GEOMETRY IN POTSDAM

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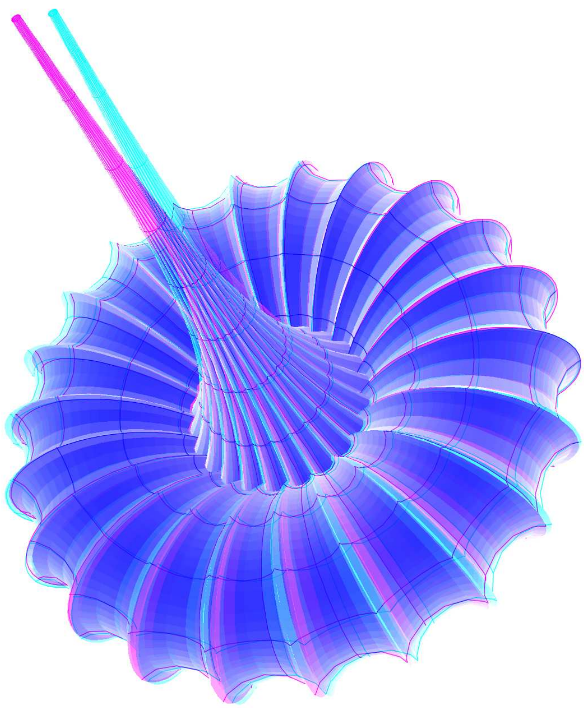
Differential Geometry

Summer Term 2023



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Preface

These are the lecture notes of an introductory course on differential geometry that were given various times at the University of Potsdam. It introduces the mathematical concepts necessary to describe and analyze curved spaces of arbitrary dimension. Important concepts are manifolds, vector fields, semi-Riemannian metrics, curvature, geodesics, Jacobi fields and much more. The focus is on Riemannian geometry but, as we move along, we also treat more general semi-Riemannian geometry such as Lorentzian geometry which is central for applications in General Relativity. We also make a connection to classical geometry when we apply differential geometry to derive the laws of trigonometry on spaces of constant curvature. One fundamental result of Riemannian geometry that we show towards the end of the course is the Bonnet-Myers theorem. It roughly states that the larger the curvature of a space, the smaller the space itself must be.

The lecture course did not require prior attendance of a course on elementary differential geometry treating curves and surfaces but such a course would certainly help to develop the right intuition.

It is my pleasure to thank all those who helped to improve the manuscript by suggestions, corrections or by work on the LaTeX code. My particular thanks go to Andrea Röser who wrote the first version in German language and created many pictures in wonderful quality, to Volker Branding who translated the manuscript into English, to Ramona Ziese who improved the layout and to Matthias Ludewig for pointing out various flaws.

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Christian Bär

1 Manifolds

1.1 Topological manifolds

Reminder. Let M be a set. A system of sets $\mathscr{O} \subset \mathscr{P}(M)$ is called a **topology** on M, if

1. $\emptyset, M \in \mathscr{O}$.
2. If $U_i \in \mathscr{O}$, $i \in I$, then also $\bigcup\limits_{i\in I} U_i \in \mathscr{O}$.
3. If $U_1, U_2 \in \mathscr{O}$, then also $U_1 \cap U_2 \in \mathscr{O}$.

The pair $(M, \mathscr{O})$ is called a **topological space**. By abuse of language, one often speaks about the topological space M rather than $(M, \mathscr{O})$.

A subset $U \subset M$ is called **open in** M if $U \in \mathscr{O}$. A subset $A \subset M$ is called **closed** if $M \setminus A \in \mathscr{O}$.

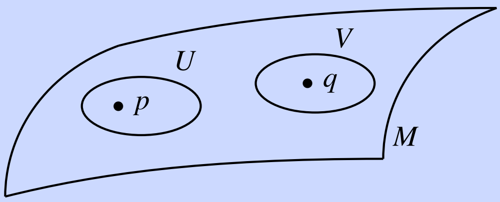
If both $(M, \mathscr{O}_M)$ and $(N, \mathscr{O}_N)$ are topological spaces, a map $f : M \to N$ is called **continuous**, if

$$f^{-1}(V) \in \mathscr{O}_M \quad \text{for all } V \in \mathscr{O}_N.$$

In other words, preimages of open sets have to be open. A bijective continuous map $f : M \to N$, whose inverse f^{-1} is also continuous, is called a **homeomorphism**. Two topological spaces M and N are called **homeomorphic**, if there exists a homeomorphism between them.

Definition 1.1.2. Let M be a topological space with topology $\mathscr{O}$. Then M is called an n**-dimensional topological manifold**, if the following holds:

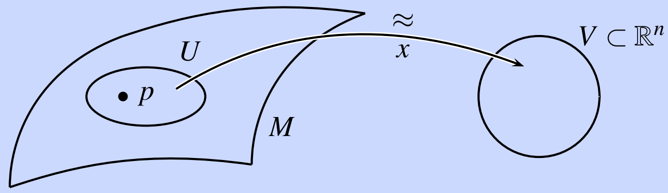
1. M is **Hausdorff**, that is, for all $p, q \in M$ with $p \neq q$ there exist open sets $U, V \subset M$ with $p \in U$, $q \in V$ and $U \cap V = \emptyset$.



2. The topology of M has a **countable basis**, that is, there exists a countable subset $\mathscr{B} \subset \mathscr{O}$, such that for every $U \in \mathscr{O}$ there are $B_i \in \mathscr{B}$, $i \in I$ with

$$U = \bigcup_{i\in I} B_i.$$

3. M is **locally homeomorphic** to $\mathbb{R}^n$, that is, for all $p \in M$ exists an open subset $U \subset M$ with $p \in U$, an open subset $V \subset \mathbb{R}^n$ and a homeomorphism $x : U \to V$.



Remark 1.1.3. The first two conditions in the definition are more of a technical nature and are sometimes neglected. The important fact is that a topological manifold is locally homeomorphic to $\mathbb{R}^n$. Loosely speaking manifolds look locally like Euclidean space. If the topology on M is induced by a metric, then the first condition is satisfied automatically. If M is given as a subset of $\mathbb{R}^N$ with the subset topology, then both conditions 1 and 2 are satisfied automatically.

Examples 1.1.4. (1) Euclidean space $M = \mathbb{R}^n$ itself is an n-dimensional topological manifold:

(i), (ii) Obvious.

(iii) Holds true with $U = M$, $V = \mathbb{R}^n$ and $x = \mathrm{id}$.

(2) Let $M \subset \mathbb{R}^n$ be an open subset. Then M is an n-dimensional topological manifold.

(i), (ii) Obvious.

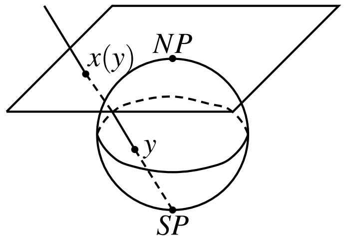
(iii) Holds true with $U = M$, $V = M$ and $x = \mathrm{id}$.

(3) The standard sphere $M = S^n = \{\, y \in \mathbb{R}^{n+1} : \|y\| = 1 \,\}$ is an n-dimensional topological manifold.

(i), (ii) Obvious, since S^n is a subset of $\mathbb{R}^{n+1}$.

(iii) We construct two homeomorphisms with the help of the *stereographic projection*.

We define $U_1 := S^n \setminus \{SP\}$ with $SP := (-1,0,\ldots,0) \in \mathbb{R}^{n+1}$ and set $V_1 := \mathbb{R}^n$. Furthermore, we define



$$\begin{aligned} x : U_1 &\longrightarrow V_1, \\ y = (y^0, \underbrace{y^1, \ldots, y^n}_{=:\hat{y}}) &\longmapsto x(y) = \frac{2}{1+y^0} \cdot \hat{y}. \end{aligned}$$

The map x is continuous and bijective. The inverse map y is given by

$$\begin{aligned} y : V_1 &\longrightarrow U_1, \\ x &\longmapsto y(x) = \frac{1}{4+\|x\|^2}(4-\|x\|^2, 4x), \end{aligned}$$

and is also continuous. Hence, x is an homeomorphism.

Analogously, we define the homeomorphism, which omits the north pole: Let now $U_2 := S^n \setminus \{NP\}$ with $NP := (1,0,\ldots,0) \in \mathbb{R}^{n+1}$ and $V_2 := \mathbb{R}^n$. Then

$$\begin{aligned} \tilde{x} : U_2 &\longrightarrow V_2, \\ y &\longmapsto \tilde{x}(y) = \tilde{x}(y^0, \underbrace{y^1, \ldots, y^n}_{=:\hat{y}}) = \frac{2}{1-y^0} \cdot \hat{y}. \end{aligned}$$

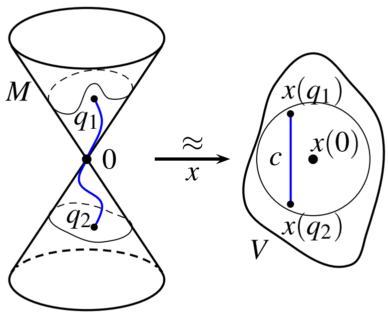
We have seen that the sphere S^n is an n-dimensional topological manifold.

(4) All n-dimensional submanifolds of $\mathbb{R}^N$ in the sense of Analysis 3 are n-dimensional topological manifolds.

(5) **Non-Example.** We consider $M := \{(y^1,y^2,y^3) \in \mathbb{R}^3 \mid (y^1)^2 = (y^2)^2 + (y^3)^2\}$, the double cone.

Since $M \subset \mathbb{R}^3$, both (i) and (ii) are satisfied.

But M is *not* a 2-dimensional manifold. If it were, then there would exist an open subset $U \subset M$ with $0 \in U$, an open subset $V \subset \mathbb{R}^2$ and a homeomorphism $x : U \to V$ with $x(0) = 0$.



W. l. o. g. assume $V = B_r(x(0))$ with $r > 0$. Choose $q_1, q_2 \in U$ with $q_1^1 > 0$ and $q_2^1 < 0$. Furthermore, choose a continuous path $c : [0,1] \to V$ with

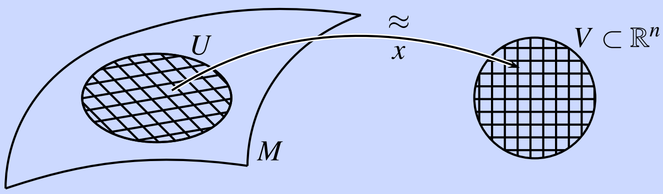
$$c(0) = x(q_1), c(1) = x(q_2) \quad \text{and} \quad c(t) \neq x(0) \text{ for all } t \in [0,1].$$

Define the continuous path $\tilde{c} := x^{-1} \circ c : [0,1] \to U$. Then

$$\tilde{c}(0) = q_1, \tilde{c}(1) = q_2,$$

that is, we have $\tilde{c}^1(0) > 0$ while $\tilde{c}^1(1) < 0$. Applying the mean value theorem we find, that there exists a $t \in (0,1)$ with $\tilde{c}^1(t) = 0$. Then $\tilde{c}(t) = (0,0,0)$ and consequently $c(t) = x(\tilde{c}(t)) = x(0)$, which contradicts the choice of c. Hence, M is not a 2-dimensional topological manifold.

Definition 1.1.5. If M is an n-dimensional topological manifold, the homeomorphisms $x : U \to V$ are called **charts** (or **local coordinate systems**) of M.



After choosing a local coordinate system $x : U \to V$ every point $p \in U$ is uniquely characterized by its coordinates $(x^1(p), \dots, x^n(p))$.

In a 0-dimensional manifold M every point $p \in M$ has an open neighborhood U, which is homeomorphic to $\mathbb{R}^0 = \{0\}$. Consequently $\{p\} = U$ is an open subset of M for all $p \in M$, that is, M carries the discrete topology. Since there exists a countable basis for the topology on M and the topology is discrete in addition, M has to be countable itself.

Thus we get:

Proposition 1.1.6. *A topological space M is a 0-dimensional topological manifold, if and only if M is countable and carries the discrete topology.*

Definition 1.1.7. We call a topological manifold M **connected**, if for every two points $p, q \in M$ there exists a continuous map $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$.

Given two points, there has to be a continuous curve in M which connects both. Usually, in Topology one calls this *path-connected*, which is in the case of manifolds equivalent to being *connected*. We do not want to go deeper into this subject at this point.

Remark 1.1.8. Following Proposition 1.1.6 every connected 0-dimensional manifold M is given by a single point: $M = \{\text{point}\}$.

In dimension 1 there are only a few connected manifolds:

Proposition 1.1.9. *Every connected* 1*-dimensional topological manifold is homeomorphic to* $\mathbb{R}$ *or to* S^1.

A proof of this fact can be found in the appendix of [M65]. Thus, the only compact, connected topological manifold of dimension 1 is S^1.

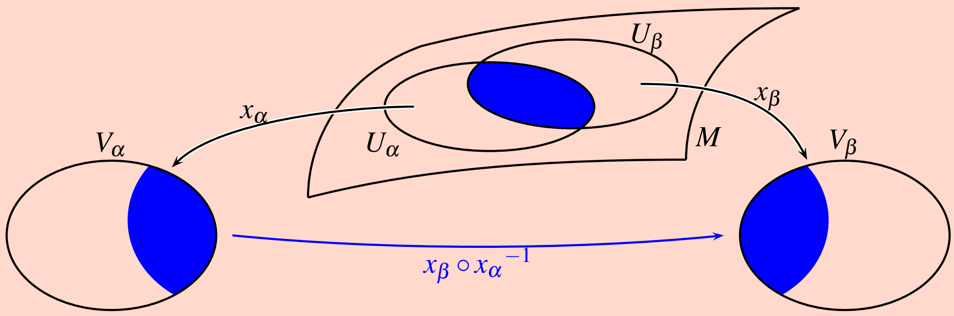
Theorem 1.1.10. *Let M and A be sets. For all $\alpha \in A$ assume that $U_\alpha \subset M$ and $V_\alpha \subset \mathbb{R}^n$ are subsets and that $x_\alpha : U_\alpha \to V_\alpha$ are bijective maps. Suppose the following holds:*

(i) $\bigcup_{\alpha \in A} U_\alpha = M$,

(ii) $x_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ *is open for all* $\alpha, \beta \in A$ *and*

(iii) $x_\beta \circ {x_\alpha}^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$ *is continuous for all* $\alpha, \beta \in A$.

Then M carries a unique topology for which all U_α are open sets and all x_α are homeomorphisms.



Proof. We first show *uniqueness:*
Let $\mathscr{O}$ be a topology on M containing the U_α and such that the x_α are homeomorphisms. If $W \in \mathscr{O}$, then also $W \cap U_\alpha \in \mathscr{O}$ and $x_\alpha(W \cap U_\alpha)$ is open for all $\alpha \in A$.

