz homomorphism*** $h : \pi_n(X,x) \to H_n(X;\mathbb{Z})$ is defined by $h([f]) := f_*([c])$.

Remark 4.2.18. By the homotopy invariance of homology, the element $f_*([c])$ only depends on the homotopy class of f, but not on the particular map f. So the map h is well-defined. To show that it is indeed a group homomorphism requires some more work (to be done in a lecture course on algebraic topology).

Remark 4.2.19. For any group G, we denote by $[G, G]$ the normal subgroup generated by all elements of the form $ghg^{-1}h^{-1}$, where g, $h \in G$. Then $G^{\text{abel}} := G/[G, G]$ is an abelian group called the ***abelianization*** of G.
If G is abelian, then we have $[G, G] = \{e\}$, so $G^{\text{abel}} = G$.

Example 4.2.20. For $G = \mathbb{Z} * \mathbb{Z} := \{a^{k_1}b^{l_1} \ldots a^{k_n}b^{l_n} \mid k_i, l_i \in \mathbb{Z}\}$, we have $G^{\text{abel}} = \mathbb{Z}^2$.

Remark 4.2.21. Since the homology groups are abelian, the Hurewicz homomorphism h vanishes on $[\pi_n(X, x), \pi_n(X, x)]$. Hence it descends to a map $h : \pi_n(X, x)^{\text{abel}} \to H_n(X; \mathbb{Z})$ also called the ***Hurewicz homomorphism***.

Theorem 4.2.22 (Hurewicz)
Let X be a path connected topological space, $x \in X$. Let $\pi_k(X, x) = \{0\}$ for $k = 1, \ldots, m-1$. Then the Hurewicz homomorphism $h : \pi_m(X, x)^{\text{abel}} \to H_m(X; \mathbb{Z})$ is an isomorphism.

Remark 4.2.23. If $m \geq 2$, then $\pi_m(X, x)$ is abelian, so $\pi_m(X, x)^{\text{abel}}$ can be replaced by $\pi_m(X, x)$ in Hurewicz's theorem.

Example 4.2.24. Using Hurewicz' theorem, we can determine the lower homotopy groups of spheres S^n from the homology groups. The spheres S^n, $n \geq 2$, are simply connected. From Hurewicz's theorem, we deduce

$$\pi_2\left(S^n, NP\right) \cong H_2\left(S^n; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}, & n = 2 \\ \{0\}, & \text{otherwise} \end{cases}$$

For $n = 2$, we are done with Hurewicz' theorem, but for $n \geq 3$, we can apply it once again to obtain

$$\pi_3\left(S^n, NP\right) \cong H_3\left(S^n; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z}, & n = 3 \\ \{0\}, & \text{otherwise} \end{cases}$$

Proceeding inductively in this way, we obtain for $m \leq n$, $n \geq 2$:

$$\pi_m\left(S^n, NP\right) \cong H_m\left(S^n; \mathbb{Z}\right) \cong \begin{cases} \mathbb{Z} & : \ m = n \\ \{0\} & 1 \leq m < n \end{cases}$$

In Hurewicz' theorem, the vanishing of the lower homotopy groups is indeed necessary: we have already seen that $\pi_3(S^2; NP) \cong \mathbb{Z}$, whereas $H_3(S^2;\mathbb{Z}) \cong \{0\}$.

Remark 4.2.25. Let M and N be simply connected topological manifolds of dimension ≥ 2. Then the connected sum $M\sharp N$ is also simply connected.

4.3 Orientations and the fundamental class

Throughout this section, let R be a ring with unit.
For any $r > 0$ and $x \in \mathbb{R}^n$, $n \geq 2$, the map

$$F_{x,r} : S^{n-1} \to \dot{B}(x,r) := B(x,r) \backslash \{x\}, \quad y \mapsto x + \frac{r}{2} y,$$

is a homotopy equivalence. Hence we have an isomorphism

$$(F_{x,r})_* : H_{n-1}\left(S^{n-1}; R\right) \to H_{n-1}\left(\dot{B}(x,r); R\right).$$

For different radii $0 < r_1 < r_2$, the diagram

$$\begin{tikzcd} S^{n-1} \arrow[r, "(F_{x,r_1})_*"] \arrow[dr, "(F_{x,r_2})_*"'] & \dot{B}(x,r_1) \arrow[d, "\iota_{r_1,r_2}"] \\ & \dot{B}(x,r_2) \end{tikzcd}$$

commutes up to homotopy (here $\iota_{r_1,r_2} : \dot{B}(x,r_1) \hookrightarrow \dot{B}(x,r_2)$ denotes the natural inclusion). Hence $(F_{x,r_2})_* = (\iota_{r_1,r_2})_* \circ (F_{x,r_1})_*$.

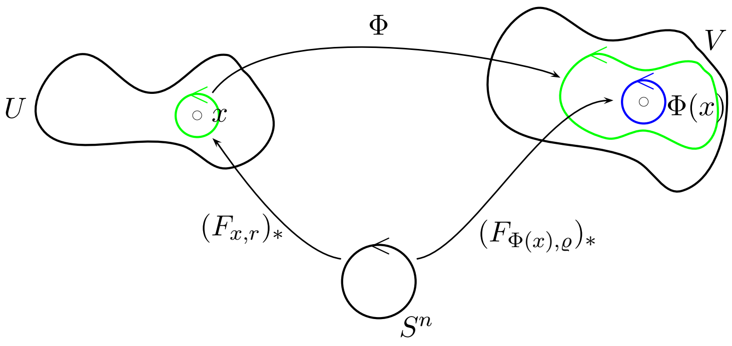
Definition 4.3.1. Let $U, V \subset \mathbb{R}^n$ be open sets. A homeomorphism $\Phi : U \to V$ is called ***R-orientation preserving*** at $x \in U$, iff for every $\varrho > 0$ with $B(\Phi(x), \varrho) \subset V$ and every $r > 0$ with $\Phi(B(x,r)) \subset B(\Phi(x), \varrho)$, the following diagram commutes:

$$\begin{tikzcd} H_{n-1}(\dot{B}(x,r);R) \arrow[rr, "(\Phi|_{\dot{B}(x,r)})_*"] & & H_{n-1}(\dot{B}(\Phi(x),\varrho);R) \\ & H_{n-1}(S^{n-1};R) \arrow[ul, "(F_{x,r})_*"] \arrow[ur, "(F_{\Phi(x),\varrho})_*"'] & \end{tikzcd}$$

commutes.

The homeomorphism $\Phi : U \to V$ is called ***R-orientation preserving***, iff it is R-orientation preserving at every point $x \in U$.

Example 4.3.2. We can illustrate the definition for $n = 2$ and $R = \mathbb{Z}$:



Example 4.3.3. For $R = \mathbb{Z}/2\mathbb{Z}$, the identity is the only automorphism of R. Therefore, the diagram above commutes always. Hence any homeomorphism is $\mathbb{Z}/2\mathbb{Z}$-orientation preserving.

Remark 4.3.4. To ensure that a homeomorphism $\Phi : U \to V$ is R-orientation preserving, it suffices to show the commutativity of the above diagram for *one* r and *one* ϱ. This is because for $0 < \varrho_1 < \varrho_2$ and $0 < r_1 < r_2$ as above, we obtain the following diagram:

$$\begin{tikzcd}
H_{n-1}\left(\dot{B}(x,r_2);R\right) \arrow[rr, "(\Phi|_{\dot{B}(x,r_2)})_*"] & & H_{n-1}\left(\dot{B}(\Phi(x),\varrho_2);R\right) \\
H_{n-1}\left(\dot{B}(x,r_1);R\right) \arrow[u, "(\iota_{r_1,r_2})_*"] \arrow[rr, "(\Phi|_{\dot{B}(x,r_1)})_*"] & & H_{n-1}\left(\dot{B}(\Phi(x),\varrho_1);R\right) \arrow[u, "(\iota_{\varrho_1,\varrho_2})_*"'] \\
& H_{n-1}\left(S^{n-1};R\right) \arrow[ul, "(F_{x,r_1})_*"] \arrow[ur, "(F_{\Phi(x),\varrho_1})_*"'] &
\end{tikzcd}$$

By construction, the upper square commutes for all r_1, r_2, ϱ_1, ϱ_2. So the lower triangle commutes iff the whole diagram commutes.

Definition 4.3.5. An atlas $\mathcal{A} \subset \{\text{homeomorphisms } \Phi : M \supset U \to V \subset \mathbb{R}^n\}$ of a topological n-manifold M is called **R-*oriented*** iff all the maps $\Phi \circ \Psi^{-1}$, for Φ, $\Psi \in \mathcal{A}$ are R-orientation preserving.
A maximal R-oriented atlas of M is called an **R-*orientation*** of M.

A pair $(M, \mathcal{A})$ consisting of a topological manifold together with an R-orientation is called an ***R-oriented manifold***. A topological manifold M is called ***R-orientable*** iff M admits an R-orientation.

