UNIVERSITÄT POTSDAM

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Homology and Cohomology

A First Contact

Version from December 12, 2006

Preface

These are the notes of a series of three lectures that I gave at the Max-Planck Institute for Graviational Physics (Einstein Institute) in October 2006. The aim was to explain the basic ideas and applications of homology and cohomology theories for manifolds. The lectures were aimed at an audience with no prior knowlegde of algebraic topology but it is assumed that the reader knows about manifolds and differential forms up to the Stokes' theorem.

Given the very short amount of time I had to be very selective and could not give complete proofs for any of the deeper results. The idea was more to give the students a first working knowledge of (co-) homology. To encourage active participation some exercises have been included in the lectures.

I am grateful to the participants of this course for lively discussion. Special thanks go to Florian Hanisch who wrote a first draft of these notes and produced most illustrations.

Potsdam, November 2006,

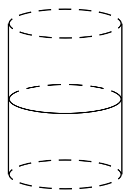
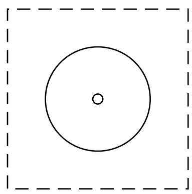
Christian Bär

Contents

1 Introduction

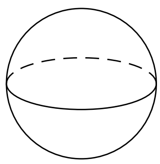
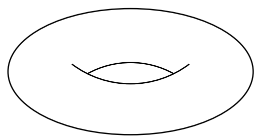
Topology, nowadays a huge field, is the science of topological spaces and continuous maps. We will restrict ourselves to manifolds and smooth maps even though much of what we will do would, when suitably modified, work as well in a larger context. We will learn about two theories, de Rham cohomology and simplicial homology. The definition of de Rham cohomology is based on analysis, more precisely, on partial differential equations and differential forms. Simplicial homology on the other hand is combinatorial in nature and based on decomposing the space into simple pieces, so-called simplices. It will turn out that both theories are dual to each other which is quite remarkable. This says in particular that non-integrability of certain partial differential equations has its only cause in the combinatorial complexity of the underlying space.
In topology one tries to answer the following type of questions:

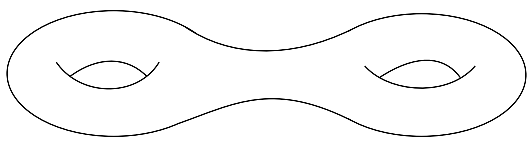
Question 1. Are the manifolds M_1 and M_2 diffeomorphic?

$M_1 = S^1 \times \mathbb{R} =$  $M_2 = \mathbb{R}^2 \setminus \{0\} =$ 

Answer. Yes, here is a diffeomorphism: $M_1 \to M_2$, $(\theta, t) \mapsto e^t \cdot \theta$.

Question 2. Are the manifolds M_1, M_2 and M_3 diffeomorphic?

$M_1 = S^2 =$  $M_2 = T^2 =$ 

$M_3 = F_2 :=$ 

Answer. No, they are pairwise non-diffeomorphic. However, at the moment we are unable to prove this. The fact that we do not find a diffeomorphism does not mean there is none.

Heuristically, the reason for M_1, M_2, and M_3 being pairwise non-diffeomorphic is that they have a different number of "holes". This argument needs to be made precise.

Question 3. Are $\mathbb{R}^2$ and $\mathbb{R}^3$ diffeomorphic?

Answer. No, because they have different dimension.
That was easy, wasn't it? But what about

Question 4. Why must diffeomorphic manifolds have equal dimension?

Indeed, this needs justification.
The aim will be to find invariants that may distinguish non-diffeomorphic manifolds. In particular, we will show that the dimension is such an invariant.

2 De Rham Cohomology

2.1 Definitions

Question 5. Does the PDE

$$\begin{cases} \frac{\partial f}{\partial x} = y^2, \\ \frac{\partial f}{\partial y} = x, \end{cases}$$

have a solution $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$?

Answer. No, because if it did have a solution f, then its differential would be

$$\omega := df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = y^2 dx + x dy$$

and hence

$$0 = ddf = d\omega = 2y dy \wedge dx + dx \wedge dy = (1 - 2y) dx \wedge dy \neq 0,$$

a contradiction.

Question 6. Does $\frac{\partial f}{\partial x} = y, \ \frac{\partial f}{\partial y} = x$ have a solution?

Answer. Again, we compute the differential of a possible solution f,

$$\tilde{\omega} := df = y dx + x dy.$$

This time $d\tilde{\omega} = dy \wedge dx + dx \wedge dy = 0$, so no contradiction arises. Indeed, $f(x, y) = xy$ is a solution.

Question 7. Does $\frac{\partial f}{\partial x} = -\frac{y}{x^2+y^2}$, $\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$ have a solution $f \in C^\infty(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$?

Answer. For the differential of a hypothetical solution f we have

$$\hat{\omega} := df = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

and hence $d\hat{\omega} = 0$. So there is no immediate contradiction. Nevertheless, there is no solution.

Exercise 1. Why not?

These examples show that in order to have a solution f to the PDE $df = \omega$ for given ω the integrability condition $d\omega = 0$ must hold. In general, this integrability condition is not sufficient however. The following lemma says that for open balls the integrability condition is the only obstruction to find solutions.

Lemma 1 (Poincaré Lemma)**.** *If M is diffeomorphic to an open ball, then for $\omega \in \Omega^k(M)$, $k \geq 1$:*

$$d\omega = 0 \qquad \Leftrightarrow \qquad \exists \eta \in \Omega^{k-1}(M) : d\eta = \omega$$

Now let M be an n-dimensional manifold and consider the sequence of exterior derivatives on differential forms

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0$$

We have $d \circ d = 0$. Put

$$Z^k(M) := \ker(d : \Omega^k(M) \to \Omega^{k+1}(M)) \quad \text{(closed } k\text{-forms)}$$
$$B^k(M) := \operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M)) \quad \text{(exact } k\text{-forms)}$$

Both $Z^k(M)$ and $B^k(M)$ are linear subspaces of $\Omega^k(M)$. Now $d^2 = 0$ implies (and is indeed equivalent to) $B^k(M) \subset Z^k(M)$. Thus we may define

Definition 2. The quotient space

$$H^k_{\mathrm{dR}}(M) := \frac{Z^k(M)}{B^k(M)}$$

is called the k^{th} *de Rham cohomology* of M.

Remark 3. If M is diffeomorphic to an open ball, then the Poincaré lemma says $Z^k(M) = B^k(M)$ for all $k \geq 1$, i. e. $H^k_{\mathrm{dR}}(M) = 0$ for all $k \geq 1$.

Remark 4. $H^1_{\mathrm{dR}}(M)$ measures the extent to which the integrability condition $d\omega = 0$ of the PDE $df = \omega$ fails to be sufficient for its solvability.

Definition 5. $b^k(M) := \dim H^k_{\mathrm{dR}}(M) \in \{0, 1, 2, \ldots, \infty\}$ is called the k^{th} *Betti number* of M.

If M is an n-dimensional manifold, $n \geq 1$, then the vector spaces $\Omega^k(M)$, $Z^k(M)$, and $B^k(M)$ are always infinite dimensional for $0 < k \leq n$. The quotient space $H^k_{\mathrm{dR}}(M)$ is sometimes finite dimensional, sometimes infinite dimensional. It will turn out that for compact M the Betti numbers are always finite.
In the simplest case, $k = 0$, de Rham cohomology is easily understood. Clearly we have

$$Z^0(M) = \left\{ f \in \Omega^0(M) \middle| \, df = 0 \right\} = \{\text{locally constant functions on } M\}$$

and

$$B^0(M) = 0.$$

Therefore

$$H^0_{\mathrm{dR}}(M) = Z^0(M) = \{\text{locally constant functions on } M\}$$

and hence

$$b^0(M) = \# \text{ connected components of } M.$$

So far we have associated to each manifold certain vector spaces, its de Rham cohomologies. Now we also associate something to smooth maps between manifolds. Let $f : M \to N$ be a smooth map. There is a linear map $f^* : \Omega^k(N) \to \Omega^k(M)$, called *pull-back*. Locally, it is given by

$$f^*\Big(\sum_{i_1 \cdots i_k} \omega_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}\Big) = \sum_{i_1 \cdots i_k} (\omega_{i_1 \cdots i_k} \circ f) \, df^{i_1} \wedge \cdots \wedge df^{i_k}$$

where $f^i = x^i \circ f$ denotes the i^{th} component of f with respect to local coordinates $x^1, \ldots, x^n$ on N. A somewhat tedious computation yields

Lemma 6. *The following diagram commutes:*

$$\begin{tikzcd}
\Omega^k(N) \arrow[r, "f^*"] \arrow[d, "d"] & \Omega^k(M) \arrow[d, "d"] \\
\Omega^{k+1}(N) \arrow[r, "f^*"] & \Omega^{k+1}(M)
\end{tikzcd}$$

Corollary 7. *We have* $f^*(Z^k(N)) \subset Z^k(M)$ *and* $f^*(B^k(N)) \subset B^k(M)$ *and we thus get a linear map*

$$\begin{aligned} f^* : H^k_{\mathrm{dR}}(N) &\to H^k_{\mathrm{dR}}(M), \\ [\omega] &\mapsto [f^*\omega]. \end{aligned}$$

Here and in the following $[\omega]$ denotes the cohomology class of the closed form ω. We have associated to any smooth map between two manifolds linear maps between their cohomologies in all degrees. It should be emphasized that the direction "gets reversed". While f maps M to N the corresponding linear maps f^* map $H^k_{\mathrm{dR}}(N)$ to $H^k_{\mathrm{dR}}(M)$. This is why de Rham cohomology is called cohomology rather than homology.