Conversely, if $W \subset M$ is a subset such that $x_\alpha(W \cap U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in A$, then $W \cap U_\alpha$ is also open in U_α for all α. Since U_α is open in M, the set $W \cap U_\alpha$ is open in M. By (i), $W = \bigcup_{\alpha \in A}(W \cap U_\alpha)$ is also open in M. We have shown that $W \in \mathscr{O}$ if and only if $x_\alpha(W \cap U_\alpha)$ is open in $\mathbb{R}^n$ for all α,

$$\mathscr{O} = \{W \subset M \,|\, x_\alpha(W \cap U_\alpha) \subset \mathbb{R}^n \text{ open for all } \alpha \in A\}.$$

Now we show *existence:*
We use the criterion for openness derived in the uniqueness part of the proof to define the topogoly. We set:

$$\mathscr{O} := \{W \subset M \,|\, x_\alpha(W \cap U_\alpha) \subset \mathbb{R}^n \text{ open for all } \alpha \in A\}.$$

Now we have to check that this $\mathscr{O}$ is a topology and that it has the desired properties:

(a) $\mathscr{O}$ is a topology because

  (i) The empty set $\emptyset$ is open in M because $x_\alpha(\emptyset \cap U_\alpha) = x_\alpha(\emptyset) = \emptyset$ is open in $\mathbb{R}^n$ for all α. Observe that the case $\alpha = \beta$ in (ii) shows that $V_\alpha = x_\alpha(U_\alpha)$ is open in $\mathbb{R}^n$. Now we see that $M \in \mathscr{O}$ because $x_\alpha(M \cap U_\alpha) = x_\alpha(U_\alpha) = V_\alpha$ is open in $\mathbb{R}^n$ for all α.

  (ii) Assume $W_i \in \mathscr{O}$ for $i \in I$. Then $\bigcup_{i \in I} W_i \in \mathscr{O}$ because

$$x_\alpha\left(\left(\bigcup_{i \in I} W_i\right) \cap U_\alpha\right) = x_\alpha\left(\bigcup_{i \in I}(W_i \cap U_\alpha)\right) = \bigcup_{i \in I} \underbrace{x_\alpha(W_i \cap U_\alpha)}_{\text{open in } \mathbb{R}^n}$$

  is open in $\mathbb{R}^n$ for all $\alpha \in A$.

  (iii) The conclusion $W_1, W_2 \in \mathscr{O} \Rightarrow W_1 \cap W_2 \in \mathscr{O}$ follows similarly.

(b) We have to show $U_\beta \in \mathscr{O}$ for all $\beta \in A$. This is obvious because $x_\alpha(U_\beta \cap U_\alpha) \subset \mathbb{R}^n$ is open for all $\alpha \in A$ by assumption.

(c) The map x_β is continuous for all $\beta \in A$ because:

  Let $Y \subset V_\beta$ be open. Then we have for all $\alpha \in A$:

$$\begin{aligned} x_\alpha(x_\beta{}^{-1}(Y) \cap U_\alpha) &= x_\alpha(x_\beta{}^{-1}(Y \cap x_\beta(U_\alpha \cap U_\beta))) \\ &= \underbrace{(x_\alpha \circ x_\beta{}^{-1})}_{\substack{=(x_\beta \circ x_\alpha{}^{-1})^{-1} \\ \text{continuous}}} \underbrace{(Y \cap \underbrace{x_\beta(U_\alpha \cap U_\beta)}_{\text{open}})}_{\text{open}} \text{ is open in } \mathbb{R}^n. \end{aligned}$$

  Thus $x_\beta{}^{-1}(Y) \in \mathscr{O}$.

(d) The map $x_\beta{}^{-1}$ is continuous because:

  Let $W \subset U_\beta$ be open. Then $W \in \mathscr{O}$. For all $\alpha \in A$ the set $x_\alpha(W \cap U_\alpha)$ is open, in particular for $\alpha = \beta$

  $(x_\beta{}^{-1})^{-1}(W) = x_\beta(W) = x_\beta(W \cap U_\beta)$ is an open set. □

Example 1.1.11 (Real-projective space). We define the *real-projective space* by

$$M = \mathbb{R}\mathrm{P}^n := \mathrm{P}(\mathbb{R}^{n+1}) := \{L \subset \mathbb{R}^{n+1} \,|\, L \text{ is one-dimensional vector-subspace }\}.$$

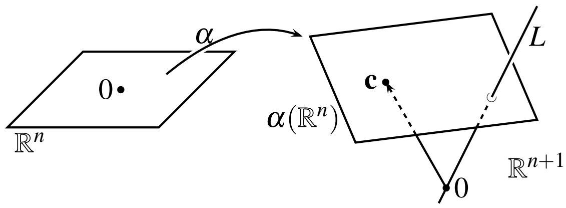
We will use Theorem 1.1.10 to equip $\mathbb{R}\mathrm{P}^n$ with the structure of an n-dimensional topological manifold. We set

$$A := \{\text{affine-linear embeddings } \alpha : \mathbb{R}^n \to \mathbb{R}^{n+1} \text{ with } 0 \notin \alpha(\mathbb{R}^n)\}.$$

Since α is affine-linear there exist a $B \in \mathrm{Mat}(n \times (n+1), \mathbb{R})$ and a $c \in \mathbb{R}^{n+1}$ such that

$$\alpha(x) = Bx + c$$

for all $x \in \mathbb{R}^n$. Since α is an embedding, B has maximal rank, $\mathrm{rank}(B) = n$.



Consequently, $\alpha(\mathbb{R}^n)$ is an affine-linear hyperplane. Set

$$U_\alpha := \{L \in \mathbb{R}\mathrm{P}^n \,|\, L \cap \alpha(\mathbb{R}^n) \neq \emptyset\}.$$

For $L \in U_\alpha$ the space $L \cap \alpha(\mathbb{R}^n)$ consists of exactly one point, because otherwise we would have $L \subset \alpha(\mathbb{R}^n)$ and hence $0 \in \alpha(\mathbb{R}^n)$, a contradiction. Moreover, we have

$$\mathbb{R}\mathrm{P}^n \setminus U_\alpha = \{L \,|\, L \subset B(\mathbb{R}^n) \text{ one-dimensional subspace}\} \tag{1.1}$$

where $\alpha(x) = Bx + c$. For $\alpha \in A$ set $V_\alpha := \mathbb{R}^n$ and

$$x_\alpha : U_\alpha \to V_\alpha, \quad x_\alpha(L) := \alpha^{-1}(L \cap \alpha(\mathbb{R}^n)).$$

Then x_α is a bijective map and we have

$${x_\alpha}^{-1}(v) = \mathbb{R} \cdot \alpha(v).$$

In the following we check the assumptions of Theorem 1.1.10:

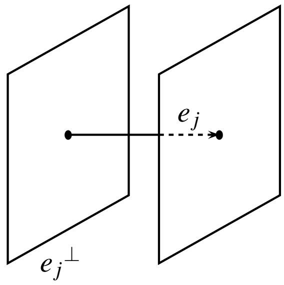
(i) We show: $\bigcup\limits_{\alpha \in A} U_\alpha = M$.

To this end, let $e_0, \dots, e_n \in \mathbb{R}^{n+1}$ be the standard basis. For $j = 1, \dots, n$ we define:

$$\alpha_j(v) := v^1 e_0 + \ldots + v^j e_{j-1} + e_j + v^{j+1} e_{j+1} + \ldots + v^n e_n.$$

Assume there existed an

$$L \in \mathbb{R}\mathrm{P}^n \setminus \bigcup_{j=0}^{n} U_{\alpha_j} \quad = \quad \bigcap_{j=0}^{n} (\mathbb{R}\mathrm{P}^n \setminus U_{\alpha_j}).$$

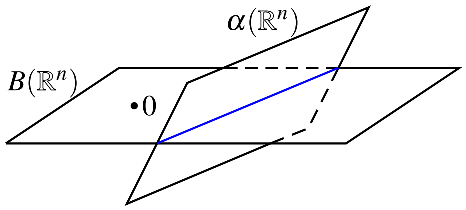


Then

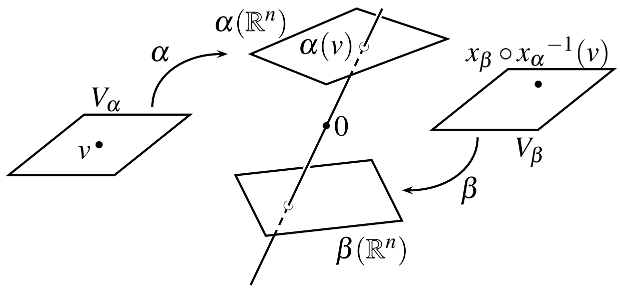
$$L \subset \bigcap_{j=0}^{n} e_j{}^\perp = \{0\}.$$

This is a contradiction, consequently $\bigcup_{j=0}^{n} U_{\alpha_j} = \mathbb{R}\mathrm{P}^n$ and hence $\bigcup_{\alpha \in A} U_\alpha = \mathbb{R}\mathrm{P}^n$.

(ii) We observe that $x_\alpha(U_\alpha \cap U_\beta)$ is the complement of an affine-linear subspace in $\mathbb{R}^n$. More precisely, by (1.1), $x_\alpha(U_\alpha \cap U_\beta) = \alpha^{-1}(\alpha(\mathbb{R}^n) \setminus B(\mathbb{R}^n))$ where we have written $\beta(x) = Bx + c$. Since affine-linear subspaces are closed, $x_\alpha(U_\alpha \cap U_\beta)$ is open.



(iii) We show that $x_\beta \circ x_\alpha{}^{-1} : v \mapsto \beta^{-1}(\mathbb{R} \cdot \alpha(v) \cap \beta(\mathbb{R}^n))$ is continuous for all $\alpha, \beta \in A$.



Write $\alpha(v) = Bv + c$ and $\beta(w) = Dw + f$. Now $w = x_\beta \circ x_\alpha{}^{-1}(v)$ is equivalent to $x_\beta^{-1}(w) = x_\alpha^{-1}(v)$, hence to $\mathbb{R} \cdot \beta(w) = \mathbb{R} \cdot \alpha(v)$. Therefore $w = x_\beta \circ x_\alpha{}^{-1}(v)$ is equivalent to the existence of $t \in \mathbb{R}$ such that $Dw + f = t \cdot (Bv + c)$. For the left-hand-side we write $Dw + f = (D, f) \begin{pmatrix} w \\ 1 \end{pmatrix}$. Note that (D, f) is an invertible $(n+1) \times (n+1)$-matrix because otherwise we could write f as a linear combination of the columns of D and hence 0 would lie in the image of β. Thus we get

$$\begin{pmatrix} w \\ 1 \end{pmatrix} = t \cdot (D, f)^{-1} \cdot (Bv + c). \tag{1.2}$$

Taking the scalar product with $e_{n+1} = (0, \cdots, 0, 1)^\top$ yields

$$1 = \left\langle e_{n+1}, \begin{pmatrix} w \\ 1 \end{pmatrix} \right\rangle = t \cdot \langle e_{n+1}, (D, f)^{-1} \cdot (Bv + c) \rangle. \tag{1.3}$$

Inserting (1.3) into (1.2) gives us

$$\begin{pmatrix} w \\ 1 \end{pmatrix} = \langle e_{n+1}, (D,f)^{-1} \cdot (Bv+c) \rangle^{-1} \cdot (D,f)^{-1} \cdot (Bv+c). \tag{1.4}$$

This shows that the components of w are rational functions of the components of v. In particular, they are continuous.

By Theorem 1.1.10, $\mathbb{R}\mathrm{P}^n$ has exactly one topology for which the U_α are open and the x_α are homeomorphisms. We still need criteria ensuring that this topology is Hausdorff and has a countable basis. Once we know this, we have turned $\mathbb{R}\mathrm{P}^n$ into an n-dimensional topological manifold.

Proposition 1.1.12 (First Addition to Theorem 1.1.10). *If in Theorem 1.1.10 there exists a countable subset $A_1 \subset A$ with*

$$\bigcup_{\alpha \in A_1} U_\alpha = M$$

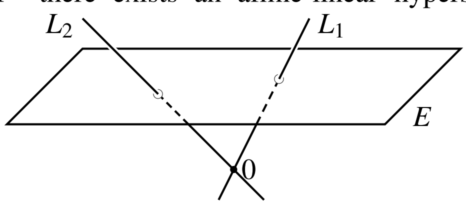
then the resulting topology has a countable basis.

Example 1.1.11 continued. For $\mathbb{R}\mathrm{P}^n$ the finite set $A_1 := \{\alpha_0, \dots, \alpha_n\}$ does the job. Consequently, the topology of $\mathbb{R}\mathrm{P}^n$ has a countable basis.

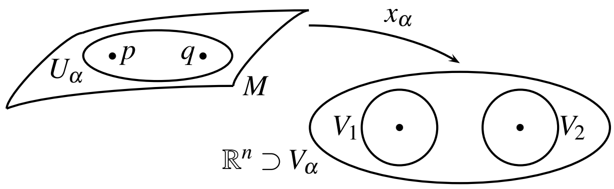
Proof of Proposition 1.1.12. The topology resulting from A has all the properties of the topology resulting from A_1. Since the topology is unique, A and A_1 give the same topology on M.
Without loss of generality we may therefore assume that $A_1 = A$ is countable. Now the topology of each $V_\alpha \subset \mathbb{R}^n$ has a countable basis $\mathscr{B}_\alpha$. Then ${x_\alpha}^{-1}(\mathscr{B}_\alpha)$ is a countable basis of the topogoly of U_α. Finally, $\bigcup_{\alpha \in A} {x_\alpha}^{-1}(\mathscr{B}_\alpha)$ is a countable basis of M. □

Proposition 1.1.13 (Second Addition to Theorem 1.1.10). *If in Theorem 1.1.10 for any two points $p, q \in M$ there is an $\alpha \in A$ such that $p, q \in U_\alpha$, then the topology of M is Hausdorff.*

Example 1.1.11 continued. For $L_1, L_2 \in \mathbb{R}\mathrm{P}^n$ there exists an affine-linear hypersurface E with $L_1 \cap E \neq \emptyset$ and $L_2 \cap E \neq \emptyset$. By Proposition 1.1.13, $\mathbb{R}\mathrm{P}^n$ is Hausdorff. Summarizing, we see that $\mathbb{R}\mathrm{P}^n$ is a n-dimensional topological manifold.