Remark 4.3.6. Any topological manifold is $\mathbb{Z}/2\mathbb{Z}$-orientable.

Remark 4.3.7. For a differentiable manifold, orientability in the differentiable sense coincides with $\mathbb{Z}$-orientability.

Remark 4.3.8. Let M be a topological manifold, and $x \in M$. On the set K_x of all charts of M sending $x \in M$ to $0 \in \mathbb{R}^n$, we have the equivalence relation

$$\Phi \sim \Psi :\Leftrightarrow \Phi \circ \Psi^{-1} \text{ is } R\text{-orientation preserving at } 0.$$

We set $\tilde{M}_x := K_x/\sim$ and $\tilde{M} := \bigsqcup_{x \in M} \tilde{M}_x$. Equipped with an appropriate topology, $\tilde{M}$ is a covering of M, called the ***R-orientation covering***.
If M is simply connected, we know from Lemma 4.1.31 that the identity $\mathrm{id}_M : M \to M$ can be lifted to a continuous section $\widetilde{\mathrm{id}}_M : M \to \tilde{M}$ of the R-orientation covering. Any such lift provides us with an R-orientation of M, since it attaches to each point $x \in M$ charts such that the corresponding chart changes are R-orientation preserving.
Hence a simply connected topological manifold M is R-orientable for any ring R.

Remark 4.3.9. Let X be a topological manifold, $n \geq 2$, and $x \in M$. Take $X_0 := \dot{M} := M \backslash \{x\}$ and take X_1 to be an open neighborhood B of x homeomorphic to a ball in $\mathbb{R}^n$. Then $X_0 \cap X_1 \cong \dot{B} \simeq S^{n-1}$. The Mayer-Vietoris sequence then reads:

$$H_n\left(\dot{M}; R\right) \oplus \underbrace{H_n(\{p\}; R)}_{\cong \{0\}} \to H_n(M; R) \xrightarrow{\partial} H_{n-1}\left(\dot{B}; R\right) \to H_{n-1}\left(\dot{M}; R\right) \oplus \underbrace{H_{n-1}(\{p\}; R)}_{\cong \{0\}}$$

It is a fact that for a compact, connected, R-oriented manifold M, the boundary homomorphism $\partial : H_n(M; R) \to H_{n-1}(\dot{B}; R)$ is an isomorphism (this is an instance of Poincaré duality).
Further, an R-orientation provides us with an isomorphism

$$H_{n-1}\left(\dot{B}; R\right) \xrightarrow{\cong} H_{n-1}\left(S^{n-1}; R\right) \cong R\,.$$

Hence we obtain a distinguished isomorphism $H_n(M; R) \xrightarrow{\cong} R$.

Definition 4.3.10. Let M be a compact, connected, R-oriented topological n-manifold. The ***(R)-fundamental class*** $[M]$ of M is the homology class $[M] \in H_n(M;R)$ that is mapped to $1 \in R$ under the isomorphism just constructed.

Remark 4.3.11. If M admits a triangulation T, then the formal sum over the simplices of T, appropriately parametrized, represents the fundamental class $[M] \in H_n(M;R)$.

Definition 4.3.12. Let X be a topological space and M an R-oriented, connected, compact topological k-manifold. A class $\alpha \in H_k(X;R)$ is ***represented*** by a continuous map $f : M \to X$ iff $\alpha = f_*([M])$.

Example 4.3.13

1. Let $X = M$. Then the identity map $f = \mathrm{id}_M$ represents the fundamental class $\alpha = [X]$, since $f_*([M]) = (\mathrm{id}_M)_*([M]) = [M] = [X]$.

2. Let X be path connected and $M = \{p\}$. Then any map $f : \{p\} \to X$ represents a generator of $H_0(X;R) \cong R$.

3. Let $X = \mathbb{C}P^n$ and $M = \mathbb{C}P^{n-1}$. Let $\iota : M \hookrightarrow X$ be the inclusion of $\mathbb{C}P^{n-1}$ into $\mathbb{C}P^n$ as in Example 4.2.14. From the Mayer-Vietoris sequence, we know that ι induces isomorphisms $\iota_* : H_{2n-2}(M) \to H_{2n-2}(X)$. Hence ι represents a generator of $H_{2n-2}(\mathbb{C}P^n)$. Similarly, by restricting to the lower dimensional complex projective spaces, the maps $\iota : \mathbb{C}P^k \hookrightarrow \mathbb{C}P^n$ represent generators of $H_{2k}(\mathbb{C}P^n)$ for $k = 1, \ldots, n$.

4. Let $X = S^2 \times S^2$, and let $(p_1, p_2) \in X$. Then the inclusion maps
$$\begin{aligned} f_1 : S^2 &\to X, \qquad x \mapsto (x, p_2) \\ f_2 : S^2 &\to X, \qquad x \mapsto (p_1, x) \end{aligned}$$
represent the two generators of $H_2(S^2 \times S^2; R) \cong R^2$. From Example 4.2.15, we know that the inclusion map $\iota : (S^2 \times S^2)\backslash\{-(p_1, p_2)\} =: (S^2 \times S^2)^{\cdot} \hookrightarrow S^2 \times S^2$ induces an isomorphism $\iota_* : H_2((S^2 \times S^2)^{\cdot}) \to H_2(S^2 \times S^2)$. To use the Mayer-Vietoris sequence, we cover $S^2 \times S^2$ by $X_0 := S^2 \times (S^2\backslash\{-(p_2)\}) \simeq S^2 \times \{p_2\}$ and

$X_1 := (S^2\backslash\{-(p_2)\}) \times S^2 \simeq \{p_1\} \times S^2$ so that $X_0 \cap X_1 \cong D^2 \times D^2 \simeq \{(p_1, p_2)\}$. The Mayer-Vietoris sequence then reads:

$$\underbrace{H_2\big(\{(p_1,p_2)\}\big)}_{\cong\{0\}} \to H_2\big(S^2{\times}\{p_2\}\big){\oplus}H_2\big(\{p_1\}{\times}S^2\big) \overset{\cong}{\to} H_2\Big(\big(S^2{\times}S^2\big)^{\cdot}\Big) \to \underbrace{H_1\big(\{(p_1,p_2)\}\big)}_{\cong\{0\}} .$$

The inclusion thus yields an isomorphism

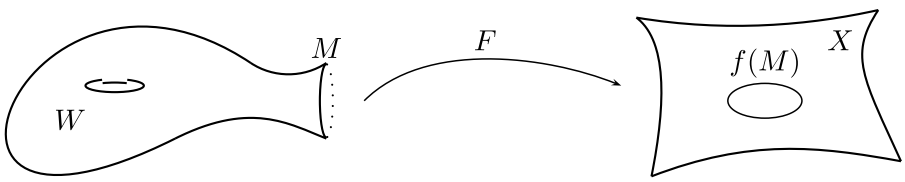
$$\begin{tikzcd} H_2\big(S^2 \times \{p_2\}\big) \oplus H_2\big(\{p_1\} \times S^2\big) \arrow[r] & H_2\big(S^2 \times S^2\big) \\ H_2\big(S^2\big) \oplus H_2\big(S^2\big) \arrow[u] \arrow[ur, "((f_1)_*,(f_2)_*)"'] & \end{tikzcd}$$

Hence the inclusions f_1, f_2 represent the two generators of $H_2(S^2 \times S^2)$ as claimed.

Especially, we are now able to identify the homology of $S^2 \times S^2$ as:

$$H_k\left(S^2 \times S^2; R\right) \cong \begin{cases} R & : \ k = 0, 4 \\ R^2 & : \ k = 2 \\ \{0\} & : \ k = 1, 3 \end{cases}$$

Remark 4.3.14. Let $\alpha \in H_k(X; R)$ be represented by a continuous map $f : M \to X$. Let W for a compact, connected, R-oriented topological $(k+1)$-manifold with $\partial W = M$. Further assume that there exists a continuous extension $F : W \to X$ of f, i.e. $F|_{\partial W} = f$. Then $\alpha = 0 \in H_k(X; R)$.



This follows from the fact (which is another instance of Poincaré duality) that the inclusion $\iota : M \hookrightarrow W$ of the boundary represents $0 \in H_k(W; R)$. Using this fact, we have:

$$\alpha = f_*([M]) = (F \circ \iota)_*([M]) = F_*\big(\underbrace{\iota_*([M])}_{=0}\big) = 0\,.$$

5 4-dimensional Manifolds

5.1 The intersection form

Throughout this section, let X be a compact, oriented, simply connected, differentiable 4-manifold. Our (preliminary) goal is to find "good" representants for elements in $H_2(X;\mathbb{Z})$.