Lemma 8 (Functoriality Properties)**.**

(1) $M \xrightarrow{f} N \xrightarrow{g} P \quad \Rightarrow \quad (g \circ f)^* = f^* \circ g^*$

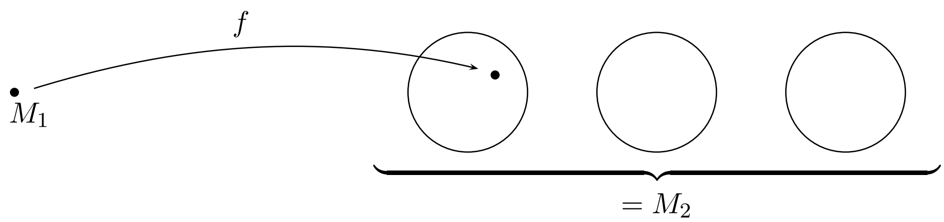
(2) $(\mathrm{id}_M)^* = \mathrm{id}_{H^k_{\mathrm{dR}}(M)}$

The proof of this lemma is not too hard. While (2) is trivial (1) is a consequence of the chain rule for the differential of the composition of two maps.

Example 9. Let us determine the induced map in a simple example. Let

$$\begin{aligned} M_1 &= \{pt\} \quad \text{and} \\ M_2 &= D^2_1 \sqcup D^2_2 \sqcup D^2_3 \end{aligned}$$

where the D^2_j are disjoint 2-dimensional open disks. The map f sends M_1 to a point in the first component D^2_1, say.



Denote by $\omega_j : M_2 \to \mathbb{R}$ the function, which takes the value 1 on D^2_j and 0 otherwise. Then $(\omega_1, \omega_2, \omega_3)$ is a basis for $Z^0(M_2) = H^0_{\mathrm{dR}}(M_2)$. For the linear map

$$f^* : \underbrace{H^0_{\mathrm{dR}}(M_2)}_{\cong \mathbb{R}^3} \to \underbrace{H^0_{\mathrm{dR}}(M_1)}_{\cong \mathbb{R}^1}$$

we have $f^*\omega_1 = \omega_1 \circ f = 1$ and $f^*\omega_2 = f^*\omega_3 = 0$. Therefore f^* is given by the matrix $(1, 0, 0)$. Observe that the precise point to which f sends M_1 does not matter; it is only important that M_1 is mapped to the first component. If f maps M_1 to the second component D_2^2, then the induced map on 0^{th} de Rham cohomology is given by the matrix $(0, 1, 0)$, similarly for the third component.

Remark 10. If M is the disjoint union of l manifolds, $M = M_1 \sqcup \cdots \sqcup M_l$, then we get an isomorphism

$$\begin{gathered} H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(M_1) \oplus \cdots \oplus H^k_{\mathrm{dR}}(M_l), \\ [\omega] \mapsto [\omega|_{M_1}] \oplus \cdots \oplus [\omega|_{M_l}]. \end{gathered}$$

Definition 11. Smooth maps $f, g : M \to N$ are called *homotopic* ($f \simeq g$), if there exists a smooth map $F : M \times [0, 1] \to N$ such that

$$\begin{aligned} f(x) &= F(x, 0) \\ g(x) &= F(x, 1) \end{aligned}$$

for all $x \in M$.

This means that the map f can be smoothly deformed into the map g.

Example 12. Let $M = N = D^n$ be n-dimensional open balls, let $f = \mathrm{id}_M$ and let g be the constant map $g = 0$. Then $f \simeq g$ because $F(x, t) := (1 - t)\, x$ defines a homotopy. Clearly, F is smooth in x and t and for $t \in [0, 1]$ and $x \in D^n$ we have $F(x, t) \in D^n$.

Lemma 13. *If $f \simeq g : M \to N$, then $f^* = g^* : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$ for all k.*

Example 9 is a good illustration for this homotopy invariance of the maps induced in cohomology.

Definition 14. Two manifolds M and N are called *homotopy equivalent* ($M \simeq N$) if there exist smooth maps $f : M \to N$ and $g : N \to M$ such that $g \circ f \simeq \mathrm{id}_M$ and $f \circ g \simeq \mathrm{id}_N$.

Clearly, if M and N are diffeomorphic ($M \approx N$), then they are also homotopy equivalent. The converse is not true:

Example 15. $D^n \simeq \{pt\}$ because we have the maps

$$\begin{aligned} f : \{pt\} &\to D^n & \qquad g : D^n &\to \{pt\} \\ pt &\mapsto 0 & x &\mapsto pt \end{aligned}$$

satisfying $g \circ f = \mathrm{id}_{\{pt\}}$ and $f \circ g = 0 \simeq \mathrm{id}_{D^n}$ by Example 12. This example shows drastically that the dimension of a manifold is not a homotopy invariant, i. e. homotopy invariant manifolds may have different dimensions. In order to distinguish homotopy inequivalent spaces we need other invariants. Here they are:

Corollary 16. *If $M \simeq N$, then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$ and therefore $b^k(M) = b^k(N)$ for all k.*

Proof. Choose $f : M \to N$ and $g : N \to M$ such that $g \circ f \simeq \mathrm{id}_M$ and $f \circ g \simeq \mathrm{id}_N$. Then by functoriality $g^* \circ f^* = (f \circ g)^* = (\mathrm{id}_N)^* = \mathrm{id}_{H^k_{\mathrm{dR}}(N)}$ and $f^* \circ g^* = \cdots = \mathrm{id}_{H^k_{\mathrm{dR}}(M)}$. Therefore f^* is an isomorphism with $(f^*)^{-1} = g^*$. □

Example 17. Observe that

$$D^n \simeq pt \quad \Rightarrow \quad H^k_{\mathrm{dR}}(D^n) \cong H^k_{\mathrm{dR}}(pt) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

The fact that $H^k_{\mathrm{dR}}(D^n) = 0$ for $k \geq 1$ is precisely the statement of the Poincaré lemma.

2.2 Mayer-Vietoris Sequence

Next we develop a tool to compute the de Rham cohomology of more complicated spaces by decomposing them into simpler parts. We start with some algebraic remarks. A sequence of vector spaces and linear maps

$$0 \longrightarrow V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{l-1}} V_l \longrightarrow 0 \tag{1}$$

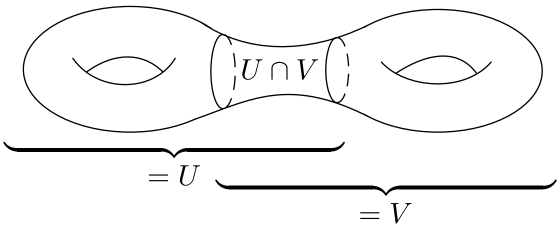
is called *exact* if $\ker(\varphi_{k+1}) = \mathrm{im}(\varphi_k)$ for all k. In particular, φ_1 is injective and φ_{l-1} is surjective. In this case, we have for finite dimensional spaces V_j

$$\sum_{j=1}^{l} (-1)^j \dim V_j = 0. \tag{2}$$

This can be proven by induction on l. In case $l = 1$ we have $0 \to V_1 \to 0$ hence $V_1 = 0$. For $l = 2$ we have $0 \to V_1 \xrightarrow{\varphi} V_2 \to 0$ so that φ must be injective and surjective. Thus $V_1 \cong V_2$ and therefore $\dim V_1 = \dim V_2$.

If we write (2) in the form $\sum_{j \text{ even}} \dim V_j = \sum_{j \text{ odd}} \dim V_j$, then it makes sense and is true also for possibly infinite dimensional spaces. In particular, if all but one of the vector spaces in the exact sequence (1) are finite dimensional, then they must all be finite dimensional.

Back to topology let M be a manifold and U, V open subsets of M such that $U \cup V = M$.



The inclusion maps

$$
\begin{aligned}
j_U &: U \hookrightarrow M & \qquad\qquad i_U &: U \cap V \hookrightarrow U \\
j_V &: V \hookrightarrow M & \qquad\qquad i_V &: U \cap V \hookrightarrow V
\end{aligned}
$$

induce maps on differential forms. The pull-back of a differential form along an inclusion map is of course nothing but the restriction of the differential form.