Proof of Proposition 1.1.13. Let $p, q \in M$ with $p \neq q$. Choose an $\alpha \in A$ with $p, q \in U_\alpha$. Since $\mathbb{R}^n$ is Hausdorff, we can choose $V_1, V_2 \subset V_\alpha$ open with $x_\alpha(p) \in V_1$, $x_\alpha(q) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then ${x_\alpha}^{-1}(V_1)$ and ${x_\alpha}^{-1}(V_2)$ separate p and q.



□

We summarize:

Corollary 1.1.14. *Let M and A be sets and let $A_1 \subset A$ be a countable subset. For all $\alpha \in A$ assume that $U_\alpha \subset M$ and $V_\alpha \subset \mathbb{R}^n$ are subsets and that $x_\alpha : U_\alpha \to V_\alpha$ are bijective maps. Suppose the following holds:*

(i) $\bigcup_{\alpha \in A_1} U_\alpha = M$*;*

(ii) $x_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ *is open for all* $\alpha, \beta \in A$*;*

(iii) $x_\beta \circ {x_\alpha}^{-1} : x_\alpha(U_\alpha \cap U_\beta) \to x_\beta(U_\alpha \cap U_\beta)$ *is continuous for all* $\alpha, \beta \in A$*;*

(iv) for any two points $p, q \in M$ *there is an* $\alpha \in A$ *such that* $p, q \in U_\alpha$*.*

Then M carries a unique topology which turns M into an n-dimensional topological manifold such that the $x_\alpha : U_\alpha \to V_\alpha$ are charts.

Example 1.1.15 (Complex-projective space). In complete analogy to the real-projective space we define *complex-projective space* by

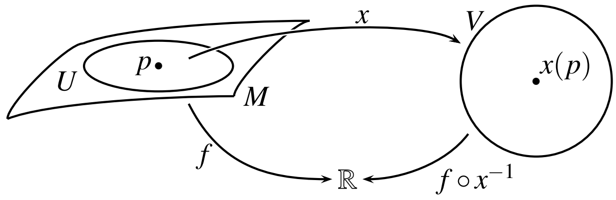
$$\mathbb{C}\mathrm{P}^n := \mathrm{P}(\mathbb{C}^{n+1}) := \{L \subset \mathbb{C}^{n+1} \,|\, L \text{ is one-dimensional complex subspace } \}.$$

Like in the real case we obtain charts $x_\alpha : U_\alpha \to \mathbb{C}^n = \mathbb{R}^{2n}$. This turns $\mathbb{C}\mathrm{P}^n$ into a $2n$-dimensional topological manifold.

1.2 Differentiable manifolds

For a topological manifold M, like for any topological space, it makes sense to speak about continuous functions $f : M \to \mathbb{R}$. In a course on differential geometry we will certainly need to differentiate functions. But *what does differentiability of f mean?*

Attempt of a definition. The function f is called *differentiable* at $p \in M$ if for some chart $x : U \to V$ with $p \in U$ the function $f \circ x^{-1} : V \to \mathbb{R}$ is differentiable in $x(p)$.



This is, in principle, a very reasonable definition. It means that f is differentiable on M if it is differentiable on $\mathbb{R}^n$ when expressed in coordinates. But there is a problem with this definition. If $y : \tilde{U} \to \tilde{V}$ is another chart with $p \in \tilde{U}$, then near $y(p)$ we have

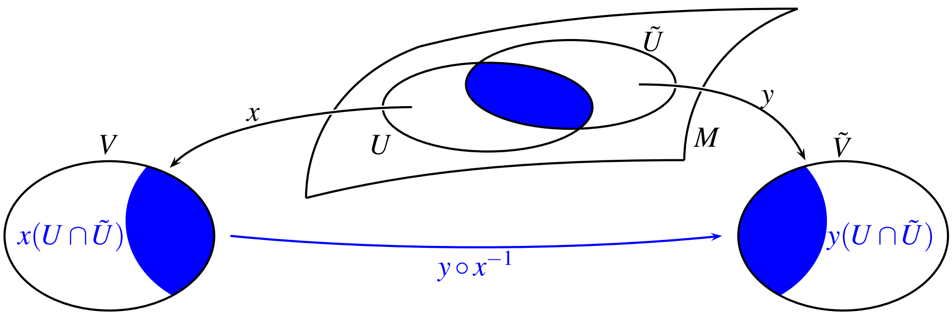
$$f \circ y^{-1} = \underbrace{(f \circ x^{-1})}_{\substack{\text{diff'able} \\ \text{at } x(p)}} \circ \underbrace{(x \circ y^{-1})}_{\substack{\text{only} \\ \text{continuous}}}.$$

This concept of differentiability depends on the choice of chart x and this is really bad because on a general topological manifold there are no preferred coordinate systems. The sad truth is that there is no reasonable concept of differentiable functions on a topological manifold.
But there is one thing we can do, we can refine the notion of a manifold. If $x \circ y^{-1}$ were a diffeomorphism and not only a homeomorphism, then the differentiability of $f \circ x^{-1}$ would imply the differentiability of $f \circ y^{-1}$. We enforce this by making the following definition.

Definition 1.2.1. Let M be an n-dimensional topological manifold. Two charts $x : U \to V$ and $y : \tilde{U} \to \tilde{V}$ of M are called **C^∞-compatible** if

$$y \circ x^{-1} : x(U \cap \tilde{U}) \to y(U \cap \tilde{U})$$

is a C^∞-diffeomorphism.



Definition 1.2.2. A set of charts $x_\alpha : U_\alpha \to V_\alpha$ of M, $\alpha \in A$, is called **atlas** of M, if

$$\bigcup_{\alpha \in A} U_\alpha = M.$$

An atlas $\mathscr{A}$ is called a **C^∞-atlas** if any two charts in $\mathscr{A}$ are C^∞-compatible.

Example 1.2.3. (1) Let $M = U \subset \mathbb{R}^n$ be open. Then $\mathscr{A} := \{\mathrm{id} : U \to U\}$ is a C^∞-atlas.

(2) Let $M = S^n$ and $\mathscr{A} := \{(x : U_1 \to V_1), (\tilde{x} : U_2 \to V_2)\}$, where $U_1 := S^n \setminus \{SP\}$, $U_2 := S^n \setminus \{NP\}$ and $V_1 := V_2 := \mathbb{R}^n$, compare Example 1.1.4.3. Furthermore, let

$$\begin{aligned} x(y) &= \frac{2}{1+y^0}\hat{y}, \quad \text{where } y = \left(y^0, \hat{y}\right) \in \mathbb{R}^{n+1}, \\ y(x) &= \frac{1}{4+\|x\|^2}\left(4-\|x\|^2, 4x\right) \quad \text{and} \\ \tilde{x}(y) &= \frac{2}{1-y^0}\hat{y}. \end{aligned}$$

Then we have for $v \in x(U_1 \cap U_2) = x(S^n \setminus \{SP, NP\}) = \mathbb{R}^n \setminus \{0\}$:

$$\begin{aligned} \tilde{x} \circ x^{-1}(v) &= \tilde{x}\left(\frac{4-\|v\|^2}{4+\|v\|^2}, \frac{4v}{4+\|v\|^2}\right) \\ &= \frac{2}{1-\frac{4-\|v\|^2}{4+\|v\|^2}} \frac{4v}{4+\|v\|^2} \\ &= \frac{8v}{4+\|v\|^2-4+\|v\|^2} \\ &= \frac{4v}{\|v\|^2}. \end{aligned}$$

Hence $\tilde{x} \circ x^{-1}$ is C^∞ on $\mathbb{R}^n \setminus \{0\} = x(S^n \setminus \{SP, NP\}) = x(U_1 \cap U_2)$. Similarly one sees that $x \circ \tilde{x}^{-1}$ is smooth. This shows that x and $\tilde{x}$ are C^∞-compatible. Hence $\mathscr{A}$ is a C^∞-atlas.

(3) Let $M = \mathbb{R}\mathrm{P}^n$, $\mathscr{A} := \{x_\alpha : U_\alpha \to \mathbb{R}^n \mid x_\alpha$ is an affine-linear embedding $\mathbb{R}^n \to \mathbb{R}^{n+1}$ of maximal rank and $0 \notin \alpha(\mathbb{R}^n)\}$, compare Example 1.1.11. All changes of charts $x_\beta \circ {x_\alpha}^{-1}$ are rational functions and hence C^∞. Therefore $\mathscr{A}$ is a C^∞-atlas.

(4) Analogously, for $M = \mathbb{C}\mathrm{P}^n$ as in Example 1.1.15, the resulting atlas is also a C^∞-atlas.

Remark 1.2.4. If $\mathscr{A}$ is a C^∞-atlas of M then

$$\mathscr{A}_{\max} := \{\text{charts } x \text{ of } M \mid x \text{ is } C^\infty\text{-compatible with all charts in } \mathscr{A}\}$$

also is a C^∞-atlas of M. The reason is this:
If x and $\tilde{x}$ are two charts of M, which are C^∞-compatible with all charts in $\mathscr{A}$, then also x and $\tilde{x}$ are C^∞-compatible with each other.
Namely, for any $p \in x(U \cap \tilde{U})$ there exists a chart $y : \tilde{\tilde{U}} \to \tilde{\tilde{V}}$ in $\mathscr{A}$ with $x^{-1}(p) \in \tilde{\tilde{U}}$. Near p we then have:

$$\tilde{x} \circ x^{-1} = \underbrace{\left(\tilde{x} \circ y^{-1}\right)}_{C^\infty} \circ \underbrace{\left(y \circ x^{-1}\right)}_{C^\infty}.$$

Hence $\tilde{x} \circ x^{-1}$ is C^∞ and similarly for $x \circ \tilde{x}^{-1}$.

Definition 1.2.5. An C^∞-atlas $\mathscr{A}_{\max}$ is called **maximal** (or also **differentiable structure**), if every chart that is C^∞-compatible with all charts in $\mathscr{A}_{\max}$, is already contained in $\mathscr{A}_{\max}$.

According to Remark 1.2.4, every C^∞-atlas $\mathscr{A}$ is contained in exactly one maximal C^∞-atlas $\mathscr{A}_{\max}$.

Definition 1.2.6. A pair $(M, \mathscr{A}_{\max})$, where M is an n-dimensional topological manifold and $\mathscr{A}_{\max}$ a differentiable structure on M, is called an n**-dimensional differentiable manifold**.

Definition 1.2.7. Let M and N be differentiable manifolds, let $p \in M$ and let $k \in \mathbb{N} \cup \{\infty\}$. A continuous map $f : M \to N$ is called k**-times continuously differentiable** (or C^k) **near** p, if for one (and therefore for every other) chart

$$(x : U \to V) \in \mathscr{A}_{\max}(M) \quad \text{with } p \in U$$

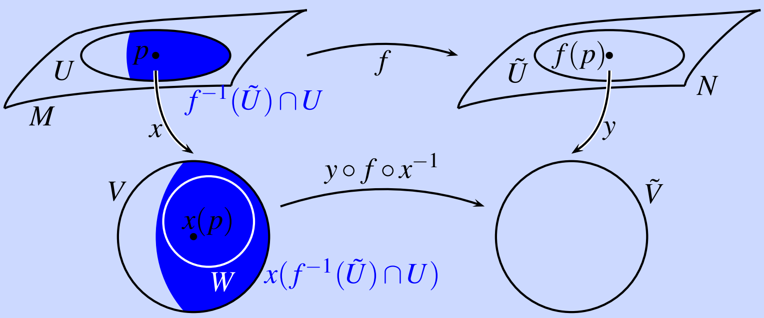
and for one (and therefore for every other) chart

$$(y : \tilde{U} \to \tilde{V}) \in \mathscr{A}_{\max}(N) \quad \text{with } f(p) \in \tilde{U}$$

there exists a neighborhood $W \subset x(f^{-1}(\tilde{U}) \cap U)$ of $x(p)$, such that

$$y \circ f \circ x^{-1} : x\big(f^{-1}(\tilde{U}) \cap U\big) \to \tilde{V}$$

is C^k on W.



Example 1.2.8. (1) Let $M = S^n$ with the differentiable structure given by

$$\mathscr{A} = \{(x : U_1 \to V_1), (\tilde{x} : U_2 \to V_2)\}$$

as in Example 1.2.3.2. We show that

$$f : S^n \to S^n, \quad f(y) = -y,$$

is C^∞ near NP. In fact, f is C^∞ on all of S^n. We compute

$$\begin{aligned}
\mathbb{R}^n \ni v \quad &\overset{x^{-1}}{\longmapsto} \quad x^{-1}(v) = \left(\frac{4-\|v\|^2}{4+\|v\|^2}, \frac{4v}{4+\|v\|^2} \right) \\
&\overset{f}{\longmapsto} \quad \left(-\frac{4-\|v\|^2}{4+\|v\|^2}, \frac{-4v}{4+\|v\|^2} \right) \\
&\overset{\tilde{x}}{\longmapsto} \quad -\frac{2}{1+\dfrac{4-\|v\|^2}{4+\|v\|^2}} \cdot \frac{4v}{4+\|v\|^2} = -\frac{8v}{8} = -v
\end{aligned}$$

Consequently, $\tilde{x} \circ f \circ x^{-1}(v) = -v$ and in particular $\tilde{x} \circ f \circ x^{-1}$ is C^∞ on $\mathbb{R}^n$. Thus, we may consider $W = \mathbb{R}^n$.

This argument shows that f is smooth near all points except SP because SP is the only point not contained in the chart U_1. Interchanging the two charts one sees similarly that f is also smooth near SP. Hence f is smooth on all of S^n.

(2) We consider the atlases $\mathscr{A}_1 := \{x = \mathrm{id} : \mathbb{R} \to \mathbb{R}\}$ on $M = \mathbb{R}$ with differentiable structure $\mathscr{A}_{1,\max}$ and $\mathscr{A}_2 := \{\tilde{x} : \mathbb{R} \to \mathbb{R}\}$ with $\tilde{x}(t) = t^3$ and differentiable structure $\mathscr{A}_{2,\max}$.

Now $\tilde{x} \circ x^{-1}(t) = t^3$ is C^∞, but $x \circ \tilde{x}^{-1}(t) = \sqrt[3]{t}$ is *not*.
Consequently, x and $\tilde{x}$ are *not* C^∞-compatible and therefore the differentiable structures are different:

$$\mathscr{A}_{1,\max} \neq \mathscr{A}_{2,\max}.$$

- Is $\mathrm{id} : (\mathbb{R}, \mathscr{A}_{1,\max}) \to (\mathbb{R}, \mathscr{A}_{2,\max})$ a C^∞-map?

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d, "x=\mathrm{id}"'] & \mathbb{R} \arrow[d, "\tilde{x}"] \\ \mathbb{R} \arrow[r, "t \mapsto t^3"] & \mathbb{R} \end{tikzcd} \quad \Rightarrow \quad \tilde{x} \circ \mathrm{id} \circ x^{-1} \text{ is } C^\infty \text{ and therefore also id.}$$

- Is $\mathrm{id} : (\mathbb{R}, \mathscr{A}_{2,\max}) \to (\mathbb{R}, \mathscr{A}_{1,\max})$ a C^∞-map?