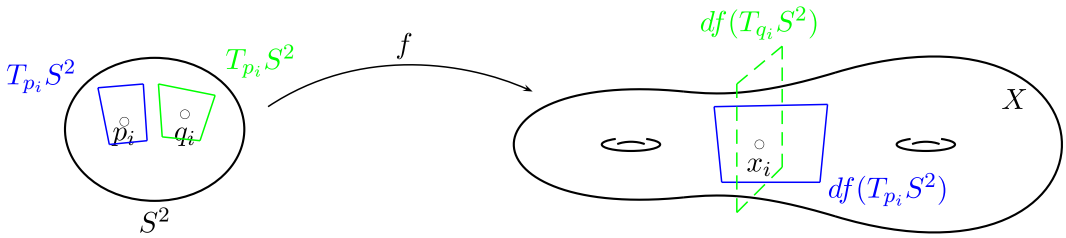
Step 1: Hurewicz's theorem
Since X is simply connected, the Hurewicz homomorphism $\pi_2(X,x) \to H_2(X;\mathbb{Z})$ is an isomorphism. Hence any $\alpha \in H_2(X;Z)$ can be represented by a continuous map $f : S^2 \to X$.

Step 2: Smoothing
Using standard mollifiers, we can deform a continuous map $f : S^2 \to X$ into a smooth map: there exists a homotopy $F : S^2 \times I \to X$ with $F(\cdot,0) = f$ such that for any $t > 0$, the map $f_t := F(\cdot,t) : S^2 \to X$ is smooth. Hence any $\alpha \in H_2(X;Z)$ can be represented by a smooth map $f : S^2 \to X$.

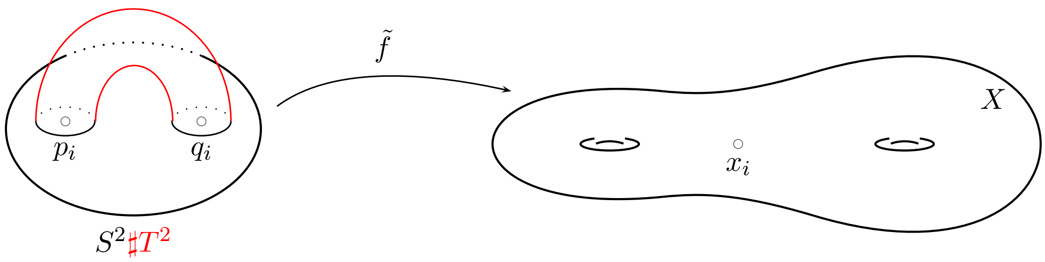
Step 3: Transversality
Due to work of R. Thom, any continuous map $f : S^2 \to X$ can be deformed into an immersion with finitely many ***double points*** $x_i = f(p_i) = f(q_i) \in X$, $i = 1,\dots,N$, (with $p_i \neq q_i \in S^2$ and $x_i \neq x_j$ for $i \neq j$), such that

$$df\left(T_{p_i}S^2\right) \oplus df\left(T_{q_i}S^2\right) = T_{x_i}X,$$

i.e. f is ***transversal***.

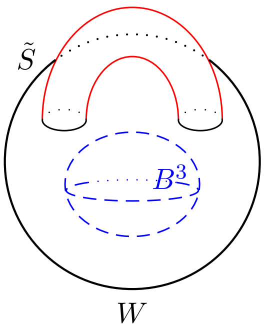


Step4: Removal of double points
For a double point $x_i \in X$, $p_i \neq q_i \in S^2$ with $f(p_i) = f(q_i) = x_i$, choose a coordinate system $\Phi : U(x_i) \to \mathbb{R}^4 = \mathbb{C}^2$ on a neighbourhood of x_i such that $\Phi(x_i) = 0$, $f(U(p_i)) = \mathbb{R}^2 \times \{0\}$ and $f(U(q_i)) = \{0\} \times \mathbb{R}^2$ for some small neighbourhoods $U(p_i)$ of p_i and $U(q_i)$ of q_i. Now take $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ and connect the two Hopf circles $S^1_1 := (\mathbb{C} \times \{0\}) \cap S^3$ and $S^1_2 := (\{0\} \times \mathbb{C}) \cap S^3$ by a cylinder Z.
Now remove $f^{-1}(\Phi^{-1}(B(0,1) \subset \mathbb{R}^4))$ from S^2 and replace it by attaching a handle as depicted schematically below. Then we may extend the map $f : S^2 \to X$ to the handle by connecting the values of f on the boundary circles of the removed discs along the colored lines in the above picture. This yields a map $\tilde{f} : S^2 \sharp T^2 \to X$, which coincides with f outside the modified regions.



Proceeding in this way for all the double points x_i, $i = 1, \ldots, N$, we obtain a new map $\tilde{f} : \tilde{S} \to X$ from a surface $\tilde{S}$ with N handles.

To show that this procedure does not change the homology class represented by f, we fill up the surface $\tilde{S}$ to get a handle body $\tilde{W}$ with $\partial\tilde{W} = \tilde{S}$. Removing from $\tilde{W}$ a (sufficiently small) ball $B^3(0,\varrho)$, we obtain a compact 3-manifold $W := \tilde{W} \backslash B^3(0,\varrho)$ with $\partial W = S^2 \sqcup \overline{S^2(\varrho)}$ (Here $\overline{S^2(\varrho)}$ denotes the sphere with the orientation reversed).



Outside the area where we performed the modifications of the surface, we may extend f resp. $\tilde{f}$ to a map $F : W \to X$ simply by setting F constant along the radial lines. To extend F on the full (solid) handle, we just repeat the attaching procedure for a family of circles $S^3(r)$, $r > 0$. By construction, we may now extend f along the depicted coloured lines on the (solid) handle such that the nerve of the handle gets mapped under the new map $F : W \to X$ to the point x_i.
This way, we obtain a map $F : W \to X$ such that $F|_{\partial W} = f \sqcup \tilde{f} : \overline{S^2} \times \tilde{S} \to X$. Thus $f \sqcup \tilde{f}$ represents $0 \in H_2(X;\mathbb{Z})$, which yields that f and $\tilde{f}$ represent the same homology class in $H_2(X;\mathbb{Z})$.

Remark 5.1.1. As a consequence from the above discussion, we note that we can represent any $\alpha \in H_2(X;\mathbb{Z})$ by an embedding $f : S \to X$, where S is a compact, connected, oriented surface.

Definition 5.1.2. Let X be as above, and let $S_1, S_2 \subset X$ be embedded, compact, oriented surfaces with transversal (or empty) intersection. Then we set

$$S_1 \cdot S_2 := \sum_{p \in S_1 \cap S_2} \varepsilon(p)\,,$$

where $\epsilon(p) := +1$, if the orientation of $T_pS_1 \oplus T_pS_2 = T_pX$ induced from the orientations of S_1 and S_2 coincides with the one on X, and $\varepsilon(p) := -1$ otherwise.
The ***intersection form*** of X is the map

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}\,, \qquad ([S_1]\,,[S_2]) \mapsto S_2 \cdot S_2\,.$$

Remark 5.1.3. The intersection form is well-defined, bilinear and symmetric.

Example 5.1.4. For $X = S^2 \times S^2$, we have $H_2(X;Z) \cong \mathbb{Z}^2$ with generators $[S_1]$, $[S_2]$, where $S_1 := S^2 \times \{p_2\}$ and $S_2 := \{p_1\} \times S^2$. We then have $S_1 \cdot S_2 = 1$ for the standard orientation of S^2 and the product orientation of X. To compute the self-intersections of $[S_1]$ (and similarly of $[S_2]$), we need another representant of $[S_1]$ intersecting S_1 transversally (this is of course not the case for S_1 itself). Note that $[S_1] = [S_1']$, where $S_1' := S^2 \times \{p_2'\}$ (and similarly for S_2). This is because for any $p_2' \in S^2$, we have $S_2 \times \{p_2\} \sqcup \overline{S^2 \times \{p_2'\}} = \partial(S^2 \times c)$, where $c : I \to S^2$ is a curve joining p_2 and p_2'. Now for $p_2 \neq p_2'$, we have $S_1 \cap S_1' = \emptyset$, so $Q_X([S_1],[S_1]) = S_1 \cdot S_1' = 0$ and similarly $Q_X([S_2],[S_2]) = 0$. Hence in the basis $[S_1],[S_2]$, the intersection form $Q_{S^2\times S^2}$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Example 5.1.5. For $X = \mathbb{C}P^2$, we have $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$ with generator $[\mathbb{C}P^1]$. To compute the self-intersection of the generator, we need two transversal representants. So let $j_1, j_2 : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be two distinct complex linear embeddings. They induce two distinct embeddings $\iota_1, \iota_2 : \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$ by $\ell \mapsto j_\nu(\ell)$, $\nu = 0, 1$. Since they are homotopic, they induce the same homology class in $H_2(\mathbb{C}P^2;\mathbb{Z})$. Denoting $\mathbb{C}P^1_\nu := \iota_\nu(\mathbb{C}P^1) \subset \mathbb{C}P^2$, $\nu = 0, 1$, we have $j_1(\mathbb{C}^2) \cap j_2(\mathbb{C}^2) = \ell_0 \subset \mathbb{C}^3$ and hence $\mathbb{C}P^1_1 \cap \mathbb{C}P^1_2 = \{\ell_0\}$. This yields

$\varepsilon(\ell_0) = 1$ with respect to the natural orientations on $\mathbb{C}P^1_\nu$ and $\mathbb{C}P^2$ induced by the complex structure. Hence $Q_{\mathbb{C}P^2} = (1)$.