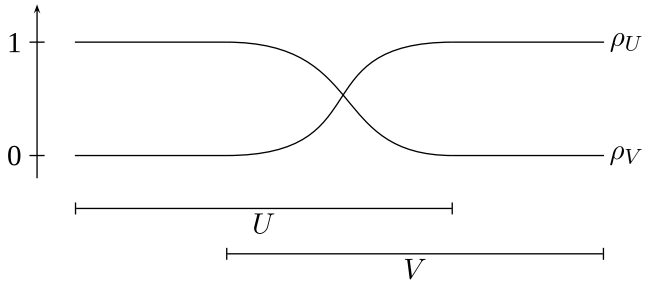
Lemma 18. *The sequence*

$$
0 \longrightarrow \Omega^k(M) \xrightarrow{j_U^* \oplus j_V^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i_U^* - i_V^*} \Omega^k(U \cap V) \longrightarrow 0
$$

is exact for every k.

Proof. Since $U \cup V = M$, $j_U^* \oplus j_V^*$ is injective because any differential form on M is determined by its restrictions to U and V. This shows exactness at $\Omega^k(M)$.
Furthermore, it is obvious from the definition that $\mathrm{im}(j_U^* \oplus j_V^*) = \ker(i_U^* - i_V^*)$. Namely, two forms on U and V respectively are restrictions of a form on M if and only if they conincide on the intersection $U \cap V$. This proves exactness at $\Omega^k(U) \oplus \Omega^k(V)$.
The only non-trivial part of the proof is to show exactness at $\Omega^k(U \cap V)$. We have to show that every form on $U \cap V$ can be written as the difference of a form on U and one on V. Choose a partition of unity subordinated to U, V, that is, two smooth functions $\rho_U, \rho_V : M \to [0,1]$ such that

$$
\begin{aligned}
\rho_U|_{U \setminus V} &= 0 & \qquad\qquad \rho_U|_{V \setminus U} &= 1 \\
\rho_V|_{U \setminus V} &= 1 & \qquad\qquad \rho_V|_{V \setminus U} &= 0 \\
\rho_U + \rho_V &= 1 \text{ on all of } M
\end{aligned}
$$



Given $\omega \in \Omega^k(U \cap V)$ the form $\rho_U \cdot \omega$ can be extended smoothly by 0 to all of U. This way we obtain $\omega' \in \Omega^k(U)$ with $i_U^*\omega' = \rho_U \cdot \omega$. Similarly, we get $\omega'' \in \Omega^k(V)$ with $i_V^*\omega'' = \rho_V \cdot \omega$. Then we have on $U \cap V$

$$\omega = (\rho_U + \rho_V)\omega = i_U^*\omega' - i_V^*(-\omega'')$$

showing that $i_U^* - i_V^*$ is onto. □

This lemma has an important consequence.

Theorem 19 (Mayer-Vietoris)**.** *Let M be an n-dimensional manifold. Then there is an exact sequence of de Rham cohomologies:*

$$0 \longrightarrow H^0_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^0_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^0} H^1_{\mathrm{dR}}(M) \longrightarrow \cdots$$

$$\cdots \longrightarrow H^{n-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^{n-1}} H^n_{\mathrm{dR}}(M) \xrightarrow{j_U^* \oplus j_V^*} H^n_{\mathrm{dR}}(U) \oplus H^n_{\mathrm{dR}}(V) \xrightarrow{i_U^* - i_V^*} H^n_{\mathrm{dR}}(U \cap V) \longrightarrow 0$$

Here the *connecting homomorphism* $\delta^k : H^k_{\mathrm{dR}}(U \cap V) \to H^{k+1}_{\mathrm{dR}}(M)$ is defined by the following procedure using the exact sequence in Lemma 18:

(1) For a class in $H^k_{\mathrm{dR}}(U \cap V)$ choose a representative $\omega \in Z^k(U \cap V)$, i. e., the class is given by $[\omega]$.

(2) Choose a preimage $\varphi = (\varphi_1, \varphi_2) \in \Omega^k(U) \oplus \Omega^k(V)$ of ω, i. e., $\omega = i_U^*\varphi_1 - i_V^*\varphi_2$.

(3) Apply d, $d\varphi = (d\varphi_1, d\varphi_2) \in \Omega^{k+1}(U) \oplus \Omega^{k+1}(V)$.

(4) Choose a preimage $\eta \in \Omega^{k+1}(M)$ of $d\varphi$, i. e., $d\varphi_1 = j_U^*\eta$ and $d\varphi_2 = j_V^*\eta$.

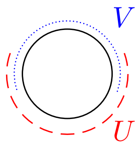
(5) Then η turns out to be closed, $\eta \in Z^{k+1}(M)$, and we take its cohomology class in $H_{\mathrm{dR}}^{k+1}(M)$,
$$\delta^k([\omega]) := [\eta]$$

Of course, one has to check that this definition is meaningful. More precisely, one has to show that all choices can be made (e. g. in step (4) one must show that $d\varphi$ lies in the image of the map $j_U^* \oplus j_V^*$) and one must make sure that the resulting cohomology class $[\eta]$ in the end is independent of the choices. This being done one has to prove exactness of the Mayer-Vietoris sequence. All this is entirely algebraic and straightforward and uses only the fact that de Rham cohomology is defined using the exterior differential on forms and Lemmas 6 and 18. This method of proof is also known as *abstract nonsense*.

Example 20. We use the Mayer-Vietoris sequence to determine the Betti numbers of the spheres.

(1) Since $S^0 = \{-1, 1\}$ is 0-dimensional and has two connected components we have
$$b^k(S^0) = \begin{cases} 2 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

(2) To treat S^1 we cover it by two subsets U and V both being diffeomorphic to an open interval D^1 such that their intersection consists of two connected components both being diffeomorphic to open intervals.

$S^1 =$ 

$\qquad U \approx V \approx D^1 \qquad U \cap V \approx D^1 \sqcup D^1$

Thus $b^0(S^1) = b^0(U) = b^0(V) = 1$ and $b^0(U \cap V) = 2$. Moreover, $b^1(U) = b^1(V) = b^1(D^1) = 0$. From Remark 10 we conclude $b^1(U \cap V) = b^1(D^1 \sqcup D^1) = b^1(D^1) + b^1(D^1) = 0$. The only Betti number that needs to be computed is $b^1(S^1)$. This is done using the Mayer-Vietoris-sequence,

$$\begin{tikzcd}
 & \mathbb{R} & \mathbb{R}\oplus\mathbb{R} & \mathbb{R}\oplus\mathbb{R} & \\
0 \arrow[r] & H^0_{\mathrm{dR}}(S^1) \arrow[r] \arrow[u, phantom, "\cong" rotate] & H^0_{\mathrm{dR}}(U)\oplus H^0_{\mathrm{dR}}(V) \arrow[r] \arrow[u, phantom, "\cong"] & H^0_{\mathrm{dR}}(U\cap V) \arrow[u, phantom, "\cong"] \arrow[dll, rounded corners] & \\
 & H^1_{\mathrm{dR}}(S^1) \arrow[r] & H^1_{\mathrm{dR}}(U)\oplus H^1_{\mathrm{dR}}(V) \arrow[r] \arrow[d, equal] & H^1_{\mathrm{dR}}(U\cap V) \arrow[r] \arrow[d, equal] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since the alternating some of the dimensions of the vector spaces in an exact sequence is 0 we have

$$\begin{aligned}
0 &= b^0(S^1) - (b^0(U) + b^0(V)) + b^0(U\cap V) \\
&\quad -b^1(S^1) + (b^1(U) + b^1(V)) - b^1(U\cap V) \\
&= 1 - (1+1) + 2 - b^1(S^1) + (0+0) - 0 \\
&= 1 - b^1(S^1).
\end{aligned}$$

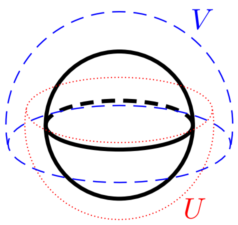
Therefore $b^1(S^1) = 1$ and we have

$$b^k(S^1) = \begin{cases} 1 & \text{if } k = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

(3) For the Betti numbers of S^n with $n \geq 2$ we will get

$$b^k(S^n) = \begin{cases} 1 & \text{if } k = 0,n \\ 0 & \text{otherwise} \end{cases}$$

We show this by induction on n. Cover S^n by two n-disks



$$\begin{aligned}
S^n &= U \cup V \\
U &\approx V \approx D^n \simeq pt \\
U \cap V &\approx S^{n-1} \times (-\epsilon, \epsilon) \simeq S^{n-1}
\end{aligned}$$