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d, "\tilde{x}"'] & \mathbb{R} \arrow[d, "x=\mathrm{id}"] \\ \mathbb{R} \arrow[r, "t \mapsto \sqrt[3]{t}"] & \mathbb{R} \end{tikzcd} \quad \Rightarrow \quad x \circ \mathrm{id} \circ \tilde{x}^{-1} \text{ is } \textit{not } C^\infty \text{ and the same holds true for id.}$$

Summarizing we see that id is a homeomorphism from $(\mathbb{R}, \mathscr{A}_{1,\max})$ to $(\mathbb{R}, \mathscr{A}_{2,\max})$ which is smooth but its inverse is not.

Definition 1.2.9. Let M and N be differentiable manifolds. A homeomorphism $f : M \to N$ is called a C^k**-diffeomorphism**, if f and f^{-1} are both C^k. Instead of C^∞-diffeomorphism we simply say **diffeomorphism**. If there exists a diffeomorphism $f : M \to N$, we say that M and N are **diffeomorphic**.

Example 1.2.8.2 continued. Let $M = (\mathbb{R}, \mathscr{A}_{1,\max})$ with $\mathscr{A}_{1,\max} = \{x = \mathrm{id} : \mathbb{R} \to \mathbb{R}\}$ and $N = (\mathbb{R}, \mathscr{A}_{2,\max})$ with $\mathscr{A}_{2,\max} = \{\tilde{x} : \mathbb{R} \to \mathbb{R}, \tilde{x}(t) = t^3\}$. We have seen that $\mathrm{id} : M \to N$ is not a diffeomorphism. But $f : M \to N$, $f(t) = \sqrt[3]{t}$ is a diffeomorphism because

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "x"'] & N \arrow[d, "\tilde{x}"] \\ \mathbb{R} \arrow[r, "\mathrm{id}"] & \mathbb{R} \end{tikzcd}$$

Thus M and N are diffeomorphic.

Question. Is every differentiable structure on $\mathbb{R}^n$ diffeomorphic to the standard structure $\mathscr{A}_{\max}$, the one induced by $\mathscr{A} = \{x = \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n\}$?
The answer is quite surprising. For $n = 0, 1, 2, 3$ and also for $n \geq 5$ it is YES. But for $n = 4$ it turns out to be NO. There exist uncountably many differentiable structures on $\mathbb{R}^4$ which are pairwise not diffeomorphic (so-called *exotic structures*). The proof of these facts is far beyond the scope of our lecture course.

Remark 1.2.10. In 1956 John Milnor showed that there exist exotic n-dimensional spheres for $n \geq 7$. These are differentiable manifolds which are homeomorphic to S^n but not diffeomorphic. But in every dimension there are only finitely many.

1.3 Tangent vectors

Question. What is the *derivative* at a point of a differentiable map between differentiable manifolds**?**
The vague answer is: It is the linear approximation of the map at that point. But what do we mean by the *linear approximation* in a point of a differentiable manifold? For this to make sense we first need a concept of "linear approximation" of a manifold at a given point.

Definition 1.3.1. Let M be a differentiable manifold and $p \in M$.
A **tangent vector** on M at the point p is an equivalence class of differentiable curves $c : (-\varepsilon, \varepsilon) \to M$ with $\varepsilon > 0$ and $c(0) = p$, where two such curves $c_1 : (-\varepsilon_1, \varepsilon_1) \to M$ and $c_2 : (-\varepsilon_2, \varepsilon_2) \to M$ are called equivalent, if for a chart $x : U \to V$ with $p \in U$ we have:

$$\frac{d}{dt}(x \circ c_1)|_{t=0} = \frac{d}{dt}(x \circ c_2)|_{t=0}.$$

Remark 1.3.2. This definition does not depend on the choice of the chart $x : U \to V$. Namely, if $y : \tilde{U} \to \tilde{V}$ is another chart with $p \in \tilde{U}$ then we get by the chain rule

$$\frac{d}{dt}(y \circ c)|_{t=0} = \frac{d}{dt}\Big(\big(y \circ x^{-1}\big) \circ (x \circ c)\Big)|_{t=0} = D\big(y \circ x^{-1}\big)|_{x(p)}\left(\frac{d}{dt}(x \circ c)|_{t=0}\right). \tag{1.5}$$

Therefore the condition

$$\frac{d}{dt}(x \circ c_1)|_{t=0} = \frac{d}{dt}(x \circ c_2)|_{t=0}$$

is equivalent to the condition

$$\frac{d}{dt}(y \circ c_1)|_{t=0} = \frac{d}{dt}(y \circ c_2)|_{t=0}.$$

Notation 1.3.3. We denote the equivalence class of c by $\dot{c}(0)$.

Definition 1.3.4. The set

$$T_pM := \{\dot{c}(0) \,|\, c : (-\varepsilon, \varepsilon) \to M \text{ differentiable with } c(0) = p\}$$

is called **tangent space** of M at the point p.

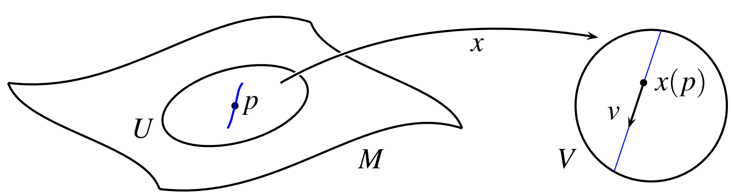
Lemma 1.3.5. *Let M be an n-dimensional differentiable manifold, let $p \in M$ and let $x : U \to V$ be a chart of M with $p \in U$. Then the map*

$$dx|_p : T_pM \to \mathbb{R}^n, \quad \dot{c}(0) \mapsto \frac{d}{dt}(x \circ c)|_{t=0},$$

is well defined and bijective.

Proof. Well-definedness and injectivity are clear from to the definition of the equivalence relation that defines $\dot{c}(0)$. To show surjectivity let $v \in \mathbb{R}^n$ and set $c(t) := x^{-1}(x(p) + tv)$. Choose $\varepsilon > 0$ so small that $x(p) + tv \in V$ whenever $|t| < \varepsilon$. Then we have

$$dx|_p(\dot{c}(0)) = \frac{d}{dt}\Big(x \circ x^{-1}\big(x(p) + tv\big)\Big)|_{t=0} = \frac{d}{dt}\big(x(p) + tv\big)|_{t=0} = v.$$



This shows surjectivity and concludes the proof. □

Definition 1.3.6. We equip T_pM with the unique vector space structure for which $dx|_p$ becomes a linear isomorphism. In other words, for $a, b \in \mathbb{R}$ and $c_1 : (-\varepsilon_1, \varepsilon_1) \to M$, $c_2 : (-\varepsilon_2, \varepsilon_2) \to M$ we set:

$$a \cdot \dot{c}_1(0) + b \cdot \dot{c}_2(0) := (dx|_p)^{-1}\Big(a \cdot dx|_p\big(\dot{c}_1(0)\big) + b \cdot dx|_p\big(\dot{c}_2(0)\big)\Big).$$

Lemma 1.3.7. *The vector space structure on T_pM does not depend on the choice of chart $x : U \to V$.*

Proof. Let $y : \tilde{U} \to \tilde{V}$ be another chart with $p \in \tilde{U}$. We have to show that the map $dy|_p : T_pM \to \mathbb{R}^n$ is also linear with respect to the vector space structure induced by x. This holds true since by (1.5)

$$dy|_p = \underbrace{D\big(y \circ x^{-1}\big)|_{x(p)}}_{\text{linear}} \circ \underbrace{dx|_p}_{\text{linear}}$$

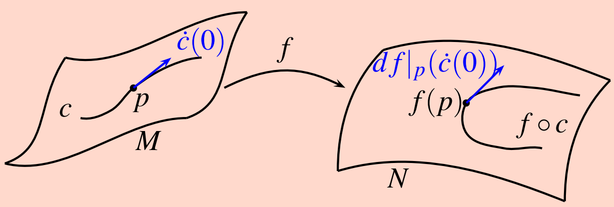
is the composition of two linear maps. □

We may think of the tangent space T_pM as the linear approximation to M at p. Now we can define the differential of a differentiable map between manifolds.

Lemma 1.3.8. *Let M and N be differentiable manifolds, let $p \in M$, and let $f : M \to N$ be differentiable near p. Then the map*

$$df|_p : T_pM \to T_{f(p)}N, \quad \dot{c}(0) \mapsto (f \circ c)\dot{}(0),$$

is well defined and linear.



Proof. We choose a chart $x : U \to V$ of M with $p \in U$ and a chart $y : \tilde{U} \to \tilde{V}$ of N with $f(p) \in \tilde{U}$.

We compute, using the chain rule,

$$\begin{aligned} dy|_{f(p)}((f\circ c)^{\cdot}(0)) &= (y\circ f\circ c)^{\cdot}(0) \\ &= \Big(\big(y\circ f\circ x^{-1}\big)\circ(x\circ c)\Big)^{\cdot}(0) \\ &= D\big(y\circ f\circ x^{-1}\big)|_{x(p)}\cdot\big((x\circ c)^{\cdot}(0)\big) \\ &= D\big(y\circ f\circ x^{-1}\big)|_{x(p)}\cdot dx|_p(\dot c(0)). \end{aligned}$$

Consequently, we have

$$df|_p = \big(dy|_{f(p)}\big)^{-1}\circ D(y\circ f\circ x^{-1})|_{x(p)}\circ dx|_p.$$

In particular, $df|_p$ is well defined (independently of the choice of c) and linear. □

Definition 1.3.9. The map $df|_p$ is called the **differential** of f at the point p.

Remark 1.3.10. If $U\subset M$ is an open subset, then the differential of the inclusion map $\iota : U\hookrightarrow M$ is the *canonical isomorphism* $d\iota : T_pU\to T_pM$, given by

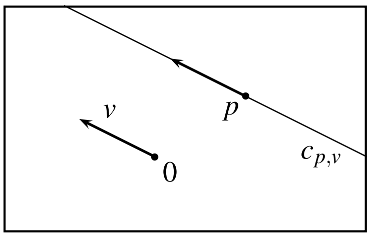
$$\dot c(0)\mapsto(\iota\circ c)^{\cdot}(0)=\dot c(0).$$

We will identify tangent spaces via this isomorphism and simply write $T_pU = T_pM$.

Remark 1.3.11. If M is an n-dimensional $\mathbb{R}$-vector space, then M and T_pM are *canonically* isomorphic via

$$\begin{aligned} M &\to T_pM, \\ v &\mapsto \dot c_{p,v}(0), \end{aligned}$$

where $c_{p,v}(t) := p+tv$.



Remark 1.3.12. For a chart $x : U\to V$ the differential $dx|_p$ has two meanings which are related by this canonical isomorphism. The following diagram commutes:

$$\begin{tikzcd}
\dot c(0) \arrow[rrr, mapsto] \arrow[dd, mapsto] \arrow[dr, phantom, "\in"] & & & (x\circ c)^{\cdot}(0) \arrow[d, phantom, "\in"] \\
& T_pU \arrow[rr, "dx|_p"] \arrow[d, "dx|_p"', "\cong"] & & T_{x(p)}V = T_{x(p)}\mathbb{R}^n \\
\frac{d}{dt}(x\circ c)|_{t=0} \arrow[r, phantom, "\in"] & \mathbb{R}^n \arrow[urr, "\cong", "\text{canonical isomorphism}"'] & &
\end{tikzcd}$$

Theorem 1.3.13 (Chain Rule). *Let M, N and P be differentiable manifolds and let $p \in M$. Assume $f : M \to N$ and $g : N \to P$ are differentiable near p and near $f(p)$, respectively. Then the following holds:*

$$d(g \circ f)|_p = dg|_{f(p)} \circ df|_p.$$

Proof. For a curve $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$ we have:

$$\begin{aligned} d(g \circ f)|_p(\dot{c}(0)) &= \frac{d}{dt}\big((g \circ f) \circ c\big)|_{t=0} \\ &= \frac{d}{dt}\big(g \circ (f \circ c)\big)|_{t=0} \\ &= dg|_{f(p)}\big((f \circ c)\dot{}(0)\big) \\ &= dg|_{f(p)}\big(df|_p\big(\dot{c}(0)\big)\big). \end{aligned}$$ □

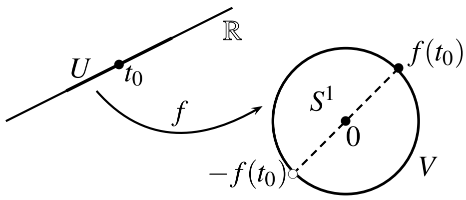
This proof of the chain rule was very simple. One may wonder why the proof of the chain rule that one remembers from one's course on calculus of several variables required a lot more work. The reason for the simplicity here is that one has already built the chain rule into the definition of the differential of a map.

Definition 1.3.14. Let M and N be differentiable manifolds. Let $k \in \mathbb{N} \cup \{\infty\}$. A surjective C^k-map $f : M \to N$ is called a **local C^k-diffeomorphism**, if for all $p \in M$ there exists an open neighborhood U of p in M and an open neighborhood V of $f(p)$ in N, such that

$$f|_U : U \to V$$

is a C^k-diffeomorphism.

Example 1.3.15. Let $f : \mathbb{R} \to \mathrm{S}^1$, $f(t) = e^{it}$. Then f is not injective (in particular, not a diffeomorphism), but it is a local diffeomorphism:
For $t_0 \in \mathbb{R}$ choose $U := (t_0 - \pi, t_0 + \pi)$ and $V := \mathrm{S}^1 \setminus \{-f(t_0)\}$.



Remark 1.3.16. If $f : M \to N$ is a local C^k-diffeomorphism, then

$$df|_p : T_pM \to T_{f(p)}N$$

is an isomorphism. In particular, we have $\dim(T_pM) = \dim(T_{f(p)}N)$ and therefore also $\dim M = \dim N$.