Remark 5.1.6. Let X be as above and denote by $\bar{X}$ the same manifold with the orientation reversed. Then we have $Q_{\bar{X}} = -Q_X$.

Remark 5.1.7. For $X = X_1 \sharp X_2$, we have isomorphisms

$$\begin{tikzcd} & H_2(\dot{X}_1;\mathbb{Z}) \oplus H_2(\dot{X}_2;\mathbb{Z}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_2(X_1;\mathbb{Z}) \oplus H_2(X_2;\mathbb{Z}) & & H_2(X;\mathbb{Z}) \end{tikzcd}$$

Taking representants of homology classes completely inside the parts $\dot{M}_1$ and $\dot{M}_2$, we see that the intersections of the two parts are completely independent. Hence $Q_X = Q_{X_1} \oplus Q_{X_2}$, which is represented by the matrix $\begin{pmatrix} Q_{X_1} & 0 \\ 0 & Q_{X_2} \end{pmatrix}$.

Example 5.1.8. For $X := k\mathbb{C}P^2 \sharp l\overline{\mathbb{C}P^2} := \underbrace{\mathbb{C}P^2 \sharp \cdots \sharp \mathbb{C}P^2}_{k} \sharp \underbrace{\overline{\mathbb{C}P^2} \sharp \cdots \sharp \overline{\mathbb{C}P^2}}_{l}$, we have

$$Q_X = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right)$$

where 1 occurs k times and -1 occurs l times.

Remark 5.1.9. Since X is simply connected, we have $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^r$ for some $r \in \mathbb{N}$, so we do not need to worry about torsion elements.
In general, if $Q : H \times H \to \mathbb{Z}$ is a symmetric bilinear form on a finitely generated $\mathbb{Z}$-module H, then for a torsion element $a \in H$ (i.e. $\exists\, k \in \mathbb{N}$ with $k \cdot a = 0$), we have for any $b \in H$:

$$k \cdot Q(a, b) = Q(k \cdot a, b) = Q(0, b) = 0\,.$$

Hence $Q(a, b) = 0$ for all b and Q descends to a symmetric bilinear form

$$H/\text{Torsion} \times H/\text{Torsion} \to \mathbb{Z}\,.$$

Then one would consider this form on $H/\text{Torsion}$, a free $\mathbb{Z}$-module of finite rank.

Definition 5.1.10. A symmetric bilinear form Q on a free $\mathbb{Z}$-module $H \cong \mathbb{Z}^r$ is called ***unimodular*** iff there exists a basis $e_1, \dots, e_r$ of H such that

$$\det\big(Q(e_i, e_j)\big)_{i,j=1,\dots,r} = \pm 1\,. \tag{5.1}$$

Remark 5.1.11. If Q is a unimodular, then the equation (5.1) holds for any basis of H: If $f_1, \dots, f_r$ is another basis, we may write $(f_1, \dots, f_r) = (e_1, \dots, e_r)\cdot A$ with a matrix $A \in \mathrm{GL}(r; \mathbb{Z})$. Then $1 = \det(A \cdot A^{-1}) = \underbrace{\det(A)}_{\in\mathbb{Z}} \cdot \underbrace{\det(A^{-1})}_{\in\mathbb{Z}}$, hence $\det(A) = \det(A^{-1}) = \pm 1$.
Then we have $Q(e_i, e_j)_{i,j} = A^t \cdot Q(f_k, f_l)_{f,k} \cdot A$, so

$$\det\big(Q(e_i, e_j)_{i,j}\big) = \underbrace{\det A^2}_{=1} \cdot \det\big(Q(f_k, f_l)_{f,k}\big)\,.$$

Remark 5.1.12. For X as above, the intersection form Q_X is unimodular (this is yet another instance of Poincaré duality).

Definition 5.1.13. The ***rank*** of a symmetric bilinear form Q on a free $\mathbb{Z}$-module H is defined as the dimension of H and is denoted by $\mathrm{rk}(Q)$.
Over $\mathbb{R}$ or $\mathbb{Q}$, the form Q can be diagonalized, and the ***signature*** of Q is defined as

$$\mathrm{sign}(Q) := \#\text{ positive eigenvalues of Q} - \#\text{ negative eigenvalues of Q}\,.$$

The ***signature*** of a 4-manifold X as above is defined as the signature of its intersection form: $\mathrm{sign}(X) := \mathrm{sign}(Q_X)$.
A symmetric bilinear form Q on a free $\mathbb{Z}$-module H is called ***positive/negative definite*** iff $Q(a,a) \gtrless 0$ for all $a \in H$, and ***indefinite*** otherwise.
We say that Q has ***even parity*** (or the ***parity*** of Q is ***even***) iff $\forall\, a \in H$: $Q(a,a) \in 2\mathbb{Z}$, and that Q has ***odd parity*** (or the ***parity*** of Q is ***odd***) otherwise.

Remark 5.1.14. A symmetric bilinear form Q on a free $\mathbb{Z}$-module $H \cong \mathbb{Z}^r$ has even parity iff for any basis of H, all diagonal elements in the matrix representation of H are even. Indeed, if Q has even parity and $e_1, \dots, e_r$ is a basis of H, then $Q(e_i, e_i) \in 2\mathbb{Z}$.

Conversely, let $(Q_{ij})_{i,j=1,\dots,r} = (Q(e_i, e_j))_{i,j=1,\dots,r}$ be the matrix representation of Q with $Q_{ii} \in 2\mathbb{Z}$. Then for any $H \ni a = \sum_{i=1}^{r} a_i \cdot e_i$, $a_i \in \mathbb{Z}$, we find, using the symmetry of Q:

$$\begin{aligned} Q(a,a) &= \begin{pmatrix} a_1 & \dots & a_r \end{pmatrix} \cdot (Q_{ij}) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \\ &= \sum_{i,j=1}^{r} Q_{ij} a_i a_j \\ &= \underbrace{2 \sum_{i<j} Q_{ij} a_i a_j}_{\in 2\mathbb{Z}} + \sum_{i=1}^{r} \underbrace{Q_{ii}}_{\in 2\mathbb{Z}} a_i^2 \\ &\in 2\mathbb{Z}\,. \end{aligned}$$

Example 5.1.15

1. The intersection form $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has even parity, is indefinite and has signature 0.

2. The intersection form $Q_{\mathbb{C}P^2} = (1)$ has odd parity, is positive definite and has signature 1.

3. The intersection form $Q_{\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has odd parity, is indefinite and has signature 0.

Note that over $\mathbb{Q}$ or $\mathbb{R}$ the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are equivalent, but over $\mathbb{Z}$, they are not, since they have different parity.

Remark 5.1.16. Let Q_1, Q_2 be symmetric bilinear forms on a free $\mathbb{Z}$-module H. Then we have:

1. The form $Q_1 \oplus Q_2$ is unimodular iff Q_1 and Q_2 are unimodular.

2. For the signature, we have: $\operatorname{sign}(Q_1 \oplus Q_2) = \operatorname{sign}(Q_1) + \operatorname{sign}(Q_2)$. Hence for 4-manifolds X_1 and X_2 as above, we have:

$$\operatorname{sign}(X_1 \sharp X_2) = \operatorname{sign}(X_1) + \operatorname{sign}(X_2)\,. \tag{5.2}$$

3. The form $Q_1 \oplus Q_2$ is positive (resp. negative) definite iff both Q_1 and Q_2 are positive (resp. negative) definite.

Example 5.1.17. Here is another important example: The symmetric bilinear form E_8 on $\mathbb{Z}^8$, represented by the matrix

$$\left(\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{array}\right) \tag{5.3}$$

has even parity, is positive definite and $\operatorname{sign}(E_8) = 8$.