For $k \geq 2$ the following piece of the Mayer-Vietoris sequence

$$\begin{tikzcd}
H^{k-1}_{\mathrm{dR}}(U)\oplus H^{k-1}_{\mathrm{dR}}(V) \arrow[r] \arrow[d, equal] & H^{k-1}_{\mathrm{dR}}(U\cap V) \arrow[r] \arrow[d, phantom, "\cong"] & H^k_{\mathrm{dR}}(S^n) \arrow[r] & H^k_{\mathrm{dR}}(U)\oplus H^k_{\mathrm{dR}}(V) \arrow[d, equal] \\
0 & H^{k-1}_{\mathrm{dR}}(S^{n-1}) & & 0
\end{tikzcd}$$

yields an isomorphism $H^{k-1}(S^{n-1}) \cong H^k(S^n)$ and therefore computes all $b^k(S^n)$ inductively for $k \geq 2$. Since S^n is connected for $n \geq 1$ we have $b^0(S^n) = 1$. Finally, to determine $b^1(S^n)$ for $n \geq 2$ we look at the initial part of the Mayer Vietoris sequence

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{R} \oplus \mathbb{R} \arrow[d, phantom, "\cong" rotate=-90] & \mathbb{R} \arrow[d, phantom, "\cong" rotate=-90] \\
0 \arrow[r] & H^0_{\mathrm{dR}}(S^{n-1}) \arrow[r] & H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \arrow[r] & H^0_{\mathrm{dR}}(U \cap V) \arrow[dll, rounded corners] \\
 & H^1_{\mathrm{dR}}(S^n) \arrow[r] & H^1_{\mathrm{dR}}(U) \oplus H^1_{\mathrm{dR}}(V) \arrow[d, phantom, "=" rotate=-90] & \\
 & & 0 &
\end{tikzcd}$$

From the alternating sum formula for the dimensions in exact sequences we conclude $b^1(S^n) = 0$.

These computations show that the Betti numbers of spheres of different dimensions are different. Hence we have

Corollary 21. *The following statements are equivalent:*

(1) $S^n \simeq S^m$

(2) $S^n \approx S^m$

(3) $n = m$

This reasoning cannot work for $\mathbb{R}^n$ instead of S^n because $\mathbb{R}^n \simeq \mathbb{R}^m \simeq \{pt\}$ for any n and m. But

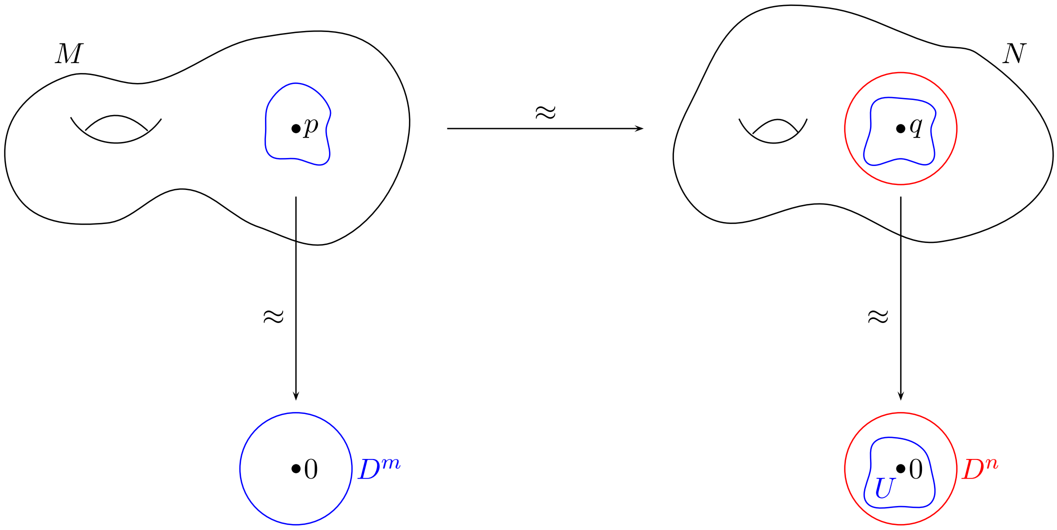
$$\begin{aligned}
\mathbb{R}^n \approx \mathbb{R}^m \quad &\Rightarrow \quad S^{n-1} \simeq \mathbb{R}^n \setminus \{0\} \approx \mathbb{R}^m \setminus \{0\} \simeq S^{m-1} \\
&\Rightarrow \quad S^{n-1} \simeq S^{m-1} \\
&\Rightarrow \quad n-1 = m-1
\end{aligned}$$

So Euclidean spaces are always homotopy equivalent but they are diffeomorphic only if they have equal dimension. Now we are ready to show that in general diffeomorphic manifolds must have the same dimension.

Corollary 22. *If M and N are diffeomorphic, then* $\dim(M) = \dim(N)$.

Proof. Write $\dim(M) = m$, $\dim(N) = n$ and suppose $M \approx N$. We fix a point $q \in N$ and choose a chart about q, i. e., an open neighborhood of q diffeomorphic to

D^n. We may and will assume that q maps to the origin $0 \in D^n$. Let $p \in M$ be the preimage of q under the diffeomorphism from M to N. We choose a neighborhood of p in M diffeomorphic to D^m. Without loss of generality we assume that this neighborhood is so small that it maps into the chart about q. This yields an open subset $U \subset D^n$ containing the origin such that $U \approx D^m$.



Denoting $\dot{D}^n := D^n \setminus \{0\}$, we have

$$
\begin{aligned}
D^n &= U \cup \dot{D}^n \\
\dot{D}^n &= D^n \setminus \{0\} \simeq S^{n-1} \\
U \cap \dot{D}^n &= U \setminus \{0\} \approx D^m \setminus \{0\} \simeq S^{m-1}
\end{aligned}
$$

For $k \geq 1$ the Mayer-Vietoris sequence

$$
\begin{array}{ccccccc}
H^k_{\mathrm{dR}}(D^n) & \longrightarrow & H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(\dot{D}^n) & \longrightarrow & H^k_{\mathrm{dR}}(U \cap \dot{D}^n) & \longrightarrow & H^{k+1}_{\mathrm{dR}}(D^n) \\
\| & & \| \qquad \wr\| & & \wr\| & & \| \\
0 & & 0 \qquad H^k_{\mathrm{dR}}(S^{n-1}) & & H^k_{\mathrm{dR}}(S^{m-1}) & & 0
\end{array}
$$

yields $H^k(S^{m-1}) \cong H^k(S^{n-1})$, hence $m - 1 = n - 1$. $\square$

Remark 23. Corollary 22 can be shown more directly by analytic methods. If there is a diffeomorphism $f : M \to N$, then its differential df at a point $p \in M$ maps

the tangent space T_pM isomorphically onto the tangent space $T_{f(p)}N$. Therefore the tangent spaces have equal dimension and so do the manifolds.
The argument given here does not use the definition of de Rham cohomology but only certain properties such as the Mayer-Vietoris sequence. It therefore works also with many other cohomology theories having the same properties. In particular, it can be used to prove that homeomorphic topological manifolds have the same dimension. In this context the analytic methods would not be available.

Exercise 2. Let M be an n-dimensional manifold, $n \geq 2$, and let $\dot{M} := M \setminus \{pt\}$. Show that

$$b^k(M) = b^k(\dot{M}) \text{ for } k \neq n, n-1$$

provided all Betti numbers are finite. Moreover, show that either

$$b^n(\dot{M}) = b^n(M) - 1 \text{ and } b^{n-1}(M) = b^{n-1}(\dot{M}) \text{ or}$$
$$b^{n-1}(\dot{M}) = b^{n-1}(M) + 1 \text{ and } b^n(M) = b^n(\dot{M}).$$

Hint: Apply the Mayer-Vietoris sequence to $M = \dot{M} \cup D^n$.

Exercise 3. Show by example that both cases occur.

2.3 Poincaré Duality & Künneth Formula

The pairing

$$\begin{aligned} \Omega^k(M) \times \Omega^l(M) &\to \Omega^{k+l}(M), \\ (\omega, \eta) &\mapsto \omega \wedge \eta, \end{aligned}$$

yields a bilinear map on cohomology,

$$\begin{aligned} H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(M) &\to H^{k+l}_{\mathrm{dR}}(M), \\ ([\omega], [\eta]) &\mapsto [\omega \wedge \eta]. \end{aligned}$$

First, one needs to check that if ω and η are closed, so is $\omega \wedge \eta$. This follows from

$$d(\omega \wedge \eta) = \underbrace{d\omega}_{=0} \wedge \eta + (-1)^k \omega \wedge \underbrace{d\eta}_{=0} = 0.$$

Then one checks that altering the closed forms by exact ones alters the wedge procduct also by an exact form. This is a consequence of

$$\begin{aligned} (\omega + d\varphi) \wedge (\eta + d\psi) &= \omega \wedge \eta + \omega \wedge d\psi + d\varphi \wedge \eta + d\varphi \wedge d\psi \\ &= \omega \wedge \eta + (-1)^k d(\omega \wedge \psi) + d(\varphi \wedge \eta) + d(\varphi \wedge d\psi) \\ &= \omega \wedge \eta + d((-1)^k \omega \wedge \psi + \varphi \wedge \eta + \varphi \wedge d\psi). \end{aligned}$$

This bilinear map is super-commutative, i. e., $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.
If M is compact and oriented, $n = \dim M$, then

$$\begin{aligned} H^n_{\mathrm{dR}}(M) &\to \mathbb{R}, \\ ([\omega]) &\mapsto \int_M \omega, \end{aligned}$$

is a well-defined linear map because by Stokes' theorem we have

$$\int_M (\omega + d\varphi) = \int_M \omega + \int_{\partial M = \emptyset} \varphi = \int_M \omega$$

Theorem 24 (Poincaré Duality)**.** *If M is a compact and oriented manifold, then the bilinear map*

$$\begin{aligned} H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) &\to \mathbb{R} \\ ([\omega], [\eta]) &\mapsto \int_M \omega \wedge \eta \end{aligned}$$

is non-degenerate.