Proof. W.l o.g. let f be a C^k-diffeomorphism. For a curve $c:(-\varepsilon,\varepsilon)\to M$ with $c(0)=p$ we have:

$$d(\mathrm{id}_M)|_p\big(\dot c(0)\big)=(\mathrm{id}_M\circ c)^{\dot{}}(0)=\dot c(0)$$

and hence

$$d(\mathrm{id}_M)_p=\mathrm{id}_{T_pM}.$$

Applying the chain rule we find:

$$\mathrm{id}_{T_pM}=d(\mathrm{id}_M)|_p=d\big(f^{-1}\circ f\big)|_p=df^{-1}|_{f(p)}\circ df|_p.$$

Analogously, we can derive $df|_p\circ df^{-1}|_{f(p)}=\mathrm{id}_{T_{f(p)}N}$. Therefore we get:

$$df^{-1}|_{f(p)}=(df|_p)^{-1}.$$ □

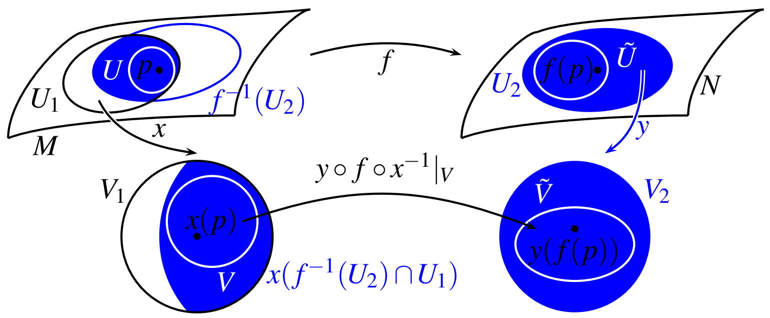
The converse of the last statement is also true:

Theorem 1.3.17 (Inverse Function Theorem). *Let M and N be differentiable manifolds and let $p\in M$. Let $f:M\to N$ be a C^k-map, $k\geq 1$.*
If $df|_p:T_pM\to T_{f(p)}N$ is an isomorphism, then there exists an open neighborhood U of p in M and an open neighborhood $\tilde U$ of $f(p)$ in N, such that

$$f|_U:U\to\tilde U$$

is a C^k-diffeomorphism.

Proof. Choose a chart $x:U_1\to V_1$ of M with $p\in U_1$ and a chart $y:U_2\to V_2$ of N with $f(p)\in U_2$.



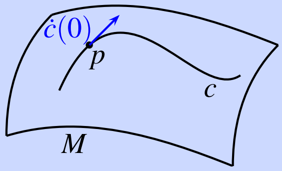
On $x(U_1\cap f^{-1}(U_2))$ the map $y\circ f\circ x^{-1}$ is defined. Since $df|_p$ is invertible, we also have that $D(y\circ f\circ x^{-1})|_{x(p)}$ is invertible.
The "classical" inverse function theorem says that there exists an open neighborhood $V\subset x(U_1\cap f^{-1}(U_2))$ of $x(p)$ and an open neighborhood $\tilde V\subset V_2$ of $y(f(p))$, such that

$$y\circ f\circ x^{-1}|_V:V\to\tilde V$$

is a C^k-diffeomorphism. With $U := x^{-1}(V)$ and $\tilde{U} := y^{-1}(\tilde{V})$ it follows that $f|_U : U \to \tilde{U}$ is a C^k-diffeomorphism. □

1.4 Directional derivatives and derivations

Definition 1.4.1.
Let M be a differentiable manifold, let $p \in M$ and and let $\dot{c}(0) \in T_pM$. For a function $f : M \to \mathbb{R}$, differentiable near p, we call



$$\partial_{\dot{c}(0)} f := df|_p(\dot{c}(0)) = \frac{d}{dt}(f \circ c)|_{t=0} \in \mathbb{R}$$

the **directional derivative** of f in the direction $\dot{c}(0)$.

Notation 1.4.2. For $U \subset M$ open and $k \in \mathbb{N} \cup \{\infty\}$, we write

$$C^k(U) := \{f : U \to \mathbb{R} \,|\, f \text{ is } C^k\}.$$

For $\alpha \in \mathbb{R}$, $f \in C^k(U)$ and $g \in C^k(\tilde{U})$ we set

$$\begin{aligned} \alpha \cdot f &\in C^k(U), & (\alpha \cdot f)(q) &:= \alpha \cdot f(q) \\ f + g &\in C^k(U \cap \tilde{U}), & (f+g)(q) &:= f(q) + g(q) \\ f \cdot g &\in C^k(U \cap \tilde{U}), & (f \cdot g)(q) &:= f(q) \cdot g(q) \end{aligned}$$

and

$$C_p^\infty := \bigcup_{\substack{U \text{ open} \\ p \in U}} C^\infty(U).$$

Definition 1.4.3. A map $\partial : C_p^\infty \to \mathbb{R}$ is called **derivation** at p if the following conditions are satisfied:

(i) *Locality:* If $\tilde{U} \subset U$ is open, $p \in \tilde{U}$, $f \in C^\infty(U)$, then

$$\partial f = \partial(f|_{\tilde{U}}).$$

(ii) *Linearity:* If $\alpha, \beta \in \mathbb{R}$, $f, g \in C_p^\infty$, then

$$\partial(\alpha f + \beta g) = \alpha \partial f + \beta \partial g.$$

(iii) *Leibniz Rule:* For $f, g \in C_p^\infty$ we have

$$\partial(f \cdot g) = \partial f \cdot g(p) + f(p) \cdot \partial g.$$

Example 1.4.4. (1) Let $M = \mathbb{R}^n$ and $p \in M$. Then $\partial = \frac{\partial}{\partial x^i}\Big|_p$ is a derivation.

(2) Let M be an arbitrary differentiable manifold, let $p \in M$ and $\dot{c}(0) \in T_pM$. Then $\partial_{\dot{c}(0)}$ is a derivation. We check (iii):

$$\begin{aligned}
\partial_{\dot{c}(0)}(f \cdot g) &= \frac{d}{dt}\big((f \cdot g) \circ c\big)|_{t=0} \\
&= \frac{d}{dt}\big((f \circ c) \cdot (g \circ c)\big)|_{t=0} \\
&= \frac{d}{dt}(f \circ c)|_{t=0} \cdot g\big(c(0)\big) + f(c(0)) \cdot \frac{d}{dt}(g \circ c)|_{t=0} \\
&= \partial_{\dot{c}(0)} f \cdot g(p) + f(p) \cdot \partial_{\dot{c}(0)} g.
\end{aligned}$$

The other two conditions are even simpler to verify.

Remark 1.4.5. The set $\mathrm{Der}(C_p^\infty)$ of all derivations at p forms an $\mathbb{R}$-vector space via

$$(\alpha\partial_1 + \beta\partial_2)(f) = \alpha\partial_1 f + \beta\partial_2 f.$$

Lemma 1.4.6. *The map* $\partial_\cdot : T_pM \to \mathrm{Der}(C_p^\infty)$, $\dot{c}(0) \mapsto \partial_{\dot{c}(0)}$, *is linear.*

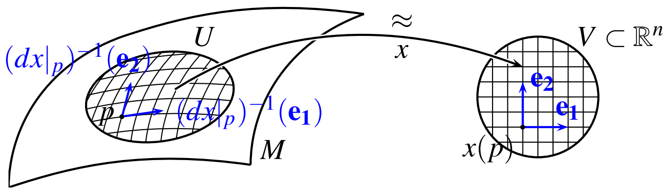
Proof. Let $x : U \to V$ be a chart of M with $p \in U$. By the definition of the vector space structure on T_pM, we have to show that $\partial \circ (dx|_p)^{-1}$ is linear. Assume $v \in \mathbb{R}^n$ and put $c(t) := x^{-1}\big(x(p) + tv\big)$. We find:

$$\begin{aligned}
(\partial_\cdot \circ (dx|_p)^{-1}(v))(f) &= df|_p\big((dx|_p)^{-1}(v)\big) \\
&= df|_p\big(\dot{c}(0)\big) \\
&= \frac{d}{dt}\big(f \circ c(t))\big)|_{t=0} \\
&= \frac{d}{dt}\big(f \circ x^{-1}(x(p) + tv)\big)|_{t=0} \\
&= \big\langle \mathrm{grad}\big(f \circ x^{-1}\big)|_{x(p)}, v\big\rangle.
\end{aligned}$$

This expression is linear in v. $\square$

Remark 1.4.7. Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$. Then $(dx|_p)^{-1}(e_1),\dots,(dx|_p)^{-1}(e_n)$ form a basis of T_pM. We find

$$\partial_{(dx|_p)^{-1}(e_j)}(f) = \langle \mathrm{grad}\big(f\circ x^{-1}\big)|_{x(p)}, e_j\rangle = \frac{\partial(f\circ x^{-1})}{\partial x^j}\bigg|_{x(p)} =: \frac{\partial f}{\partial x^j}\bigg|_p$$



For every chart x we have the derivations

$$\frac{\partial}{\partial x^1}\bigg|_p,\dots,\frac{\partial}{\partial x^n}\bigg|_p.$$

Proposition 1.4.8. *Let M be a differentiable manifold and let $p\in M$. Then the map*

$$\partial_\cdot : T_pM \to \mathrm{Der}(C_p^\infty),\ \dot c(0)\mapsto \partial_{\dot c(0)},$$

is an isomorphism. In particular, every derivation is a directional derivative and for every chart $x: U\to V$ with $p\in U$

$$\frac{\partial}{\partial x^1}\bigg|_p,\dots,\frac{\partial}{\partial x^n}\bigg|_p$$

is a basis of $\mathrm{Der}(C_p^\infty)$.

Proof. It suffices to show that the derivations

$$\frac{\partial}{\partial x^1}\bigg|_p,\dots,\frac{\partial}{\partial x^n}\bigg|_p$$

form a basis of $\mathrm{Der}(C_p^\infty)$. Namely, then we know that the linear map $\partial_\cdot$ maps the basis $(dx|_p)^{-1}(e_1),\dots,(dx|_p)^{-1}(e_n)$ of T_pM onto the basis $\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^n}\big|_p$ of $\mathrm{Der}(C_p^\infty)$ and is hence an isomorphism.

a) *Linear Independence:* Let $\sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}\Big|_p = 0$. We have to show: $\alpha^1 = \ldots = \alpha^n = 0$. Choose

$f = x^j$. Then

$$0 = \sum_{i=1}^{n} \alpha^i \underbrace{\left.\frac{\partial x^j}{\partial x^i}\right|_p}_{=\delta_i^j} = \alpha^j \text{ for } j = 1, \ldots, n.$$

b) *Generating Property:* Let $\delta \in \mathrm{Der}(C_p^\infty)$. Set $\alpha^j := \delta(x^j)$ for $j = 1 \ldots, n$. We will show that

$$\delta = \sum_{j=1}^{n} \alpha^j \cdot \left.\frac{\partial}{\partial x^j}\right|_p.$$

b1) We have

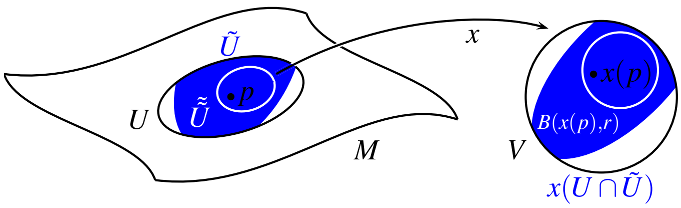
$$\delta(1) = \delta(1 \cdot 1) \overset{\textbf{(iii)}}{=} \delta(1) \cdot 1 + 1 \cdot \delta(1) = 2\delta(1)$$

and hence $\delta(1) = 0$. Now let $\alpha \in \mathbb{R}$. Then we find

$$\delta(\alpha) = \delta(\alpha \cdot 1) \overset{\textbf{(ii)}}{=} \alpha \cdot \delta(1) = 0.$$

Consequently, derivations vanish on all constant functions.

b2) Let $f \in C_p^\infty$, more precisely $f \in C^\infty(\tilde{U})$ with $p \in \tilde{U}$ open. Choose a neighborhood $\tilde{\tilde{U}}$ of p with $\tilde{\tilde{U}} \subset \tilde{U} \cap U$ and $x(\tilde{\tilde{U}}) = B(x(p), r)$.



Lemma 1.4.9 (see below) with $h = f \circ x^{-1}$ says that there exist $g_1, \ldots, g_n \in C^\infty(B(x(p), r))$ such that

$$\left(f \circ x^{-1}\right)(x) = \left(f \circ x^{-1}\right)(x(p)) + \sum_{i=1}^{n} \left(x^i - x^i(p)\right) \cdot g_i(x) \quad \text{and}$$
$$\frac{\partial (f \circ x^{-1})}{\partial x^i}(x(p)) = g_i\big(x(p)\big).$$

It follows that

$$
\begin{aligned}
\delta(f) &\overset{\text{(i)}}{=} \delta(f|_{\tilde{U}}) \\
&= \delta\Big(f(p) + \sum_{i=1}^{n} \big(x^i - x^i(p)\big)(g_i \circ x)\Big) \\
&\overset{\substack{\text{(ii)} \\ \text{(b1)}}}{=} \sum_{i=1}^{n} \delta\Big(\big(x^i - x^i(p)\big)(g_i \circ x)\Big) \\
&\overset{\text{(iii)}}{=} \sum_{i=1}^{n} \Big(\delta\big(x^i - x^i(p)\big) \cdot g_i\big(x(p)\big) + \underbrace{\big(x^i - x^i(p)\big)|_p}_{=0} \delta(g_i \circ x)\Big) \\
&\overset{\substack{\text{(ii)} \\ \text{(b1)}}}{=} \sum_{i=1}^{n} \delta\big(x^i\big) g_i\big(x(p)\big) \\
&= \sum_{i=1}^{n} \alpha^i \cdot \frac{\partial f}{\partial x^i}\bigg|_p .
\end{aligned}
$$

□

Lemma 1.4.9. *Let $h \in C^\infty(B(q,r))$. Then there exist $g_1, \ldots, g_n \in C^\infty(B(q,r))$ with*

(i) $h(x) = h(q) + \sum_{i=1}^{n} \big(x^i - q^i\big) g_i(x)$ *and*

(ii) $\dfrac{\partial h}{\partial x^i}(q) = g_i(q)$.

Proof. For $x \in B(q,r)$ set $w_x : [0,1] \to \mathbb{R}$, $w_x(t) := h(tx + (1-t)q)$. It follows that

$$
\begin{aligned}
h(x) - h(q) &= w_x(1) - w_x(0) \\
&= \int_0^1 \dot{w}_x(t)\, dt \\
&= \int_0^1 \sum_{i=1}^{n} \frac{\partial h}{\partial x^i}\bigg|_{tx+(1-t)q} \cdot (x^i - q^i)\, dt \\
&= \sum_{i=1}^{n} (x^i - q^i) \underbrace{\int_0^1 \frac{\partial h}{\partial x^i}\bigg|_{tx+(1-t)q} dt}_{=:\, g_i(x)}
\end{aligned}
$$

With this definition of the g_i, (i) holds. Moreover, (ii) follows from (i) by differentiation at q. □

At this point we have the following situation for a differentiable manifold:

$$\begin{tikzcd}
(dx|_p)^{-1}(e_j) \in T_pM \arrow[rr, "\partial."] \arrow[rr, phantom, "\cong"', shift right=2] \arrow[dr, "dx|_p"', "\cong"] & & \mathrm{Der}(C_p^\infty) \ni \left.\frac{\partial}{\partial x^i}\right|_p \\
& \underbrace{\mathbb{R}^n}_{\ni e_j} \arrow[ur, "\partial \circ (dx|_p)^{-1}"', "\cong"] &
\end{tikzcd}$$

depend on
the choice of x

From now on we identify T_pM with $\mathrm{Der}(C_p^\infty)$ via the isomorphism $\partial.$. For example, we write for $\xi \in T_pM$

$$\xi = \sum_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$$

instead of $\partial_\xi = \sum\limits_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$ and $\xi = \sum\limits_{i=1}^{n} \xi^i (dx|_p)^{-1}(e_i)$ where $(\xi^1, \dots, \xi^n)^\top = dx|_p(\xi)$.