Example 5.1.18. On the complex projective space $\mathbb{C}P^n$, represented as the base space of the U(1)-principal bundle $S^{2n+1} \to \mathbb{C}P^n = \mathrm{U}(1)\backslash S^{2n+1}$ with the U(1)-action given by scalar multiplication in $\mathbb{C}^{n+1}$, we introduce the so called ***homogeneous coordinates***: the equivalence class of $(z_0, \ldots, z_n) \in S^{2n+1}$ in $\mathbb{C}P^n$ is denoted by $[z_0 : \ldots : z_n]$.
The **K3** ***surface*** (or ***Kummer surface***) is the complex surface

$$K3 := \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\} .$$

Since the defining equation is homogeneous, it is compatible with the equivalence relation $z \sim z' :\Leftrightarrow z = \lambda \cdot z'$, $\lambda \in \mathrm{U}(1)$. Hence $K3$ is a well-defined compact complex hypersurface in $\mathbb{C}P^3$. The ***Lefschetz hyperplane theorem*** from algebraic geometry tells us that the inclusion $\iota : K3 \hookrightarrow \mathbb{C}P^3$ induces isomorphisms $\iota_\sharp : \pi_k(K3, p) \to \pi_k(\mathbb{C}P^3, p) = \{0\}$ for $k = 0, 1$. Thus $K3$ is simply connected. One can show:

$$Q_{K3} = (-E_8) \oplus (-E_8) \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Hence Q_{K3} is indefinite, has even parity and $\operatorname{sign}(K3) = -16$.

Remark 5.1.19. Let X be a simply connected, compact, oriented differentiable 4-manifold. Then Q_X has even parity iff X has a Sspin structure. In this case, one can define spinors and the Dirac operator on X.

Theorem 5.1.20 (Rochlin)
Let X be a simply connected, compact, oriented differentiable 4-manifold and suppose that Q_X has even parity. Then we have:

$$\operatorname{sign}(X) \in 16 \cdot \mathbb{Z} .$$

Sketch of proof. Let D be the Dirac operator on the spinor bundle of X, $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$. The ***Atiyah-Singer index theorem*** tells us that

$$\operatorname{ind}(D^+) := \dim(\ker D^+) - \dim(\ker D^-) = \frac{1}{8}\operatorname{sign}(X)\,,$$

hence $\operatorname{sign}(X) \in 8\mathbb{Z}$.
In dimension 4 the spinor bundle has a quaternionic structure J, i.e. a $\mathbb{C}$-antilinear automorphism satisfying $J^2 = -\mathrm{id}$. The Dirac operator D commutes with J so that both $\ker D^+$ and $\ker D^-$ are vector spaces with a quaternionic structure and therefore even-dimensional. Hence

$$\operatorname{ind}(D^+) = \underbrace{\dim(\ker D^+)}_{\in 2\mathbb{N}} - \underbrace{\dim(\ker D^-)}_{\in 2\mathbb{N}} \in 2\mathbb{Z}\,,$$

so that $\operatorname{sign}(X) = 8 \cdot \operatorname{ind}(D^+) \in 16\mathbb{Z}$. □

Example 5.1.21. Summarizing, we have the following list of examples:

X	$\operatorname{sign}(X)$	Parity
S^4	0	even
$S^2 \times S^2$	0	even
$\mathbb{C}P^2$	1	odd
$k\mathbb{C}P^2 \sharp l\overline{\mathbb{C}P^2}$	$k-l$	odd
$K3$	-16	even

5.2 Classification results

Throughout this section, let X be a simply connected, compact, oriented topological 4-manifold, and let Q be a symmetric bilinear form on a $\mathbb{Z}$-module H of finite rank.

Remark 5.2.1. The intersection form Q_X of a simply connected, compact, oriented (topological) 4-manifold can be defined in purely homological terms, without using a differentiable structure (this is another use of Poincaré duality). Further, the intersection form is homotopy invariant, i.e. if $X_1 \simeq X_2$, then $Q_{X_1} \cong Q_{X_2}$. That the converse also holds true, is a rather deep result from topology:

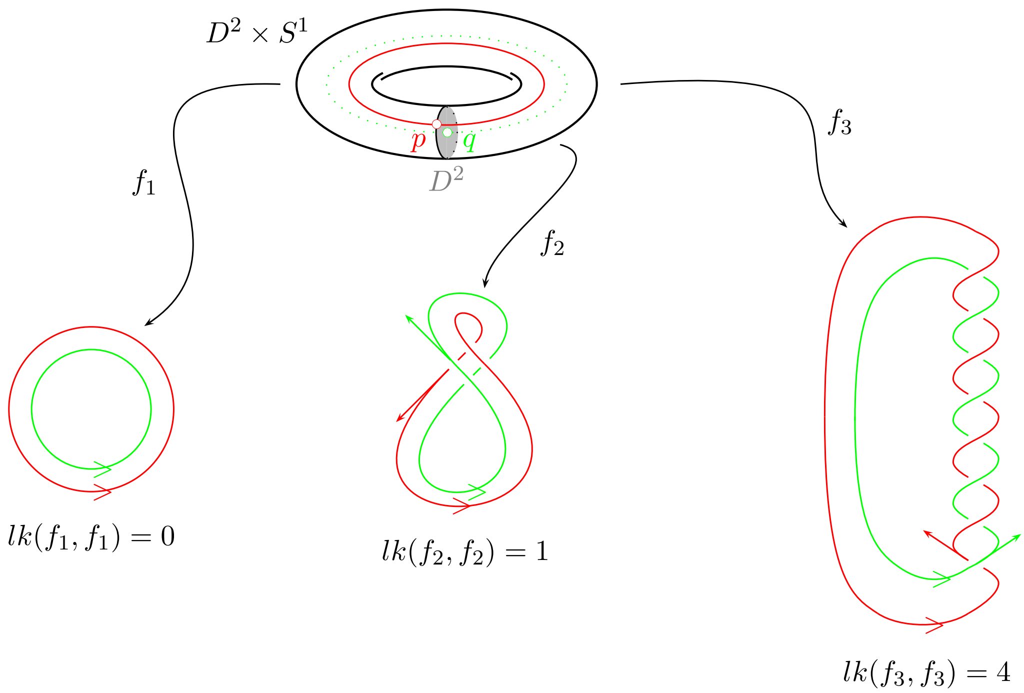
Theorem 5.2.2 (Whitehead)
Let X_1, X_2 be simply connected, compact, oriented topological 4-manifolds. Then $X_1 \simeq X_2$ iff $Q_{X_1} \cong Q_{X_2}$.

Theorem 5.2.3 (Freedman, 1982)
For any unimodular symmetric bilinear form Q on a $\mathbb{Z}$-module H of finite rank, there exists a simply connected, compact, oriented topological 4-manifold X such that $Q_X \cong Q$.
Further, if Q has even parity, then X is uniquely determined up to homeomorphism by Q. If Q has odd parity, then there are up to homeomorphism exactly two simply connected, compact, oriented topological 4-manifolds X with $Q_X \cong Q$.

Sketch of existence proof.

a) For a diffeomorphism $f : \bar{D}^2 \times S^1 \to f(\bar{D}^2 \times S^1) \subset S^3$, the ***self linking number*** is defined as:

$$\mathrm{lk}(f, f) := \mathrm{lk}\big(f(0, \cdot), f(1, \cdot)\big) .$$



For two such maps f_1, f_2 with disjoint images, the ***linking number*** is defined as:

$$\mathrm{lk}\big(f_1, f_2\big) := \mathrm{lk}\big(f_1(0, \cdot), f_2(0, \cdot)\big) .$$

For r such maps $f_1, \ldots, f_r$ with pairwise disjoint images, the ***linking matrix*** is the matrix $(\mathrm{lk}(f_i, f_j))_{i,j=1,\ldots,r}$.

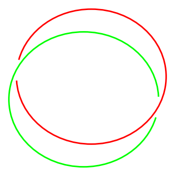
b) Now for a given Q, choose $f_1, \ldots, f_r : \bar{D}^2 \times S^1 \to S^3$ as in a) with pairwise disjoint images and linking matrix Q. Then we start the construction of X with the closed 4-ball $\bar{B}^4$ with $\partial \bar{B}^4 = S^3$. We may glue r copies of $\bar{D}^2 \times \bar{D}^2$ along the boundary component $\bar{D}^2 \times S^1$ to $\bar{B}^4$ by the diffeomorphisms

$$f_i : \bar{D}^2 \times \partial\left(\bar{D}^2\right) \to f_i\left(\bar{D}^2 \times S^1\right) \subset S^3 = \partial\left(\bar{B}^4\right)$$

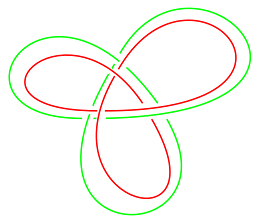
to get a new compact 4-manifold X_1 with boundary.

c) By a careful study of ∂X_1, one can find a contractible 4-manifold X_2 with $\partial X_2 \cong \partial X_1$. Then we may glue X_1 and X_2 along their boundary to get the simply connected, compact, oriented 4-manifold $X := X_1 \cup_{\partial X_i} X_2$. Since X_2 is contractible, the topology of X is determined by the topology of X_1, and one may check that indeed $Q_X \cong Q$. □

Example 5.2.4. To illustrate the dependence of the 4-manifolds X thus constructed, we consider two different knots with self linking number 1. The above construction thus realizes $X = (1)$:



Taking the unknot, one obtains the expected 4-manifold $X = \mathbb{C}P^2$.