Corollary 25. *For such M we have $b^k(M) = b^{n-k}(M)$.*

Example 26. $b^n(S^n) = b^0(S^n) = 1$. We see that the spheres have the smallest possible Betti numbers that a compact and orientable manifold can have.

Example 27. $b^n(\mathbb{R}^n) = 0 \neq 1 = b^0(\mathbb{R}^n)$, but this is not a contradiction since $\mathbb{R}^n$ is not compact.

Definition 28. For a compact manifold M the number $\chi(M) := \sum_{k=0}^n (-1)^k b^k(M)$ is called the *Euler characteristic* of M.

Example 29.

$$\chi(S^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Corollary 30. *If M is compact, orientable, and odd-dimensional, then $\chi(M) = 0$.*

Proof. Since $b^0(M) - b^n(M) = b^1(M) - b^{n-1}(M) = \cdots = 0$ by Poincaré duality we have

$$\chi(M) = b^0(M) - b^1(M) \pm \ldots + b^{n-1}(M) - b^n(M) = 0.$$

$\square$

Next we will compute the Betti numbers of a product of two manifolds M and N. Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ the canonical projections. Similarly to the pairing defined above,

$$\Omega^k(M) \times \Omega^l(N) \to \Omega^{k+l}(M \times N),$$
$$(\omega, \eta) \mapsto \pi_M^*\omega \wedge \pi_N^*\eta,$$

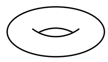
gives a bilinear map $H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(N) \to H^{k+l}_{\mathrm{dR}}(M \times N)$.

Theorem 31 (Künneth Formula)**.** *The map* $H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(N) \to H^{k+l}_{\mathrm{dR}}(M \times N)$ *yields an isomorphism*

$$H^p_{\mathrm{dR}}(M \times N) \cong \bigoplus_{k+l=p} H^k_{\mathrm{dR}}(M) \otimes H^l_{\mathrm{dR}}(N).$$

Hence

$$b^p(M \times N) = \sum_{k+l=p} b^k(M) b^l(N).$$

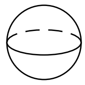
Example 32. For the 2-dimensional torus $M = T^2 = S^1 \times S^1 =$ 

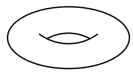
we have $b^0(T^2) = 1$ because T^2 is connected. By Poincaré duality $b^2(T^2) = b^0(T^2) = 1$. The Künneth formula gives $b^1(T^2) = b^0(S^1)b^1(S^1) + b^1(S^1)b^0(S^1) = 2$. From $b^1(S^2) = 0 \neq 2 = b^1(T^2)$ we conclude that S^2 and T^2 cannot be homotopy equivalent. In particular, they are not diffeomorphic.

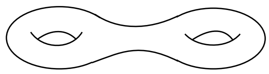
Exercise 4. Compute the following table (by induction on g):

$b^0(F_g)$	$b^1(F_g)$	$b^2(F_g)$	$\chi(F_g)$
1	$2g$	1	$2-2g$

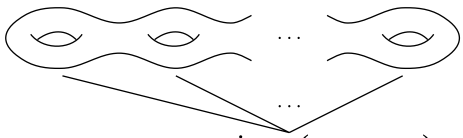
where

$F_0 \quad = \quad S^2 =$ 

$F_1 \quad = \quad T^2 =$ 

$F_2 \quad =$ 

$\vdots$

$F_g \quad =$ 

g times (g = genus)

In particular, $F_g \approx F_h \iff F_g \simeq F_h \iff g = h$.

Corollary 33. *For any two compact manifolds M and N we have*

$$\chi(M \times N) = \chi(M)\chi(N).$$

Proof. Using the Künneth formula we compute

$$\begin{aligned}
\chi(M)\chi(N) &= \Big(\sum_k (-1)^k b^k(M)\Big)\Big(\sum_l (-1)^l b^l(M)\Big) \\
&= \sum_{kl} (-1)^{k+l} b^k(M) b^l(N) \\
&= \sum_p (-1)^p \underbrace{\sum_{k+l=p} b^k(M) b^l(M)}_{=b^p(M \times N)} \\
&= \chi(M \times N).
\end{aligned}$$

$\square$

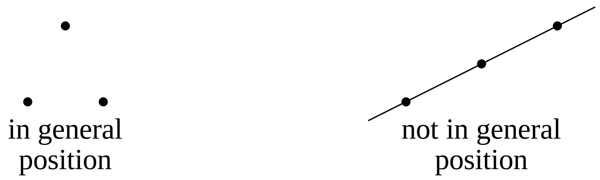
Exercise 5. Compute $b^k(T^n)$ for all k and n.

3 Simplicial Homology

Next we describe simplicial homology. Again, we will associate vector spaces to manifolds. This time they are based on decomposing the manifold into simplices.

3.1 Definitions

Definition 34. (1) $v_0, \dots, v_k \in \mathbb{R}^N$ are said to be *in general position* if they are not contained in a $(k-1)$-dimensional affine subspace of $\mathbb{R}^N$.

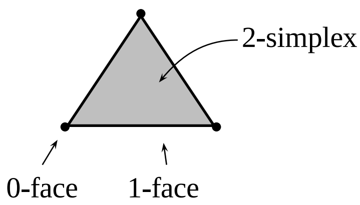


(2) If $v_0, \dots, v_k$ are in general position, then the convex hull

$$|v_0 \cdots v_k| := \left\{ \sum_j^k a_j v_j \,\middle|\, a_j \geq 0, \sum_j a_j = 1 \right\}$$

is called a k-simplex.

(3) If $\emptyset \neq \{w_0, \dots, w_l\} \subset \{v_0, \dots, v_k\}$, then $|w_0 \cdots w_l|$ is called a *face* of $|v_0 \cdots v_k|$.



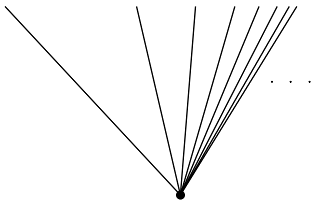
Definition 35. A set $\mathcal{K}$ of simplices in $\mathbb{R}^N$ is called a *(Euclidean) simplicial complex*, if

(1) for each simplex in $\mathcal{K}$ all faces are also in $\mathcal{K}$,

(2) for any $\sigma, \tau \in \mathcal{K}$ the intersection $\sigma \cap \tau$ is either empty or is a common face of σ and τ,

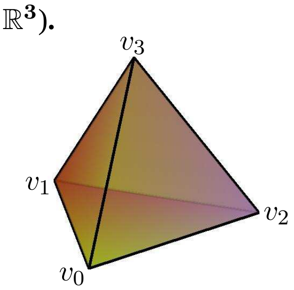
(3) for any $x \in \mathbb{R}^N$ there exits a neighborhood U of x in $\mathbb{R}^N$ which meets only finitely many simplices.

allowed



not allowed

Example 36 (Tetrahedron in $\mathbb{R}^3$)**.**



In this example

$$\mathcal{K} \;=\; \Big\{\underbrace{|v_0|, |v_1|, |v_2|, |v_3|}_{\text{0-dim. simplices}}, \underbrace{|v_0v_1|, |v_0v_2|, |v_0v_3|, |v_1v_2|, |v_1v_3|, |v_2v_3|}_{\text{1-dim. simplices}},$$
$$\underbrace{|v_0v_1v_2|, |v_0v_1v_3|, |v_0v_2v_3|, |v_1v_2v_3|}_{\text{2-dim. simplices}}\Big\}.$$

This defines a 2-dimensional simplicial complex.

Definition 37. If $\mathcal{K}$ is a simplicial complex, then we call

$$|\mathcal{K}| := \bigcup_{\sigma \in \mathcal{K}} \sigma$$

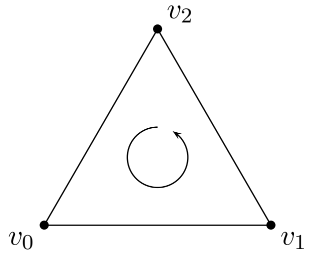
its *geometric realization*.