Question. How do the coefficients $\xi^1, \dots, \xi^n$ of a tangent vector change, if we replace the chart x by another chart y?

Let $\xi \in T_pM$, let x and y be charts, both containing p. We express ξ with respect to both charts

$$\xi = \sum_{i=1}^{n} \xi^i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{j=1}^{n} \eta^j \left.\frac{\partial}{\partial y^j}\right|_p .$$

Now we want to compute the coefficients ξ^i in terms of the η^j and vice versa. Using the Chain Rule (Theorem 1.3.13) we compute

$$\begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} = dx|_p(\xi) = (dx|_p)\left((dy|_p)^{-1}\begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix}\right) = D(x \circ y^{-1})|_{y(p)} \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix}.$$

Interchanging the roles of x and y, we also get $\begin{pmatrix} \eta^1 \\ \vdots \\ \eta^n \end{pmatrix} = D(y \circ x^{-1})|_{x(p)} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}$. Thus

$$\boxed{\eta^j = \sum_{i=1}^{n} \left.\frac{\partial(y^j \circ x^{-1})}{\partial x^i}\right|_{x(p)} \cdot \xi^i} \tag{1.6}$$

In the physics literature this transformation rule is put at the heart of the definition of a tangent vector, then usually called a *contravariant vector*. For a physicist, a contravariant vector is a vector $(\xi^1, \dots, \xi^n)$ associated to a chart which transforms as in (1.6) when the chart is changed. We have now understood that this vector is the coefficient vector of an (abstractly defined) tangent vector with respect to the basis $\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p$ of T_pM induced by the chart x.

Let us look at the special case $\xi = \frac{\partial}{\partial x^i}\big|_p$, that is, $(\xi^1, \dots, \xi^n)^\top = e_i$. By (1.6), we get

$$\begin{aligned}
\frac{\partial}{\partial x^i}\bigg|_p &= \sum_{j=1}^n \eta^j \frac{\partial}{\partial y^j}\bigg|_p \\
&= \sum_{j=1}^n \sum_{k=1}^n \xi^k \frac{\partial(y^j \circ x^{-1})}{\partial x^k}\bigg|_{x(p)} \cdot \frac{\partial}{\partial y^j}\bigg|_p \\
&= \sum_{j=1}^n \frac{\partial(y^j \circ x^{-1})}{\partial x^i}\bigg|_{x(p)} \cdot \frac{\partial}{\partial y^j}\bigg|_p,
\end{aligned}$$

hence

$$\boxed{\frac{\partial}{\partial x^i}\bigg|_p = \sum_{j=1}^n \frac{\partial(y^j \circ x^{-1})}{\partial x^i}\bigg|_{x(p)} \cdot \frac{\partial}{\partial y^j}\bigg|_p} \tag{1.7}$$

In the physics literature it is customary to use the *Einstein summation convention* meaning that when an index appears twice in an expression, once as an upper index and once as a lower index, then summation over this index is understood. So (1.7) would be written as

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial(y^j \circ x^{-1})}{\partial x^i}\bigg|_{x(p)} \cdot \frac{\partial}{\partial y^j}\bigg|_p$$

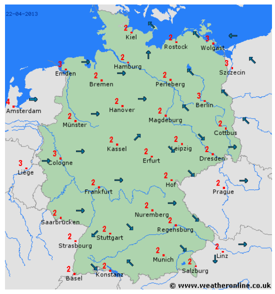
or even shorter as

$$\frac{\partial}{\partial x^i} = \frac{\partial y^j}{\partial x^i} \cdot \frac{\partial}{\partial y^j}.$$

This makes formula (1.7) easy to memorize; we simply cancel ∂y^j. In these lecture notes we will not use the Einstein summation convention unless explicitly stated otherwise. But when you do computations for yourself, the Einstein summation convention can be quite convenient and is recommended as long as you are aware of it.

1.5 Vector fields

Next we want to introduce vector fields. Vector fields are maps which associate to each point of a manifold a tangent vector in the corresponding tangent space. Hence the target space is varying and depends on the point. For this reason we first need to introduce the *tangent bundle*.


[1]

Definition 1.5.1. Let M be a differentiable manifold. Then we call

$$TM := \bigcup_{p \in M} T_pM$$

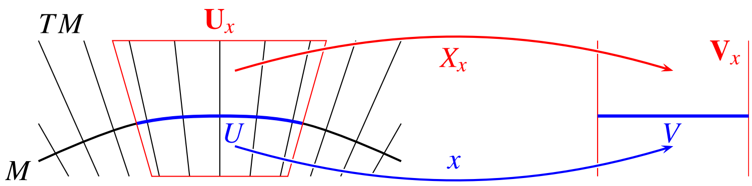
the **tangent bundle** of M.

We equip TM with the structure of a differentiable manifold. Denote the differentiable structure of M by $\mathscr{A}_{M,\max}$. Let $\pi : TM \to M$, $\pi(\xi) = p$ for $\xi \in T_pM$ be the "footpoint map". For every chart $x : U \to V$ in $\mathscr{A}_{M,\max}$ we construct a chart $X_x : \mathbf{U}_x \to \mathbf{V}_x$ of TM as follows: We set

$$\begin{aligned} \mathbf{U}_x &:= \pi^{-1}(U) \subset TM, \\ \mathbf{V}_x &:= V \times \mathbb{R}^n \subset \mathbb{R}^{2n} \quad \text{and} \\ X_x(\xi) &:= \Big(x\big(\pi(\xi)\big), dx|_{\pi(\xi)}(\xi)\Big). \end{aligned}$$

Then we have $X_x^{-1}(v,w) = (dx|_{x^{-1}(v)})^{-1}(w)$.

Schematic picture:



By construction we have:

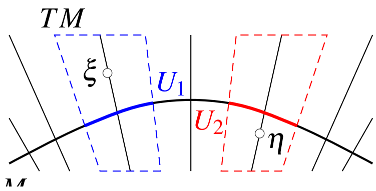
$$\bigcup_{\substack{(x:U\to V)\\ \in \mathscr{A}_{M,\max}}} \mathbf{U}_x = TM.$$

[1]Source: http://www.weatheronline.co.uk

Let x and y be charts on M. Then we have:

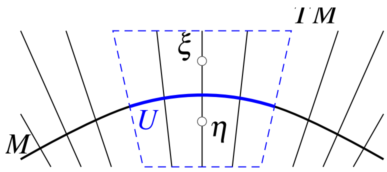
$$\begin{aligned}
X_y \circ X_x{}^{-1}(v,w) &= X_y\big((dx|_{x^{-1}(v)})^{-1}(w)\big) \\
&= \Big(y\big(\pi\big((dx|_{x^{-1}(v)})^{-1}(w)\big)\big), dy|_{\pi((dx|_{x^{-1}(v)})^{-1}(w))}\big((dx|_{x^{-1}(v)})^{-1}(w)\big)\Big) \\
&= \Big(y\big(x^{-1}(v)\big), dy|_{x^{-1}(v)}\big((dx|_{x^{-1}(v)})^{-1}(w)\big)\Big) \\
&= \Big(\big(\underbrace{y\circ x^{-1}}_{C^\infty}\big)(v), \underbrace{D\big(y\circ x^{-1}\big)|_v}_{C^\infty}\cdot w\Big).
\end{aligned}$$

Hence $X_y \circ X_x{}^{-1}$ is a C^∞-diffeomorphism, in particular, it is a homeomorphism. By Theorem 1.1.10, TM carries exactly one topology, for which the X_x are homeomorphisms.
We show: *The topology of TM has a countable basis.* Since the topology of M has a countable basis, M has a countable C^∞-atlas. Then the corresponding (countably many) charts of TM suffice to cover TM. By Proposition 1.1.12 the topology of TM has a a countable basis.
We show: *TM is Hausdorff.* Let $\xi, \eta \in TM$ with $\xi \neq \eta$. We consider two cases.
Case 1: $\pi(\xi) \neq \pi(\eta)$.
Since M is Hausdorff there exists an open neighborhood U_1 of $\pi(\xi)$ and an open neighborhood U_2 of $\pi(\eta)$ such that $U_1 \cap U_2 = \emptyset$. The sets $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are open neighborhoods of ξ and η with



$$\pi^{-1}(U_1) \cap \pi^{-1}(U_2) = \emptyset.$$

Case 2: $\pi(\xi) = \pi(\eta)$.
Let $x : U \to V$ be a chart of M with $\pi(\xi) = \pi(\eta) \in U$. Then we have $\xi, \eta \in \pi^{-1}(U) = \mathbf{U}_x$. The proof of Proposition 1.1.13 shows that we can separate ξ and η.



We summarize: The tangent bundle TM carries a unique topology turning it into a $2n$-dimensional topological manifold with atlas

$$\mathscr{A}_{TM} = \{X_x : \mathbf{U}_x \to \mathbf{V}_x \,|\, (x : U \to V) \in \mathscr{A}_{M,\max}\}.$$

Since the changes of charts $X_x \circ X_y{}^{-1}$ are not only homeomorphisms but C^∞-diffeomorphisms, we find that $\mathscr{A}_{TM}$ is a C^∞-atlas. Hence $(TM, \mathscr{A}_{TM,\max})$ becomes a $2n$-dimensional differentiable manifold.

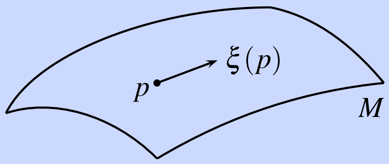
Remark 1.5.2. The footpoint map $\pi : TM \to M$ is expressed in the charts $x : U \to V$ of M and $X_x : \mathbf{U}_x \to \mathbf{V}_x$ of TM by

$$x \circ \pi \circ X_x^{-1} : V \times \mathbb{R}^n \to V, \quad (v,w) \mapsto v.$$

In particular, π is a smooth map.

Definition 1.5.3. A map $\xi : M \to TM$ is called a **vector field** on M, if for every $p \in M$ we have

$$\pi\big(\xi(p)\big) = p.$$



Remark 1.5.4. Let $x : U \to V$ be a chart of M. A vector field ξ on U is characterized by coefficient functions

$$\xi^1, \dots, \xi^n : V \to \mathbb{R}$$

for which

$$\xi(p) = \sum_{i=1}^{n} \xi^i\big(x(p)\big) \left.\frac{\partial}{\partial x^i}\right|_p .$$

Since a vector field is a map from the differentiable manifold M to the differentiable manifold TM we know what it means that the vector field is C^k. We investigate how this can be characterized in terms of the coefficient functions. For the chart x of M we consider the corresponding chart X_x on TM. The commutative diagram

$$\begin{array}{rcccl}
 & M & \xrightarrow{\xi} & TM & \\
 & \cup & & \cup & \\
x^{-1}(v) \in & U & \xrightarrow{\xi|_U} & \mathbf{U}_x & \ni \xi(x^{-1}(v)) \\
 & \downarrow x & & \downarrow X_x & \\
v \in & V & \longrightarrow & V \times \mathbb{R}^n & \ni \Big(x\big(\overbrace{\pi\big(\xi(x^{-1}(v))\big)}^{=x^{-1}(v)}\big), dx|_{\pi\left(\xi(x^{-1}(v))\right)}\big(\xi(x^{-1}(v))\big)\Big) \\
 & \cap & & \cap & = \big(v, \xi^1(v), \dots, \xi^n(v)\big) \\
 & \mathbb{R}^n & & \mathbb{R}^{2n} &
\end{array}$$

shows that ξ corresponds in these coordinates to the map $v \mapsto \big(v, \xi^1(v), \dots, \xi^n(v)\big)$. Thus ξ is C^k on U if and only if the coefficient functions $\xi^1, \dots, \xi^n$ are C^k on V.

Example 1.5.5. We consider $M = \mathbb{R}^2$ with *polar coordinates*. For $\varphi_0 \in \mathbb{R}$ we set $U := \mathbb{R}^2 \setminus \mathbb{R}_{\geq 0} \cdot \begin{pmatrix} \cos\varphi_0 \\ \sin\varphi_0 \end{pmatrix}$, $V := (0,\infty) \times (\varphi_0, \varphi_0 + 2\pi)$ and $y : U \to V$ such that

$$y^{-1}(r,\varphi) := (r\cos\varphi, r\sin\varphi).$$

On U the vector field $\xi := r\frac{\partial}{\partial r}$ is defined. Using (1.7) we express this vector field in terms of Cartesian coordinates, i.e., with respective to the chart $x = \mathrm{id} : \mathbb{R}^2 \to \mathbb{R}^2$:

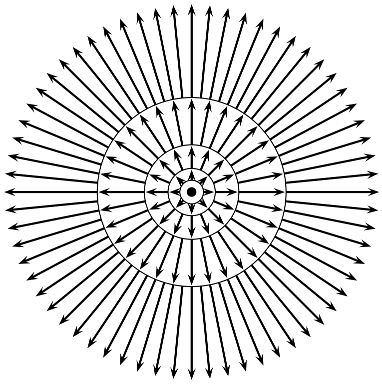
$$\begin{aligned}
\xi &= r\frac{\partial}{\partial r} \\
&= r\left(\frac{\partial x^1}{\partial r}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial r}\frac{\partial}{\partial x^2}\right) \\
&= r\left(\frac{\partial(r\cos\varphi)}{\partial r}\frac{\partial}{\partial x^1} + \frac{\partial(r\sin\varphi)}{\partial r}\frac{\partial}{\partial x^2}\right) \\
&= r\left(\cos\varphi\,\frac{\partial}{\partial x^1} + \sin\varphi\,\frac{\partial}{\partial x^2}\right) \\
&= x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}.
\end{aligned}$$

In Cartesian coordinates:

$$\begin{aligned}
\xi^1(x^1,x^2) &= x^1, \\
\xi^2(x^1,x^2) &= x^2.
\end{aligned}$$

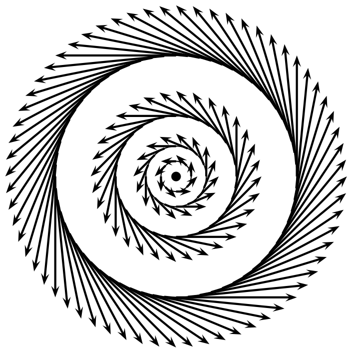
In polar coordinates:

$$\begin{aligned}
\eta^1(r,\varphi) &= r, \\
\eta^2(r,\varphi) &= 0.
\end{aligned}$$



Similarly, we can express the vector field $\frac{\partial}{\partial\varphi}$ in Cartesian coordinates:

$$\begin{aligned}
\frac{\partial}{\partial\varphi} &= \frac{\partial x^1}{\partial\varphi}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial\varphi}\frac{\partial}{\partial x^2} \\
&= -r\sin\varphi\,\frac{\partial}{\partial x^1} + r\cos\varphi\,\frac{\partial}{\partial x^2} \\
&= -x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}.
\end{aligned}$$



2 Semi-Riemannian Geometry

On topological manifolds one can consider continuous maps. In order to be able to define differentiable maps we had to add structure to a topological manifold which gave rise to differentiable manifolds. We were then able to define linear approximations to manifolds (tangent spaces) and and to maps (the differential). The concept of a differentiable manifold is what one needs to do analysis.
In order to do geometry we need to enrich our manifolds once more. We want to measure lengths of and angles between tangent vectors. This requires scalar products on the tangent spaces and leads to the concept of a Riemannian manifold.