Taking the trefoil knot yields a topological 4-manifold $X =: *\mathbb{C}P^2$, called ***fake*** $\mathbb{C}P^2$.

Remark 5.2.5. Let M_{E_8} be the simply connected compact 4-manifold with intersection form $Q = E_8$. Then $\mathrm{sign}(M_{E_8}) = 8$. If M_{E_8} were a differentiable 4-manifold, this would contradict Rochlin's theorem 5.1.20. Hence M_{E_8} is a simply connected, compact, topological 4-manifold which cannot carry a differentiable structure.

Remark 5.2.6. Such a phenomen cannot happen in dimensions lower than 4: for $k \leq 3$, any topological k-manifold carries a differentiable structure which is unique up to diffeomorphism.

Theorem 5.2.7 (Serre)
Let Q_1, Q_2 be indefinite, unimodular symmetric bilinear forms on free $\mathbb{Z}$-modules of finite rank. Then we have:

$$Q_1 \cong Q_2 \quad \Leftrightarrow \quad \begin{array}{l} \mathrm{rk}(Q_1) = \mathrm{rk}(Q_2) \\ \mathrm{sign}(Q_1) = \mathrm{sign}(Q_2) \\ Q_1 \textit{ and } Q_2 \textit{ have the same parity}. \end{array}$$

The isomorphism types of forms Q of odd parity are represented by the matrices

$$A^{\mathrm{odd}}_{k,l} := \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right),$$

where $k, l \geq 1$. For these forms, we have $\mathrm{rk}(Q_{k,l}) = k + l$ and $\mathrm{sign}(Q_{k,l}) = k - l$. The isomorphism types of forms Q of even parity are represented by

$$A^{\mathrm{even}}_{\pm k,l} := \pm k E_8 + l \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What about definite intersection forms? So far, the classification is unknown. We will see in a minute, what's the problem in showing this: there are huge numbers of them!

Lemma 5.2.8
Let Q be a unimodular, symmetric bilinear form on a free $\mathbb{Z}$-module of finite rank. Then there exists an element $w \in H$ such that for any $x \in H$, we have:

$$Q(x,x) \equiv Q(w,x) \pmod 2.$$

Proof. Since Q is unimodular, for any linear map $f : H \to \mathbb{Z}$, there exists a unique $y \in H$ such that

$$\forall\, x \in H : \ f(x) = Q(y,x).$$

Indeed, writing f, x and Q with respect to a basis of H, we obtain the equation

$$(f_1, \ldots, f_r) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} = (y_1, \ldots, y_r) \cdot \left((Q_{i,j})_{i,j=1,\ldots,r}\right) \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} .$$

The form Q is unimodular, so $\det(Q) = \pm 1$, thus the matrix $(Q_{i,j})_{i,j=1,\ldots,r}$ is invertible and we can solve for y.
Now let $\bar{H} := H/2\mathbb{Z}$. Then Q induces a symmetric bilinear form $\bar{Q} : \bar{H} \times \bar{H} \to \mathbb{Z}/2\mathbb{Z}$, and we still have: for any linear map $\bar{f} : \bar{H} \to \mathbb{Z}/2\mathbb{Z}$, there exists a unique $y_{\bar{f}} \in \bar{H}$ such that for any $\xi \in \bar{H}$, we have:

$$\bar{f}(\xi) = \bar{Q}\left(y_{\bar{f}}, \xi\right) .$$

Now we take $\bar{f} : \bar{H} \to \mathbb{Z}/2\mathbb{Z}$, $\bar{f}(\xi) := \bar{Q}(\xi, \xi)$. Although $\bar{f}$ is quadratic, it is linear, since it takes values in $\mathbb{Z}/2\mathbb{Z}$:

$$\bar{f}(\xi + \eta) = \bar{Q}(\xi + \eta, \xi + \eta) = \bar{Q}(\xi, \xi) + \underbrace{2\bar{Q}(\xi, \eta)}_{=0 \in \mathbb{Z}/2\mathbb{Z}} + \bar{Q}(\eta, \eta) = \bar{f}(\xi) + \bar{f}(\eta) .$$

Hence there is a unique $y_{\bar{f}} \in \bar{H}$ such that for any $\xi \in \bar{H}$, we have $\bar{Q}(y_{\bar{f}}, \xi) = \bar{Q}(\xi, \xi)$. Taking $w \in H$ with $\bar{w} = y_{\bar{f}} \in \bar{H}$, we are done. □

Definition 5.2.9. An element $w \in H$ as in Lemma 5.2.8 is called a ***characteristic element*** for Q.
In case $H = H_2(X; \mathbb{Z})$ for a simply connected, compact, oriented topological 4-manifold X, a surface $S \subset X$ representing a characteristic element for Q_X is called a ***characteristic surface***.

Lemma 5.2.10 (van der Blij)
Let Q be a unimodular, symmetric bilinear form on a free $\mathbb{Z}$-module H of finite rank, and let $w \in H$ be a characteristic element. Then $\operatorname{sign}(Q) \equiv Q(w, w) \pmod 8$.

Remark 5.2.11. If Q has even parity, then $w = 0$ is a characteristic element. Hence by Lemma 5.2.10, we have $\operatorname{sign}(Q) \equiv 0 \pmod 8$. Compared to Rochlin's theorem 5.1.20,

this statement is weaker in that it only shows divisibility by 8. But it also applies to topological instead of only differentiable manifolds X.
For a positive definite form rank and signature coincide. Thus if Q is positive definite and of even parity, then the rank of Q must be divisible by 8.

Remark 5.2.12. To estimate the difficulty in classifying definite symmetric bilinear forms Q, we set

$$\mathcal{Q}_{8k} := \{\text{isomorphism classes of unimodular positive definite bilinear forms of even parity and rank } 8k\,\}\,.$$

From number theory, we have the ***Minkowski-Siegel mass formula***:

$$\sum_{Q \in \mathcal{Q}_{8k}} \frac{1}{\#\mathrm{Aut}(Q)} = 2^{1-8k} \cdot \frac{B_{2k}}{(4k)!} \cdot \prod_{j=1}^{4k-1} B_j\,,$$

where B_j is the j^{th} ***Bernoulli number*** defined by:

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} - \sum_{j=1}^{\infty} (-1)^j \cdot \frac{B_j}{(2j)!} \cdot x^{2j}\,.$$

Corollary 5.2.13
Since $\#\mathrm{Aut}(Q) \geq 1$ for any Q, we obtain from the Minkowski-Segal formula the following lower bound on the number of isomorphism types of rank $8k$ unimodular, positive definite symmetric bilinear forms:

$$\#\mathcal{Q}_{8k} \geq \underbrace{2^{1-8k} \cdot \frac{B_{2k}}{(4k)!} \cdot \prod_{j=1}^{4k-1} B_j}_{=:a_{8k}}\,.$$

Just to indicate the difficulty, we list the first few values of the lower bound:

k	$\#\mathcal{Q}_{8k}$	a_{8k}	classification
1	1	$10^{-9} \cdot 1,43\ldots$	E_8
2	2	$10^{-18} \cdot 2,48\ldots$	$E_8 \oplus E_8$, Γ_{16}
3	24	$10^{-15} \cdot 7,93\ldots$	Niemeyer, 1968
4	unknown	$10^{7} \cdot 4,03\ldots$	unknown
5	unknown	$10^{51} \cdot 4,39\ldots$	unknown

5.3 Donaldson's theorem

Theorem 5.3.1 (Donaldson, 1983)
Let X be a simply connected, compact, oriented differentiable 4-manifold with positive definite intersection form Q_X. Then, over $\mathbb{Z}$,

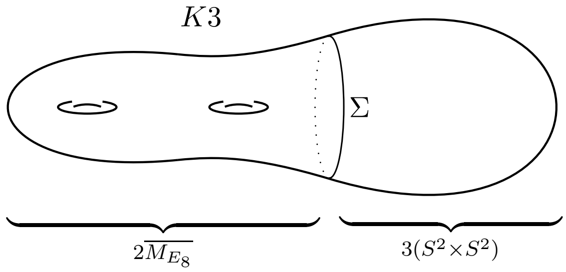
$$Q_X \cong \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Corollary 5.3.2
The topological 4-manifold X with $Q_X \cong E_8 \oplus E_8$ does not admit a differentiable structure.

Example 5.3.3 (Exotic $\mathbb{R}^4$). Using Corollary 5.3.2, one can construct an ***exotic differentiable structure*** on $\mathbb{R}^4$, i.e. a differentiable manifold M such that M is homeomorphic but not diffeomorphic to the standard $\mathbb{R}^4$. Such a manifold is called ***exotic*** $\mathbb{R}^4$ or ***fake*** $\mathbb{R}^4$.