We think of $|\mathcal{K}|$ as of the actual geometric object which we want to study while $\mathcal{K}$ itself is the combinatorial description telling us how to manufacture $|\mathcal{K}|$ out of simplices.

Let $v_0, \ldots, v_k$ be in general position. Two orderings of $\{v_0, \ldots, v_k\}$ are called *equivalent*, if they are transformed into each other by an even permutation. For example,

$$(v_1, v_0, v_2) \nsim (v_0, v_1, v_2) \sim (v_1, v_2, v_0)$$

An equivalence class of orderings of $\{v_0, \ldots, v_k\}$ is called an *orientation* of $\{v_0, \ldots, v_k\}$ (and also of $|v_0, \ldots, v_k|$).



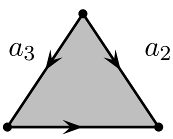
The simplex $|v_0 \cdots v_k|$ together with the orientation given by the ordering $(v_0, \ldots, v_k)$ will be denoted by $\langle v_0 \cdots v_k \rangle$. For the converse orientation we write

$$-\langle v_0 \cdots v_k \rangle := \langle v_1 v_0 v_2 \cdots v_k \rangle$$

Let $\mathcal{K}$ be a simplicial complex. Equip each simplex in $\mathcal{K}$ with an orientation and let

$$C_k(\mathcal{K}, \mathbb{R}) := \left\{ \sum_{j=1}^{m} a_j \sigma_j \,\middle|\, a_j \in \mathbb{R}, \sigma_j \in \mathcal{K} \text{ simplex of dimension } k, m \in \mathbb{N} \right\}$$

$C_k(\mathcal{K}, \mathbb{R})$ is an $\mathbb{R}$-vector-space with basis given by all k-dimensional simplices in $\mathcal{K}$. An element of $C_k(\mathcal{K}, \mathbb{R})$ is called a *k-chain*. We can think of a k-chain as a decoration of the oriented k-dimensional simplices with certain real numbers where only finitely many are allowed to be non-zero. For example,

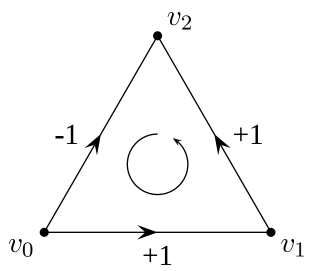


For any k define the *boundary map* as the linear map

$$\partial : C_k(\mathcal{K}, \mathbb{R}) \to C_{k-1}(\mathcal{K}, \mathbb{R})$$

given on the basis vectors by

$$\partial(\langle v_0 \cdots v_k \rangle) := \sum_{j=0}^{k} (-1)^j \langle v_0 \cdots \widehat{v_j} \cdots v_k \rangle.$$



Lemma 38. *$\partial \circ \partial : C_k(\mathcal{K}, \mathbb{R}) \to C_{k-2}(\mathcal{K}, \mathbb{R})$ is zero for all k.*

Proof.

$$\begin{aligned}
&\partial\partial\langle v_0 \cdots v_k \rangle \\
&= \partial \sum_{j=0}^{k} (-1)^j \langle v_0 \cdots \widehat{v_j} \cdots v_k \rangle = \sum_{j=0}^{k} (-1)^j \partial \langle v_0 \cdots \widehat{v_j} \cdots v_k \rangle \\
&= \sum_{j=0}^{k} (-1)^j \left(\sum_{i=0}^{j-1} (-1)^i \langle v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_k \rangle + \sum_{i=j+1}^{k} (-1)^{i-1} \langle v_0 \cdots \widehat{v_j} \cdots \widehat{v_i} \cdots v_k \rangle \right) \\
&= \sum_{i<j} (-1)^{i+j} \langle v_0 \cdots \widehat{v_i} \cdots \widehat{v_j} \cdots v_k \rangle - \sum_{j<i} (-1)^{i+j} \langle v_0 \cdots \widehat{v_j} \cdots \widehat{v_i} \cdots v_k \rangle \\
&= 0
\end{aligned}$$

□

In contrast to the case of de Rham **co**homology, where the d-Operator increases the degree of forms, the ∂-operator defined above decreases the degree of chains. We have

$$0 \longleftarrow C_0(\mathcal{K}, \mathbb{R}) \xleftarrow{\partial} C_1(\mathcal{K}, \mathbb{R}) \xleftarrow{\partial} C_2(\mathcal{K}, \mathbb{R}) \xleftarrow{\partial} \cdots$$

We define the vector space of *boundaries*,

$$B_k(\mathcal{K}, \mathbb{R}) := \operatorname{im}(\partial : C_{k+1}(\mathcal{K}, \mathbb{R}) \to C_k(\mathcal{K}, \mathbb{R})),$$

and the vector space of *cycles*,

$$Z_k(\mathcal{K}, \mathbb{R}) := \ker(\partial : C_k(\mathcal{K}, \mathbb{R}) \to C_{k-1}(\mathcal{K}, \mathbb{R})).$$

Again, $\partial \circ \partial = 0$ implies $B_k(\mathcal{K}, \mathbb{R}) \subset Z_k(\mathcal{K}, \mathbb{R})$ so that we can define

Definition 39. The vector space

$$H_k(\mathcal{K}, \mathbb{R}) := \frac{Z_k(\mathcal{K}, \mathbb{R})}{B_k(\mathcal{K}, \mathbb{R})}$$

is called the k^{th} *simplicial homology* of $\mathcal{K}$. Its dimensions

$$b_k(\mathcal{K}, \mathbb{R}) := \dim H_k(\mathcal{K}, \mathbb{R}) \in \{0, 1, 2, \ldots, \infty\}$$

are again called *Betti numbers*.

Obviously, if $\mathcal{K}$ consists of finitely many simplices, then $C_k(\mathcal{K}, \mathbb{R})$ is finite dimensional. Hence $Z_k(\mathcal{K}, \mathbb{R})$, $B_k(\mathcal{K}, \mathbb{R})$, and $H_k(\mathcal{K}, \mathbb{R})$ are then also finite dimensional.

Example 40 (Homology for the tetrahedron)**.** Let $\mathcal{K}$ be the tetrahedron from Example 36. The boundary map $\partial_1 : C_1(\mathcal{K}, \mathbb{R}) \to C_0(\mathcal{K}, \mathbb{R})$ is easily seen to be given by the matrix

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

In particular, $\mathrm{rk}(\partial_1) = 3$ and therefore $\dim B_0(\mathcal{K}, \mathbb{R}) = 3$ and $\dim Z_1(\mathcal{K}, \mathbb{R}) = \dim C_1(\mathcal{K}, \mathbb{R}) - \mathrm{rk}(\partial_1) = 6 - 3 = 3$. Thus

$$\begin{aligned} b_0(\mathcal{K}, \mathbb{R}) &= \dim H_0(\mathcal{K}, \mathbb{R}) \\ &= \dim Z_0(\mathcal{K}, \mathbb{R}) - \dim B_0(\mathcal{K}, \mathbb{R}) \\ &= \dim C_0(\mathcal{K}, \mathbb{R}) - \dim B_0(\mathcal{K}, \mathbb{R}) \\ &= 4 - 3 \quad = \quad 1. \end{aligned}$$

The boundary map $\partial_2 : C_2(\mathcal{K}, \mathbb{R}) \to C_1(\mathcal{K}, \mathbb{R})$ is given by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Thus $\mathrm{rk}(\partial_2) = 3$, hence $\dim B_1(\mathcal{K}, \mathbb{R}) = 3$. Therefore

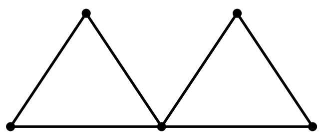
$$\begin{aligned} b_1(\mathcal{K}, \mathbb{R}) &= \dim H_1(\mathcal{K}, \mathbb{R}) \\ &= \dim Z_1(\mathcal{K}, \mathbb{R}) - \dim B_1(\mathcal{K}, \mathbb{R}) \\ &= 3 - 3 \quad = \quad 0. \end{aligned}$$

Moreover, $\dim Z_2(\mathcal{K}, \mathbb{R}) = \dim C_2(\mathcal{K}, \mathbb{R}) - \mathrm{rk}(\partial_2) = 4 - 3 = 1$ and $\dim B_2(\mathcal{K}, \mathbb{R}) = 0$, thus $b_2(\mathcal{K}, \mathbb{R}) = 1$. We have shown

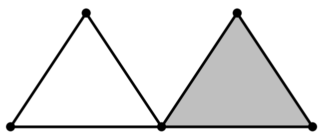
$$b_k(\mathcal{K}, \mathbb{R}) \quad = \quad \begin{cases} 1 & \text{if } k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

We observe that the Betti numbers $b_k(\mathcal{K}, \mathbb{R})$ coincide with the Betti numbers based on de Rham cohomology for the 2-sphere S^2. Moreover, S^2 and $|\mathcal{K}|$ are homeomorphic via central projection.