2.1 Bilinear forms

We start by recalling some facts about bilinear forms from linear algebra.

Definition 2.1.1. Let V be an n-dimensional $\mathbb{R}$-vector space. A **symmetric bilinear form** is a map $g : V \times V \to \mathbb{R}$ with

(i) $g(\alpha v + \beta w, z) = \alpha g(v,z) + \beta g(w,z)$ for all $v,w,z \in V$ and $\alpha, \beta \in \mathbb{R}$ and

(ii) $g(v,w) = g(w,v)$ for all $v,w \in V$.

We call g **non-degenerate** if $g(v,w) = 0$ for all $w \in V$ implies $v = 0$.

For a basis $(b_1, \dots, b_n)$ of V we set

$$g_{ij} := g(b_i, b_j) \in \mathbb{R}$$

for $i, j = 1, \dots, n$. Then $(g_{ij})_{i,j=1,\dots,n}$ is a symmetric $n \times n$-matrix. From $(g_{ij})_{i,j=1,\dots,n}$ we can reconstruct g: For $v = \sum_{i=1}^n \alpha^i b_i$ and $w = \sum_{j=1}^n \beta^j b_j$ we have:

$$g(v,w) = g\left(\sum_{i=1}^n \alpha^i b_i, \sum_{j=1}^n \beta^j b_j\right) = \sum_{i,j=1}^n \alpha^i \beta^j g_{ij}.$$

Notation 2.1.2. Let $b_1^*, \dots, b_n^*$ the dual basis of the dual space $V^* = \{\text{linear maps } V \to \mathbb{R}\}$ of $b_1, \dots, b_n$, that is $b_i^*(b_j) = \delta_{ij}$. Often, we write

$$g = \sum_{i,j=1}^{n} g_{ij}\, b_i^* \otimes b_j^*.$$

The insertion of $v, w \in V$ then means the following:

$$g(v,w) = \sum_{i,j=1}^{n} g_{ij}\, b_i^*(v) \cdot b_j^*(w) = \sum_{i,j=1}^{n} g_{ij}\, \alpha^i \beta^j.$$

Transformation of principal axes. Let g be a non-degenerate symmetric bilinear form on V. Then there exists a basis $e_1, \ldots, e_n$ of V, such that

$$g(e_i, e_j) = \begin{cases} 0 & i \neq j \\ \varepsilon_i \in \{\pm 1\} & i = j \end{cases},$$

in other words,

$$(g_{ij})_{i,j=1,\ldots,n} = \begin{pmatrix} -1 & & & & & 0 \\ & \ddots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}. \tag{2.1}$$

Such a basis is called a **generalized orthonormal basis**. We the number of -1's occurring in (2.1) the **index** of g and denote it by $\mathrm{Index}(g)$. We observe that for a non-degenerate symmetric bilinear form the following are equivalent:

(1) g is a Euclidean scalar product;

(2) g is positive definite;

(3) $\mathrm{Index}(g) = 0$.

If $\mathscr{B} = (b_1, \ldots, b_n)$ and $\tilde{\mathscr{B}} = (\tilde{b}_1, \ldots, \tilde{b}_n)$ are two bases of V, we define the transformation matrix $T = (t_i^j)_{i,j=1,\ldots,n}$ by

$$\tilde{b}_i = \sum_{j=1}^{n} t_i^j b_j.$$

Then the representing matrix of g transforms as follows:

$$\begin{aligned} g_{ij}^{(\tilde{\mathscr{B}})} &= g(\tilde{b}_i, \tilde{b}_j) \\ &= g\left(\sum_{k=1}^{n} t_i^k b_k, \sum_{l=1}^{n} t_i^l b_l\right) \\ &= \sum_{k,l=1}^{n} t_i^k t_j^l \cdot g(b_k, b_l) \\ &= \sum_{k,l=1}^{n} t_i^k t_j^l \cdot g_{kl}^{(\mathscr{B})} \end{aligned} \tag{2.2}$$

Let V and W be two finite-dimensional $\mathbb{R}$-vector spaces. Let g be a symmetric bilinear form on V and $\Phi : W \to V$ be a linear map. Then we can pull back g via Φ to W, that is, we can define a symmetric bilinear form $\Phi^* g$ on W by

$$\big(\Phi^* g\big)(w_1, w_2) := g\big(\Phi(w_1), \Phi(w_2)\big).$$

Remark 2.1.3. If g is positive definite, then $\Phi^* g$ is positive semidefinite. Namely:

$$(\Phi^* g)(w,w) = g(\Phi(w), \Phi(w)) \geq 0 \quad \forall w \in W.$$

If furthermore Φ is injective, then $\Phi^* g$ is also positive definite. Namely:

$$(\Phi^* g)(w,w) = 0 \quad \Longrightarrow \quad \Phi(w) = 0 \quad \Longrightarrow \quad w = 0.$$

Definition 2.1.4. Let g_V and g_W be symmetric bilinear forms on V and W, respectively. We call a bijective linear map $\Phi : W \to V$ an **isometry**, if

$$g_V\left(\Phi(w_1), \Phi(w_2)\right) = g_W(w_1, w_2), \quad \forall w_1, w_2 \in W,$$

that is, if $\Phi^* g_V = g_W$.

2.2 Semi-Riemannian metrics

Let M be a differentiable manifold. We consider maps g which assign to every point $p \in M$ a non-degenerate symmetric bilinear form $g|_p$ on T_pM. If $x : U \to V$ is a chart of M, we define $g_{ij}^{(x)} = g_{ij} : V \to \mathbb{R}$ by

$$g_{ij}(v) := g|_{x^{-1}(v)} \left(\frac{\partial}{\partial x^i}\bigg|_{x^{-1}(v)}, \frac{\partial}{\partial x^j}\bigg|_{x^{-1}(v)} \right).$$

Definition 2.2.1. Such a map g is called a **semi-Riemannian metric** on M, if the map depends smoothly on the base point in the following sense:
For every chart $x : U \to V$ of M the $g_{ij} : V \to \mathbb{R}$ are C^∞-functions.

Remark 2.2.2. Note the similarity of the definition of smoothness of g with the characterization of smoothness of vector fields in Remark 1.5.4. We express the vector field or semi-Riemannian metric with respect to the basis $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ of the tangent space induced by a chart and then require smoothness of the coefficient functions.

Transformation by change of charts

Let $x : U \to V$ and $y : \tilde{U} \to \tilde{V}$ be two charts of M with $p \in U \cap \tilde{U}$. By (1.7),

$$\underbrace{\left.\frac{\partial}{\partial y^i}\right|_p}_{=\tilde{b}_i} = \sum_{j=1}^{n} \underbrace{\left.\frac{\partial(x^j \circ y^{-1})}{\partial y^i}\right|_{y(p)}}_{=t_i^j} \cdot \underbrace{\left.\frac{\partial}{\partial x^j}\right|_p}_{=b_j}$$

Inserting this into (2.2) yields

$$g_{ij}^{(y)}(y(p)) = \sum_{k,l=1}^{n} \left.\frac{\partial(x^k \circ y^{-1})}{\partial y^i}\right|_{y(p)} \cdot \left.\frac{\partial(x^l \circ y^{-1})}{\partial y^j}\right|_{y(p)} \cdot g_{kl}^{(x)}(x(p)).$$

For all $v \in y(U \cap \tilde{U})$ we hence have

$$\boxed{g_{ij}^{(y)}(v) = \sum_{k,l=1}^{n} \left.\frac{\partial(x^k \circ y^{-1})}{\partial y^i}\right|_{v} \cdot \left.\frac{\partial(x^l \circ y^{-1})}{\partial y^j}\right|_{v} \cdot g_{kl}^{(x)}\left((x \circ y^{-1})(v)\right)} \tag{2.3}$$

In the physicist's short notation this formula reads as

$$\boxed{g_{ij}^{(y)} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j} \cdot \left(g_{kl}^{(x)} \circ \left(x \circ y^{-1}\right)\right).}$$

Consequence. *The condition that g is smooth does not have to be checked for all charts, if suffices to check it for a subatlas of $\mathscr{A}_{\max}(M)$ which covers M.*

Remark 2.2.4. Recall that $dx|_p : T_pM \to \mathbb{R}^n$ is a linear isomorphism for any chart $x : U \to V$ with $p \in U$. In particular, $dx^1|_p, \ldots, dx^n|_p \in (T_pM)^*$.

Definition 2.2.5. The dual space $(T_pM)^* =: T_p^*M$ is called **cotangent space** of M at p.

Lemma 2.2.6. *The $dx^1|_p, \ldots, dx^n|_p$ form the dual basis of* $\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$.

Proof. Since $dx|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = e_i$ we have $dx^j|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \delta_i^j$ for $i = 1, \ldots, n$. □

According to Notation 2.1.2 we may also write:

$$\boxed{g|_p = \sum_{i,j=1}^{n} g_{ij}\left(x(p)\right) \cdot dx^i|_p \otimes dx^j|_p}$$

In the physics literature you will find the following short version of this equation:

$$\boxed{g = g_{ij} \cdot dx^i \cdot dx^j}$$

If one changes the basis of a vector space by the transformation $\tilde{b}_i = \sum_{j=1}^{n} t_i^j b_j$, then we get

$$b_i^* = \sum_{j=1}^{n} t_j^i \tilde{b}_j^*.$$

Namely, denote the transformation matrix by $T = (t_i^j)$. Then we find:

$$\begin{aligned}
\left(\sum_{j=1}^{n} t_j^i \tilde{b}_j^*\right)(b_k) &= \left(\sum_{j=1}^{n} t_j^i \tilde{b}_j^*\right)\left(\sum_{l=1}^{n} (T^{-1})_k^l \tilde{b}_l\right) \\
&= \sum_{j,l=1}^{n} t_j^i (T^{-1})_k^l \underbrace{\tilde{b}_j^*(\tilde{b}_l)}_{=\delta_j^l} \\
&= \sum_{j=1}^{n} t_j^i (T^{-1})_k^j \\
&= \delta_k^i,
\end{aligned}$$

hence

$$\sum_{j=1}^{n} t_j^i \tilde{b}_j^* = b_i^*.$$

For $b_1^* = dx^1|_p, \dots, b_n^* = dx^n|_p$ this means:

$$\boxed{dx^i|_p = \sum_{j=1}^{n} \left.\frac{\partial(x^i \circ y^{-1})}{\partial y^j}\right|_{y(p)} \cdot dy^j|_p}$$

or, in the physicist's short notation

$$\boxed{dx^i = \frac{\partial x^i}{\partial y^j} dy^j}$$

If you have forgotten the transformation formula (2.3), you can quickly deduce it in "physics style" as follows:

$$\begin{aligned}
\boxed{g_{kl}^{(y)}} \cdot dy^k \cdot dy^l &= g_{ij}^{(x)} \cdot dx^i \cdot dx^j \\
&= g_{ij}^{(x)} \cdot \left(\frac{\partial x^i}{\partial y^k} \cdot dy^k \right) \cdot \left(\frac{\partial x^j}{\partial y^l} \cdot dy^l \right) \\
&= \boxed{\frac{\partial x^i}{\partial y^k} \cdot \frac{\partial x^j}{\partial y^l} \cdot g_{ij}^{(x)}} \cdot dy^k \cdot dy^l .
\end{aligned}$$

Comparing the coefficients in the blue boxes yields (2.3).

Example 2.2.7. Let $M \subset \mathbb{R}^n$ be open. Let β be a non-degenerate symmetric bilinear form on $\mathbb{R}^n$. For every $p \in M$ let $\Phi_p : T_pM \to \mathbb{R}^n$ the canonical isomorphism. Set $g|_p := \Phi_p^* \beta$. We check the smoothness of g in Cartesian coordinate, i.e., in the chart $x = \mathrm{id} : U = M \to V = M$.

$$\begin{aligned}
g|_p \left(\left.\frac{\partial}{\partial x^i}\right|_p , \left.\frac{\partial}{\partial x^j}\right|_p \right) &= (\Phi_p^* \beta) \left(\left.\frac{\partial}{\partial x^i}\right|_p , \left.\frac{\partial}{\partial x^j}\right|_p \right) \\
&= \beta \left(\Phi_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right), \Phi_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) \right) \\
&= \beta \left(dx|_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right), dx|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) \right) \\
&= \beta(e_i, e_j).
\end{aligned}$$

Consequently, the g_{ij} are constant, hence C^∞. In this manner, we can equip M with a semi-Riemannian metric with arbitrary index.