We start with the Kummer surface $K3$ with $Q_{K3} = -2E_8 \oplus 3\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Now let M_{E_8} be the simply connected, compact, oriented topological 4-manifold with intersection form $Q_{M_{E_8}} \cong E_8$. By Freedman's theorem 5.2.3, $K3$ is homeomorphic to

$$\overline{M_{E_8}} \sharp \overline{M_{E_8}} \sharp \left(S^2 \times S^2\right) \sharp \left(S^2 \times S^2\right) \sharp \left(S^2 \times S^2\right) =: 2\overline{M_{E_8}} \sharp 3 \left(S^2 \times S^2\right) .$$

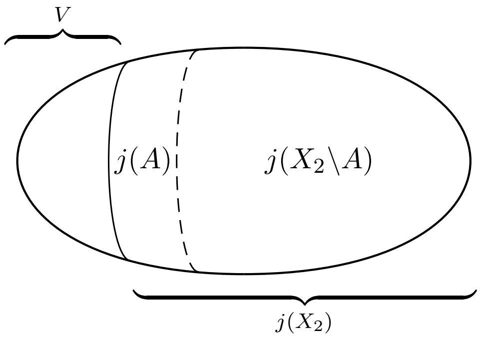


Hence there exists a topologically embedded 3-sphere $\Sigma \subset K3$, which cuts $K3$ into pieces X_1, X_2 with common boundary Σ and $X_1 \cong 2\bar{M}_{E_8} - B^4$, whereas $X_2 \cong 3(S^2 \times S^2) - B^4$. Now we take on X_2 the differentiable structure such that the embedding $X_2 \hookrightarrow K3$ is a smooth map. We need the following facts from the theory of Casson:

1. There exists a smooth embedding $j : X_2 \hookrightarrow 3(S^2 \times S^2)$. We let

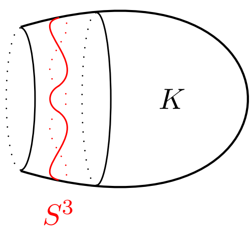
$$V := 3\big(S^2 \times S^2\big) \backslash j\big(X_2 \backslash A\big) ,$$

where $A := B' \backslash \bar{B}^4$ and $B' \supset B^4$ is some larger ball.

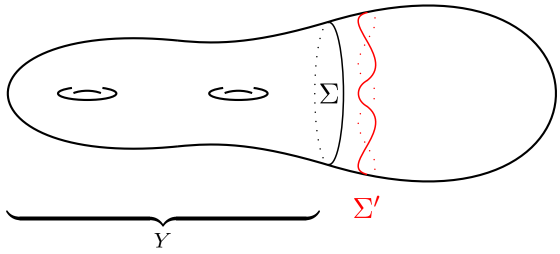


2. We then have by the Seifert-vanKampen theorem that $\pi_1(V) = \{0\}$ and by the Mayer-Vietoris sequence that $H_2(V;\mathbb{Z}) = \{0\}$. Further, V has exactly one topological end, homeomorphic to $(0,\infty) \times S^3$. The second fact we need is that this implies $V \approx B^4 \approx \mathbb{R}^4$.

We note that $K := 3(S^2 \times S^2) \backslash j(X_2)$ is a compact subset of V. Suppose there were a smoothly embeded 3-sphere $S \subset V$ surrounding K.



Then the smooth embedding $X_2 \hookrightarrow K3$ maps Σ' to a smoothly embedded 3-sphere in $K3$. Cutting $K3$ along Σ', we obtain a smooth 4-manifold Y with boundary Σ'. Now glueing a 4-ball $\bar{B}^4$ into Y along $\partial Y = \Sigma' \cong_{\text{diffeo}} S^3 = \partial \bar{B}^4$, we obtain a smooth 4-manifold Z without boundary.



By construction, Z is simply connected, compact and has intersection form $Q_Z \cong E_8 \oplus E_8$. This contradicts the Corollary 5.3.2 to the theorem of Donaldson 5.3.1. Hence there exists no smoothly embedded 3-sphere S surrounding the compact set $K \subset V \approx \mathbb{R}^4$. Since in $\mathbb{R}^4$ with the standard differentiable structure, any compact set is contained in a large ball and hence is surrounded by a smoothly embedded 3-sphere – namely the boundary of that ball – we conclude that the $V \approx \mathbb{R}^4$ cannot be diffeomorphic to the standard $\mathbb{R}^4$.

Sketch of proof of Donaldson's theorem.

a) Let X be a simply connected, compact, oriented differentiable 4-manifold with positive definte intersection form Q_X. Choose an SU(2)-principal bundle $P \to X$ such that for $c_2(P) \in H^4_{\mathrm{dR}}(X;\mathbb{R})$, we have $\int_X c_2(P) = -1$.[1] Denote by

$$\begin{aligned} \mathcal{A}(P) &:= \left\{\omega \in \mathcal{C}(P) \mid \bar{\Omega} \in \Omega^2_+(M; P \times_{\mathrm{Ad}} \mathfrak{su}(2)\right\} \\ &= \{\mathrm{SU}(2)\text{-instantons}\} \end{aligned}$$

the space of connections with self-dual curvature forms and by

$$\mathcal{M} := \mathcal{A}(P)/\mathcal{G}(P)$$

the ***moduli space*** of gauge equivalence classes of SU(2)-instantons. Analyzing the moduli space $\mathcal{M}$, one finds:

a) The moduli space $\mathcal{M}$ is a 5-dimensional manifold with finitely many singular points $p_1, \ldots, p_n \in \mathcal{M}$.

b) Any of the singular points p_i has a neighborhood U_{p_i} homeomorphic to a cone on $\mathbb{C}P^2$.

c) For every divergent sequence $[\omega_k]_{k\in\mathbb{N}} \in \mathcal{M}$, there is a subsequence such that the following holds: there exists a point $x \in M$ such that for any $r > 0$, we have

$$\int\limits_{X - B_r(x)} \left|\bar{\Omega}_k\right|^2 \mathrm{dvol} \overset{k\to\infty}{\longrightarrow} 0\,.$$

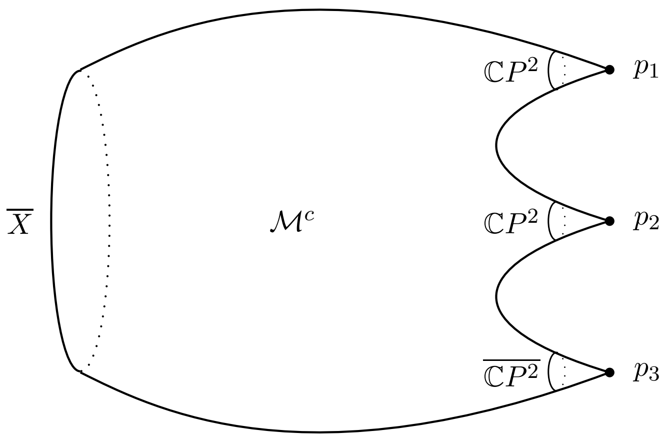
[1]This is possible, since SU(2)-principal bundles are classified up to isomorphism by their first Pontrjagin class p_1 and since $H^4_{\mathrm{dR}}(X;\mathbb{R}) \cong \mathbb{R}$. In particular, the bundle P is unique up to ismomorphism.

Note that since all the connections ω_k have self-dual curvatures $\bar{\Omega}_k$, we have:

$$\begin{aligned}
\int_X |\bar{\Omega}_k|^2 \mathrm{dvol} &= \int_X \left(|\bar{\Omega}_k^+|^2 + |\bar{\Omega}_k^-|^2\right) \mathrm{dvol} \\
&\overset{*\bar{\Omega}_k = \bar{\Omega}_k}{=} \int_X |\bar{\Omega}_k^+|^2 \mathrm{dvol} \\
\text{and} \qquad -8\pi^2 \int_X c_2(P) &= \int_X \left(|\bar{\Omega}_k^+|^2 - |\bar{\Omega}_k^-|^2\right) \mathrm{dvol} \\
&\overset{*\bar{\Omega}_k = \bar{\Omega}_k}{=} \int_X |\bar{\Omega}_k^+|^2 \mathrm{dvol} \\
&= \int_X |\bar{\Omega}_k|^2 \mathrm{dvol} \\
\text{hence} \qquad \int_X |\bar{\Omega}_k|^2 \mathrm{dvol} &= -8\pi^2 \int_X c_2(P) \\
&= 8\pi^2 \,.
\end{aligned}$$

Hence the curvatures $\bar{\Omega}_k$ concentrate at the point x.