Exercise 6. Compute b_k and basis vectors for H_k for



Exercise 7. Do the same for



3.2 De Rham's theorem

Definition 41. Let M be a manifold, let $\mathcal{K}$ be a simplicial complex. Then a homeomorphism $h : |\mathcal{K}| \to M$ such that the restriction of h to each simplex is smooth is called a *smooth triangulation* of M.

One can show that any compact (smooth) manifold can be triangulated by a finite simplicial complex.

Question 8. How does $H^k_{\mathrm{dR}}(M)$ relate to $H_k(\mathcal{K},\mathbb{R})$, where $\mathcal{K}$ is a simplicial complex which triangulates M?

Let $h : |\mathcal{K}| \to M$ be a smooth triangulation of M. Then there is a bilinear map

$$\Omega^k(M) \times C_k(\mathcal{K},\mathbb{R}) \to \mathbb{R},$$
$$(\omega,\sigma) \mapsto \int_\sigma h^*\omega.$$

It induces a bilinear map on cohomology and homology respectively. Namely:

- If $\omega = d\eta$, then by Stokes' theorem

  $$(d\eta,\sigma) \mapsto \int_\sigma h^* d\eta = \int_\sigma dh^*\eta = \int_{\partial\sigma} h^*\eta.$$

  Therefore, $(d\eta, \sum_i a_i\sigma_i) \mapsto 0$ provided $\sum_i a_i\sigma_i \in Z_k(\mathcal{K},\mathbb{R})$. This shows that the pairing is well-defined on $H^k_{\mathrm{dR}}(M) \times Z_k(\mathcal{K},\mathbb{R})$.

- Similarly, if $\sigma = \partial\tau$ and $\omega \in Z^k(M,\mathbb{R})$, then again by Stokes' theorem,

  $$(\omega,\sigma) = (\omega,\partial\tau) = \int_{\partial\tau} h^*\omega = \int_\tau dh^*\omega = \int_\tau h^*(\underbrace{d\omega}_{0}) = 0.$$

Therefore, we obtain a well-defined map

$$H^k_{\mathrm{dR}}(M) \times H_k(\mathcal{K},\mathbb{R}) \to \mathbb{R},$$
$$([\omega],[\sigma]) \mapsto \int_\sigma h^*\omega.$$

Theorem 42 (de Rham)**.** *This bilinear map is non-degenerate, i. e.,*

$$H^k_{\mathrm{dR}}(M) \to H_k(\mathcal{K},\mathbb{R})^*,$$
$$[\omega] \mapsto \left([\sigma] \mapsto \int_\sigma h^*\omega\right),$$

is an isomorphism. In particular, $b^k(M) = b_k(\mathcal{K},\mathbb{R})$.

Remark 43. If M is compact, then let $\mathcal{K}$ be a finite simplicial complex triangulating M. Then $b_k(\mathcal{K},\mathbb{R}) < \infty$, hence $b^k(M) < \infty$ for all k.

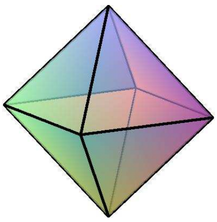
Remark 44. For the Euler characteristic of a triangulated compact manifold M we compute

$$\begin{aligned}
\chi(M) &= \sum_k (-1)^k b^k(M) \\
&= \sum_k (-1)^k b_k(\mathcal{K}, \mathbb{R}) \\
&= \sum_k (-1)^k \left(\dim \ker \partial_k - \dim \mathrm{im} \partial_{k+1}\right) \\
&= \sum_k (-1)^k \dim \ker \partial_k + \sum_k (-1)^{k+1} \dim \mathrm{im} \partial_{k+1} \\
&= \sum_k (-1)^k \dim \ker \partial_k + \sum_k (-1)^k \dim \mathrm{im} \partial_k \\
&= \sum_k (-1)^k \dim(C_k(\mathcal{K}, \mathbb{R})) \\
&= \sum_k (-1)^k \#(k\text{-dimensional simplices})
\end{aligned}$$

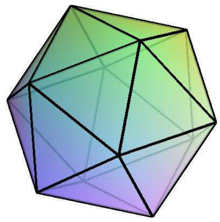
Example 45. If $\mathcal{K}$ is a simplicial complex triangulating S^2, then

$$\#\text{vertices} - \#\text{lines} + \#\text{triangles} = \chi(S^2) = 2. \tag{3}$$

In particular, if $\mathcal{K}$ is a finite 2-dimensional simplicial complex with $|\mathcal{K}| \subset \mathbb{R}^3$ such that $|\mathcal{K}|$ bounds a convex domain, then central projection yields a smooth triangulation $h : |\mathcal{K}| \to S^2$. Hence (3) holds. Besides the tetrahedron this applies e. g. to the octahedron and the icosahedron.

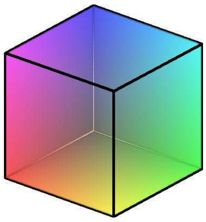


octahedron

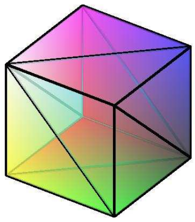


icosahedron

For other convex polyhedra like the cube the results seem not to apply because the 2-dimensional faces are squares not triangles so that the cube does not define a simplicial complex. But we can subdivide each 2-dimensional face by adding a diagonal. This yields a simplicial complex to which (3) applies.



cube



cube, subdivided

This adding of diagonals increases the number of 1-dimensional and 2-dimensional faces by the same amount. Hence this new contribution cancelles in (3). Therefore (3) applies to the (original) cube as well. Similar reasoning shows that it applies to all convex polyhedra in $\mathbb{R}^3$, e. g. to the dodecahedron. The formula

$$\#\text{vertices} - \#\text{lines} + \#\text{triangles} = 2$$

for convex polyhedra is much older than homology theory. It is known as *Euler's formula*.

Remark 46. In the definition of $H_k(\mathcal{K}, \mathbb{R})$, the coefficient ring $\mathbb{R}$ can be replaced by any commutative ring with unit. Popular choices are $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{Z}/2$. For the comparison with de Rham cohomology we have to use real coefficients because differential forms naturally form a real vector space. It should be mentioned that simplicial homology with integral coefficients contains sometimes more information than the one with real coefficients; the $H_k(\mathcal{K}, \mathbb{R})$ can always be computed out of the $H_k(\mathcal{K}, \mathbb{Z})$ but not conversely.

4 Further reading

There are many good introductions to algebraic topology. If the focus should be on manifolds - as in these notes - then I can recommend [1, 2, 3, 5]. They all introduce various (co-) homology theories on manifolds and explain lots of applications. A rather encyclopedic account of algebraic topology can be found in [4].

5 Solutions to the exercises

Exercise 1. Let $c : [0, 2\pi] \to \mathbb{R}^2$, $c(t) = (\cos(t), \sin(t))$, be the loop winding around the origin once. We compute

$$\begin{aligned}
\int_c \widehat{\omega} &= \int_0^{2\pi} \left(-\frac{\sin(t)}{\sin(t)^2 + \cos(t)^2} d\cos(t) + \frac{\cos(t)}{\sin(t)^2 + \cos(t)^2} d\sin(t) \right) \\
&= \int_0^{2\pi} \left(\sin(t)^2 + \cos(t)^2 \right) \, dt = 2\pi.
\end{aligned}$$

If $\widehat{\omega} = df$ had a solution f, then we would get

$$\int_c \widehat{\omega} = \int_c df = f(c(2\pi)) - f(c(0)) = f(1,0) - f(1,0) = 0,$$

a contradiction.

Exercise 2. Removing a point from a manifold of dimension at least 2 does not change the number of connected components, hence

$$b^0(M) = b^0(\dot{M}).$$

We will apply the Mayer-Vietoris sequence for the open cover of M by $\dot{M}$ and an n-dimensional ball D^n containing the point that has been removed. We observe that $\dot{M} \cap D^n = \dot{D}^n \simeq S^{n-1}$. For $k = 1 < n - 1$ the Mayer-Vietoris sequence yields

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, phantom, sloped, "\cong"] & \mathbb{R} \arrow[d, phantom, sloped, "\cong"] \\
0 \arrow[r] & H^0_{\mathrm{dR}}(M) \arrow[r] & H^0_{\mathrm{dR}}(\dot{M}) \oplus H^0_{\mathrm{dR}}(D^n) \arrow[r] & H^0_{\mathrm{dR}}(\dot{M} \cap D^n) \arrow[dll, rounded corners] \\
 & H^1_{\mathrm{dR}}(M) \arrow[r] & H^1_{\mathrm{dR}}(\dot{M}) \oplus H^1_{\mathrm{dR}}(D^n) \arrow[r] & H^1_{\mathrm{dR}}(\dot{M} \cap D^n) \\
 & & 0 \arrow[u, phantom, sloped, "="] & 0. \arrow[u, phantom, sloped, "="]
\end{tikzcd}$$

Therefore $0 = b^0(M) - (b^0(\dot{M}) + 1) + 1 - b^1(M) + (b^1(\dot{M}) + 0)$, hence $b^1(M) = b^1(\dot{M})$.