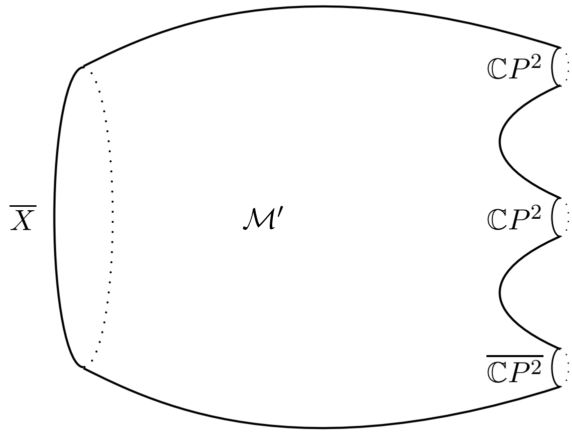
b) Now let $\mathcal{M}^c := \mathcal{M} \sqcup X$ with the topology such that a sequence $[\omega_k]_{k\in\mathbb{N}}$ as above converges to the concentration point x of their curvatures.



Cutting boundaries of the singular points $p_1, \dots, p_n$ from $\mathcal{M}^c$, we obtain a compact 5-manifold $\mathcal{M}'$ with boundary

$$\partial\mathcal{M}' = \overline{X} \sqcup \underbrace{\mathbb{C}P^2 \sqcup \ldots \sqcup \mathbb{C}P^2}_{n_+} \sqcup \underbrace{\overline{\mathbb{C}P^2} \sqcup \ldots \sqcup \overline{\mathbb{C}P^2}}_{n_-}$$

and of course $n_+ - n_- = n$ is the total number of singular points in $\mathcal{M}$.



For the signature of the boundary components, we obtain:

$$0 = \operatorname{sign}\left(\partial\mathcal{M}'\right) = \operatorname{sign}\left(\overline{X}\right) + n_+ \operatorname{sign}\left(\mathbb{C}P^2\right) - n_- \operatorname{sign}\left(\overline{\mathbb{C}P^2}\right),$$

hence $\operatorname{sign}(X) = n_+ - n_-$.

c) Next we note that the intersection form Q_X can be realized on the de Rham cohomology $H^2_{\mathrm{dR}}(X;\mathbb{Z})$ by:

$$Q_X\left([\alpha],[\beta]\right) := \int_X \alpha \wedge \beta\,.$$

Here the de Rham cohomology with integer coefficients is defined as: [2]

$$H^2_{\mathrm{dR}}(X;\mathbb{Z}) := \left\{ [\alpha] \in H^2_{\mathrm{dR}}(X;\mathbb{R}) \mid \forall\, c \in Z_{2,\mathrm{smooth}}(X;\mathbb{Z}) : \int_c \alpha \in \mathbb{Z} \right\}.$$

d) For any $\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{Z})$ with $Q_X(\alpha,\alpha) = 1$, choose a U(1)-principal bundle $L \to X$ with $c_1(L) = \alpha$. [3] Now for the total Chern class of $L \oplus L^*$, we obtain:

$$\begin{aligned}
\underbrace{1}_{\in H^0_{\mathrm{dR}}(X;\mathbb{R})} + \underbrace{c_1(L\oplus L^*)}_{\in H^2_{\mathrm{dR}}(X;\mathbb{R})} + \underbrace{c_2(L\oplus L^*)}_{\in H^4_{\mathrm{dR}}(X;\mathbb{R})} &= c(L\oplus L^*) \\
&= c(L)\cdot c(L^*) \\
&= \left(1 + c_1(L)\right)\cdot\left(1 - c_1(L)\right) \\
&= 1 - c_1(L)^2 \\
&= 1 + c_2(P)\,.
\end{aligned}$$

[2]The set of smooth singular cycles $Z_{2,\mathrm{smooth}}(X;\mathbb{Z})$ is the submodule of $Z_2(X;\mathbb{Z})$ spanned by *smooth* maps $\sigma : \Delta_2 \to X$. Here "smooth" means that σ is smooth in the interior of Δ_2 and all derivatives extend continuously to Δ_2. One can show that the embedding of the smooth singular chains resp. cycles as a subcomplex of the singular chain complex $(C_\bullet(X;\mathbb{Z}),\partial)$ induces an isomorphism on homology. Smoothness is needed here in order that the integral $\int_c \alpha$ makes sense.

[3]This is possible, since U(1)-principal bundles are classified up to isomorphism by their first Chern class c_1. In particular, the bundle L is unique up to isomorphism.

Hence any such $\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{Z})$ yields a splitting $P \cong L \oplus L^*$. Any self-dual connection on L induces a so called ***reducible*** self-dual connection on P. Such connections have more symmetries than generic self-dual connections and hence result in the singular points $p_1, \ldots, p_n$. This establishes a $1:1$-correspondence

$$\{\text{singular points in } \mathcal{M}\} \overset{1:1}{\longleftrightarrow} \left\{\text{pairs } \pm\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{Z}) \text{ with } Q_X(\alpha,\alpha) = 1\ \right\}.$$

d) Now any such $\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{Z})$ with $Q_X(\alpha,\alpha) = 1$ yields a decomposition

$$H^2_{\mathrm{dR}}(X;\mathbb{Z}) \cong \mathbb{Z}\alpha \oplus \alpha^\perp\,,\ \beta \mapsto Q_X(\beta,\alpha)\,\alpha \oplus \big(\beta - Q_X(\beta,\alpha)\alpha\big)\,.$$

Now let $n(Q)$ be the function counting the singular points in the moduli space $\mathcal{M}$ of a 4-manifold with intersection form Q, i.e.

$$n(Q) := \frac{1}{2}\#\left\{\alpha \in H^2_{\mathrm{dR}}(X;\mathbb{Z}) \mid Q(\alpha,\alpha) = 1\right\}.$$

Then since $\mathrm{rk}(Q) = \mathrm{rk}(Q|_{\alpha^\perp}) + 1$, we have $n(Q) = n(Q|_{\alpha^\perp}) + 1$. By induction over $m = \mathrm{rk}(Q)$, we find

$$n(Q) \leq \mathrm{rk}(Q)$$

with equality iff $Q \cong \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$.

Applying this inequality to the positive definite intersection form Q_X of the 4-manifold X, we obtain:

$$\mathrm{rk}(Q_X) = \mathrm{sign}(Q_X) = n_+ - n_- \leq n_+ + n_- = n(Q_X) \leq \mathrm{rk}(Q_X)\,.$$

Hence $n_- = 0$ and $\mathrm{rk}(Q_X) = n(Q_X)$ so that

$$Q_X \cong \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

as claimed. □

From the theorems of Freedman, Serre and Donaldson, we conclude that any simply connected, compact, orientable differentiable 4-manifold X is homeomorphic to either $m\,\mathbb{C}P^2 \sharp\, n\,\overline{\mathbb{C}P^2}$ or $\pm m\, M_{E_8} \sharp\, n\,(S^2 \times S^2)$ (for appropriate $m, n \in \mathbb{N}$). Note that not all topological manifolds in this list carry differentiable structures: e.g. $1M_{E_8} \sharp\, 0\,(S^2 \times S^2)$ does not, but $-2M_{E_8} \sharp\, 3\,(S^2 \times S^2) = K3$ does. It is still an open question, which manifolds of that list precisely carry differentiable structures.
Another open problem closely related to this question is the following:

Conjecture 5.3.4 ($\frac{11}{8}$-Conjecture)
If X is a simply connected, compact, oriented, differentiable 4-manifold with intersection form Q_X of even parity, then we have:

$$b_2(X) \geq \frac{11}{8} \cdot \operatorname{sign}(X)\,.$$

If the $\frac{11}{8}$-conjecture holds true, then for $X = mM_{E_8} \sharp n(S^2 \times S^2)$, we have $b_2(X) = 8m + 2n$ and $\operatorname{sign}(X) = 8m$, so that

$$8m + 2n \geq \frac{11}{8} \cdot 8m = 11m \;\Leftrightarrow\; 2n \geq 3m\,.$$

Now for $m = -2k$, we obtain $n \geq -3k$, so that for $l = n - 3k$, we find:

$$\underbrace{M_{E_8}\sharp \ldots \sharp M_{E_8}}_{-2k} \sharp \underbrace{S^2 \times S^2 \sharp \ldots \sharp S^2 \times S^2}_{3k} \sharp \underbrace{S^2 \times S^2 \sharp \ldots \sharp S^2 \times S^2}_{l=n-3k} = kK3 \sharp l(S^2 \times S^2).$$

A partial result to the above conjecture has been obtained by Furuta using ***Seiberg-Witten theory***, which is a U(1) gauge theory (with U(1) gauge fields coupled to $\mathrm{Spin}^{\mathbb{C}}$ spinor fields):

Theorem 5.3.5 (Furuta)
If X is a simply connected, compact, oriented, differentiable 4-manifold with intersection form Q_X of even parity, then we have:

$$b_2(X) \geq \frac{10}{8} \cdot \operatorname{sign}(X) + 2\,.$$

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