For $2 \leq k < n-1$ the Mayer-Vietoris sequence yields

$$\begin{tikzcd}
0 \arrow[r] & H^k_{\mathrm{dR}}(M) \arrow[r] & H^k_{\mathrm{dR}}(\dot{M}) \oplus H^k_{\mathrm{dR}}(D^n) \arrow[r] & H^k_{\mathrm{dR}}(\dot{M} \cap D^n) \\
& & \underset{\textstyle 0}{\shortparallel} & \underset{\textstyle 0}{\shortparallel}
\end{tikzcd}$$

thus $b^k(M) = b^k(\dot{M})$. To compute $b^k(M)$ for $k = n-1$ and $k = n$ we look at the final part of the Mayer-Vietoris sequence,

$$0 \longrightarrow H^{n-1}_{\mathrm{dR}}(M) \longrightarrow H^{n-1}_{\mathrm{dR}}(\dot{M}) \oplus 0 \longrightarrow \mathbb{R} \longrightarrow H^n_{\mathrm{dR}}(M) \longrightarrow H^n_{\mathrm{dR}}(\dot{M}) \oplus 0 \longrightarrow 0$$

This implies $b^{n-1}(M) - b^{n-1}(\dot{M}) + 1 - b^n(M) + b^n(\dot{M}) = 0$. It also implies that $H^{n-1}_{\mathrm{dR}}(M) \to H^{n-1}_{\mathrm{dR}}(\dot{M})$ is injective, hence $b^{n-1}(M) \leq b^{n-1}(\dot{M})$. Moreover, it shows that $H^n_{\mathrm{dR}}(M) \to H^n_{\mathrm{dR}}(\dot{M})$ is onto, thus $b^n(M) \geq b^n(\dot{M})$. Therefore,

$$\underbrace{-b^{n-1}(M) + b^{n-1}(\dot{M})}_{\geq 0} + \underbrace{b^n(M) - b^n(\dot{M})}_{\geq 0} = 1.$$

This proves the claim.

Exercise 3. For $M = \mathbb{R}^n$ we have $b^{n-1}(M) = b^n(M) = 0$ and $\dot{M} \simeq S^{n-1}$, hence $b^{n-1}(\dot{M}) = 1$ and $b^n(\dot{M}) = 0$.
For $M = S^n$ we have $b^{n-1}(M) = 0$ and $b^n(M) = 1$ while $\dot{M} \approx \mathbb{R}^n$ (via stereographic projection), hence $b^{n-1}(\dot{M}) = b^n(\dot{M}) = 0$.

Exercise 4. All surfaces F_g are connected, hence $b^0(F_g) = 1$. By Poincaré duality $b^2(F_g) = 1$. It remains to compute $b^1(F_g)$. We know the result already for $g = 0$ and $g = 1$.
To procede inductively let $g \geq 2$. We cover F_g by two open subsets U and V such that $U \approx \dot{F}_{g-1}$, $V \approx \dot{T}^2$, and $U \cap V \approx S^1 \times D^1 \simeq S^1$. Heuristically, U covers the

first $g-1$ "holes" while V covers the last "hole". The Mayer-Vietoris sequence is now given by

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r] & \mathbb{R} \arrow[dll, rounded corners] \\
& H^1_{\mathrm{dR}}(F_g) \arrow[r] & H^1_{\mathrm{dR}}(\dot{F}_{g-1}) \oplus H^1_{\mathrm{dR}}(\dot{T}^2) \arrow[r] & \mathbb{R} \arrow[dll, rounded corners] \\
& H^2_{\mathrm{dR}}(F_g) \arrow[r] & H^2_{\mathrm{dR}}(\dot{F}_{g-1}) \oplus H^2_{\mathrm{dR}}(\dot{T}^2) \arrow[r] & 0.
\end{tikzcd}
$$

Thus

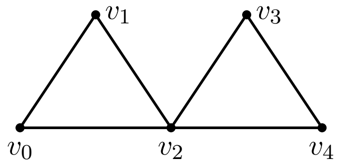
$$b^1(F_g) = b^1(\dot{F}_{g-1}) - b^2(\dot{F}_{g-1}) + b^1(\dot{T}^2) - b^2(\dot{T}^2).$$

From Exercise 2 and by induction we know that $b^1(\dot{F}_{g-1}) - b^2(\dot{F}_{g-1}) = b^1(F_{g-1}) - b^2(F_{g-1}) + 1 = 2(g-1)$. Similarly, $b^1(\dot{T}^2) - b^2(\dot{T}^2) = 2$. The result follows.

Exercise 5. We claim that $b^k(T^n)$ is given by the binomial coefficient $\begin{pmatrix} n \\ k \end{pmatrix}$. For $n = 1$ we have $T^1 = S^1$ and the result is known. We procede by induction on n using the Künneth formula.

$$
\begin{aligned}
b^k(T^n) &= b^k(T^{n-1} \times S^1) \\
&= \sum_{i+j=k} b^i(T^{n-1}) \cdot b^j(S^1) \\
&= b^k(T^{n-1}) \cdot 1 + b^{k-1}(T^{n-1}) \cdot 1 \\
&= \begin{pmatrix} n-1 \\ k \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \\
&= \begin{pmatrix} n \\ k \end{pmatrix}
\end{aligned}
$$

Exercise 6. For definiteness we give the vertices names,



Then C_0 has the basis $\langle v_0\rangle, \dots, \langle v_4\rangle$ and C_1 has the basis $\langle v_0v_1\rangle$, $\langle v_1v_2\rangle$, $\langle v_2v_0\rangle$, $\langle v_2v_3\rangle$, $\langle v_3v_4\rangle$, $\langle v_4v_2\rangle$. With respect to this basis the boundary map $\partial : C_1 \to C_0$ is given by the matrix

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

This matrix has rank 4. Its kernel has basis vectors $(1,1,1,0,0,0)^\top$ and $(0,0,0,1,1,1)^\top$. Hence $H_1 = Z_1$ has dimension $b_1 = 2$ and the basis vectors $\langle v_0v_1\rangle + \langle v_1v_2\rangle + \langle v_2v_0\rangle$ and $\langle v_2v_3\rangle + \langle v_3v_4\rangle + \langle v_4v_2\rangle$.
Moreover, $b_0 = \dim H_0 = \dim Z_0 - \dim B_0 = \dim C_0 - \dim B_0 = 5 - 4 = 1$. Since the vector $(1,1,1,1,1)^\top$ does not lie in the image of the matrix, the element $\langle v_0\rangle + \langle v_1\rangle + \langle v_2\rangle + \langle v_3\rangle + \langle v_4\rangle \in C_0$ represents a basis vector in H_0.

Exercise 7. The discussion of H_0 is the same as in Exercise 6, i. e., $b_0 = 1$ and $\langle v_0\rangle + \langle v_1\rangle + \langle v_2\rangle + \langle v_3\rangle + \langle v_4\rangle \in C_0$ represents a basis vector in H_0.
But now we also have to consider the boundary map $\partial : C_2 \to C_1$. There is only one 2-simplex, namely $\langle v_2v_3v_4\rangle$. Hence C_2 is 1-dimensional with basis $\langle v_2v_3v_4\rangle$. The boundary map $\partial : C_2 \to C_1$ satisfies

$$\begin{aligned} \partial\langle v_2v_3v_4\rangle &= \langle v_3v_4\rangle - \langle v_2v_4\rangle + \langle v_2v_3\rangle \\ &= \langle v_3v_4\rangle + \langle v_4v_2\rangle + \langle v_2v_3\rangle. \end{aligned}$$

In particular, $b_2 = \dim Z_2 = 0$ and $b_1 = \dim Z_1 - \dim B_1 = 2 - 1 = 1$. The element $\langle v_0v_1\rangle + \langle v_1v_2\rangle + \langle v_2v_0\rangle \in C_1$ represents a basis vector of H_1.

References

[1] R. Bott, L. W. Tu: *Differential Forms in Algebraic Topology*, Springer 1995

[2] W. Lück: *Algebraische Topologie*, Vieweg 2005

[3] S. Morita: *Geometry of Differential Forms*, AMS 2001

[4] E. H. Spanier: *Algebraic Topology*, Springer 1981

[5] F. W. Warner: *Foundations of Differentiable Manifolds and Lie Groups*, Springer 1983