Example 2.2.8. Let $M \subset \mathbb{R}^{n+k}$ be an n-dimensional submanifold. Then there exists a canonical injective map $\Phi_p : T_pM \to \mathbb{R}^{n+k}$, defined by

$$\dot{c}(0) \mapsto \frac{d}{dt} c|_{t=0}$$

equivalence class of the curve $c : (-\varepsilon, \varepsilon) \to M$

derivative of $c : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n+k}$

Then define $g|_p := \Phi_p^* \langle \cdot, \cdot \rangle$, where $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n+k} x^i y^i$ is the usual Euclidean scalar product, $\mathbf{x} = (x^1, \ldots, x^{n+k})^T$, $\mathbf{y} = (y^1, \ldots, y^{n+k})^T$. Since the Euclidean scalar product is positive definite and Φ_p is injective, we conclude that $g|_p$ is also positive definite for all $p \in M$. The semi-Riemannian metric on M defined in this way is called *first fundamental form*.
The charts of submanifolds correspond to local parametrizations of M, i.e. to maps $F : V \to M$ with $V \subset \mathbb{R}^n$ open, where

$$x = F^{-1} : U = F(V) \to V$$

is a chart of M. In addition, we have with $p = x^{-1}(v)$:

$$\begin{aligned} g_{ij}(v) &= g|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right) \\ &= ({\Phi_p}^* \langle \cdot, \cdot \rangle)\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right) \\ &= \left\langle \Phi_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right), \Phi_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)\right\rangle \\ &= \left\langle \frac{d}{dt}F(v + t\cdot e_i)|_{t=0}, \frac{d}{dt}F(v + t\cdot e_j)|_{t=0}\right\rangle \\ &= \left\langle \frac{\partial F}{\partial x^i}(v), \frac{\partial F}{\partial x^j}(v)\right\rangle. \end{aligned}$$

Hence $g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}\right\rangle$, in particular, the g_{ij} are smooth.

Definition 2.2.9. A semi-Riemannian metric g, for which $g|_p$ is always positive definite, is called **Riemannian metric**. A pair (M, g), consisting of a differentiable manifold M and a (semi-)Riemannian metric g on M is called **(semi-)Riemannian manifold**.
A semi-Riemannian metric g is called **Lorentzian metric**, if $g|_p$ has always index 1. The pair (M, g) is then called **Lorentzian manifold**.

Example 2.2.10. The first fundamental form of a submanifold $M \subset \mathbb{R}^{n+k}$ is a Riemannian metric. For example, for $S^n \subset \mathbb{R}^{n+1}$ we call the first fundamental form the *standard metric* g_{std} of S^n.
We express the standard metric of S^2 in the coordinates given by stereographic projection from the "south pole" $(-1, 0, 0)$. Recall from Example 1.1.4 that the inverse of this chart map is given by

$$F : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3, \quad F(x) = \frac{1}{4 + \|x\|^2}(4 - \|x\|^2, 4x).$$

One computes

$$\begin{aligned} \frac{\partial F}{\partial x^1} &= \frac{1}{(4 + \|x\|^2)^2}(-16x^1, 4(4 - (x^1)^2 + (x^2)^2), -8x^1x^2), \\ \frac{\partial F}{\partial x^2} &= \frac{1}{(4 + \|x\|^2)^2}(-16x^2, -8x^1x^2, 4(4 + (x^1)^2 - (x^2)^2)). \end{aligned}$$

Moreover,

$$g_{11} = \left\langle \frac{\partial F}{\partial x^1}, \frac{\partial F}{\partial x^1}\right\rangle = \frac{16}{(4 + \|x\|^2)^2}$$

and similarly for the other g_{ij}. The metric in these coordinates turns out to be

$$(g_{ij}) = \frac{16}{(4 + \|x\|^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Example 2.2.11. Let $M \subset \mathbb{R}^{n+1}$ be open. The *Minkowski scalar product* $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathbb{R}^{n+1}$ has index 1, where

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = -x^0 y^0 + x^1 y^1 + \cdots + x^n y^n$$

for $\mathbf{x} = (x^0, x^1, \ldots, x^n)$ and $\mathbf{y} = (y^0, y^1, \ldots, y^n)$. If $\Phi_p : T_pM \to \mathbb{R}^{n+1}$ is the canonical isomorphism, we can define a Lorentzian metric on M by

$$g_{\mathrm{Mink}}|_p := {\Phi_p}^* \langle\langle\cdot,\cdot\rangle\rangle .$$

The Lorentzian manifold $(\mathbb{R}^{n+1}, g_{\mathrm{Mink}})$ is called *Minkowski space*. The four-dimensional Minkowski space is the mathematical model for spacetime in special relativity.

Example 2.2.12. We express the Euclidean metric $g_{\mathrm{eucl}} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$ of $\mathbb{R}^2$ in polar coordinates. Here x^1 and x^2 are the Cartesian coordinates. With $x^1 = r\cos\varphi$ and $x^2 = r\sin\varphi$ we then find:

$$\begin{aligned} dx^1 &= \frac{\partial x^1}{\partial r} dr + \frac{\partial x^1}{\partial \varphi} d\varphi &&= \cos\varphi\, dr - r\sin\varphi\, d\varphi \\ dx^2 &= \frac{\partial x^2}{\partial r} dr + \frac{\partial x^2}{\partial \varphi} d\varphi &&= \sin\varphi\, dr + r\cos\varphi\, d\varphi. \end{aligned}$$

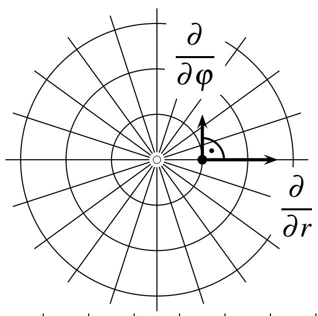
Thus

$$\begin{aligned} g_{\mathrm{eucl}} &= (\cos\varphi\, dr - r\sin\varphi\, d\varphi) \otimes (\cos\varphi\, dr - r\sin\varphi\, d\varphi) \\ &\quad + (\sin\varphi\, dr + r\cos\varphi\, d\varphi) \otimes (\sin\varphi\, dr + r\cos\varphi\, d\varphi) \\ &= \cos^2\varphi\, dr\otimes dr - r\cos\varphi\sin\varphi\, dr\otimes d\varphi - r\sin\varphi\cos\varphi\, d\varphi\otimes dr + r^2\sin^2\varphi\, d\varphi\otimes d\varphi \\ &\quad + \sin^2\varphi\, dr\otimes dr + \sin(\varphi) r\cos\varphi\, dr\otimes d\varphi + r\cos\varphi\sin\varphi\, d\varphi\otimes dr + r^2\cos^2\varphi\, d\varphi\otimes d\varphi \\ &= dr\otimes dr + r^2\, d\varphi\otimes d\varphi \end{aligned}$$

and hence

$$(g_{ij}^{\mathrm{Polar}}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

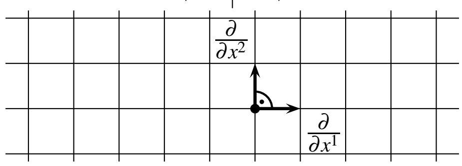
This matrix tells us:



- $\dfrac{\partial}{\partial r}$ has length 1,
- $\dfrac{\partial}{\partial \varphi}$ has length r,
- $\dfrac{\partial}{\partial r}$ and $\dfrac{\partial}{\partial \varphi}$ are orthogonal to each other

In Cartesian coordinates we have:

$$(g_{ij}^{\mathrm{Cartes}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Definition 2.2.13. Let (M, g_M) and (N, g_N) be semi-Riemannian manifolds. A local diffeomorphism $\varphi : M \to N$ is called **local isometry**, if

$$d\varphi|_p : (T_pM, g_M|_p) \to (T_{\varphi(p)}N, g_N|_{\varphi(p)})$$

for all $p \in M$ is a linear isometry.
If a local isometry is also bijective, that is, if it is a diffeomorphism, we call it an **isometry**.

Definition 2.2.14. If $\varphi : M \to N$ is a local diffeomorphism and g a semi-Riemannian metric on N, then we call the semi-Riemannian metric $\varphi^* g$ on M given by

$$(\varphi^* g)|_p := (d\varphi|_p)^*(g|_{\varphi(p)}),$$

the **pullback** of g. In other words, we have for $\xi, \eta \in T_pM$:

$$(\varphi^* g)|_p(\xi, \eta) = g|_{\varphi(p)}\big(d\varphi|_p(\xi), d\varphi|_p(\eta)\big).$$

Remark 2.2.15. The metric $\varphi^* g$ is the unique semi-Riemannian metric on M, for which φ is a local isometry.

Definition 2.2.16. Let (M, g) be a semi-Riemannian manifold. Then we call

$$\mathrm{Isom}(M, g) := \{\varphi : M \to M \text{ isometry}\}$$

the **isometry group** of M.

Remark 2.2.17. The set $\mathrm{Isom}(M, g)$ is a group with respect to composition of maps. The neutral element is id_M.

Example 2.2.18. We look for the isometries of $(\mathbb{R}^n, g_{\mathrm{eucl}})$. Let

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n, \quad \varphi(x) = Ax + b,$$

be an affine map with $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. Such a φ is called a *Euclidean motion*. We check that every Euclidean motion is an isometry of $(\mathbb{R}^n, g_{\mathrm{eucl}})$: Let $\Phi_p : T_pM \to \mathbb{R}^n$ be the canonical isomorphism; for $\xi = \Phi_p^{-1}(X) \in T_pM$ this means that $\xi = \dot{c}(0)$ where $c(t) = p + tX$. Similarly, $\eta = \Phi_p^{-1}(Y) = \dot{\tilde{c}}(0) \in T_pM$ with $\tilde{c}(t) = p + tY$. We compute:

$$\begin{aligned}\varphi^*(g_{\mathrm{eucl}}|_p)(\xi, \eta) &= g_{\mathrm{eucl}}|_p(d\varphi|_p(\xi), d\varphi|_p(\eta)) \\ &= \langle \Phi_p(d\varphi|_p(\xi)), \Phi_p(d\varphi|_p(\eta)) \rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \Phi_p((\varphi\circ c)^{\cdot}(0)), \Phi_p((\varphi\circ \tilde{c})^{\cdot}(0))\rangle \\
&= \langle \Phi_p((A(p+tX)+b)^{\cdot}(0)), \Phi_p((A(p+tY)+b)^{\cdot}(0))\rangle \\
&= \langle \Phi_p(Ap+b+tAX)^{\cdot}(0)), \Phi_p(Ap+b+tAY)^{\cdot}(0))\rangle \\
&= \langle AX, AY\rangle \\
&= \langle X, Y\rangle \\
&= \langle \Phi_p(\xi), \Phi_p(\eta)\rangle \\
&= g_{\mathrm{eucl}}(\xi,\eta).
\end{aligned}
$$

Hence $\varphi^*(g_{\mathrm{eucl}}|_p) = g_{\mathrm{eucl}}$ showing that φ is a local isometry. Since φ is bijective, it is an isometry. Summarizing, we have shown

$$\{\text{Euclidean motions}\} \subset \mathrm{Isom}(\mathbb{R}^n, g_{\mathrm{eucl}}).$$

We will see later that the inverse conclusion also holds; the isometries of $(\mathbb{R}^n, g_{\mathrm{eucl}})$ are precisely the Euclidean motions.

Example 2.2.19. To find isometries of Minkowski space $(M,g) = (\mathbb{R}^{n+1}, g_{\mathrm{Mink}})$ we define

$$
\begin{aligned}
\mathrm{O}(n,1) :=& \big\{A \in \mathrm{Mat}((n+1)\times(n+1),\mathbb{R}) \,|\, \langle\langle Ay, Az\rangle\rangle = \langle\langle y,z\rangle\rangle \ \forall y,z \in \mathbb{R}^{n+1}\big\} \\
=& \big\{A \in \mathrm{Mat}((n+1)\times(n+1),\mathbb{R}) \,|\, A^\top I_{1,n}, A = I_{1,n}\big\}
\end{aligned}
$$

where

$$
I_{1,n} = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.
$$

Now affine transformations $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\varphi(x) = Ax + b$ with $A \in \mathrm{O}(n,1)$ and $b \in \mathbb{R}^{n+1}$, are called *Poincaré transformations*. The same discussion as for Euclidean space shows

$$\{\text{Poincaré transformations}\} \subset \mathrm{Isom}(\mathbb{R}^{n+1}, g_{\mathrm{Mink}}).$$

Again, we will see later that equality holds; the isometries of Minkowski space are precisely the Poincaré transformations.

Example 2.2.20. To find isometries of the sphere $(M,g) = (S^n, g_{\mathrm{std}})$ let $A \in \mathrm{O}(n+1)$. We set $\varphi := A|_{S^n} : S^n \to S^n$. Let $\Phi_p : T_pS^n \to \mathbb{R}^{n+1}$ be as in Example 2.2.8. Then the diagram

$$
\begin{tikzcd}
T_pS^n \arrow[r, "d\varphi|_p"] \arrow[d, "\Phi_p"'] & T_{\varphi(p)}S^n \arrow[d, "\Phi_{\varphi(p)}"] \\
\mathbb{R}^{n+1} \arrow[r, "\mathbf{A}"] & \mathbb{R}^{n+1}
\end{tikzcd}
$$

commutes because:

$$\begin{tikzcd} \dot c(0) \arrow[r, mapsto] \arrow[d, mapsto] & (\varphi\circ c)^{\cdot}(0) & = & (A\circ c)^{\cdot}(0) \arrow[d, mapsto] \\ \frac{d}{dt}c|_{t=0} \arrow[r, mapsto] & \mathbf{A}\cdot\frac{d}{dt}c|_{t=0} & = & \frac{d}{dt}(\mathbf{A}\cdot c)|_{t=0} \end{tikzcd}$$

Therefore

$$\begin{aligned} g_{\mathrm{std}}(d\varphi|_p(\xi), d\varphi|_p(\eta)) &= \langle \Phi_{\varphi(p)}(d\varphi|_p(\xi)), \Phi_{\varphi(p)}(d\varphi|_p(\eta))\rangle \\ &= \langle A\Phi_p(\xi), A\Phi_p(\eta)\rangle \\ &= \langle \Phi_p(\xi), \Phi_p(\eta)\rangle \\ &= g_{\mathrm{std}}(\xi,\eta). \end{aligned}$$

This shows that φ is an isometry. Hence

$$\mathrm{O}(n+1) \subset \mathrm{Isom}(S^n, g_{\mathrm{std}}).$$

Again, it will turn out that equality holds.

2.3 Differentiation of vector fields

We know how to differentiate functions on a manifold. We also know what differentiable vector fields are. But: How do we differentiate a vector field? What is the differential of a vector field at a point in the manifold?

First attempt. Let M be a differentiable manifold and let $p \in M$. Let $\xi \in T_pM$ and let η be a differentiable vector field on M. We try to define the derivative of η in the direction ξ.

To this extent, we choose a chart $x : U \to V$ on M with $p \in U$. We write $\xi \in T_pM$ as $\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}\big|_p$ with $\xi^i \in \mathbb{R}$ and $\eta = \sum_{i=j}^n \eta^j \frac{\partial}{\partial x^j}$ where the η^j are smooth functions near $x(p)$.

The first idea that comes to one's mind is to differentiate the coefficient functions η^j in the direction ξ. This would yield the expression

$$\sum_{i,j=1}^n \xi^i \cdot \frac{\partial \eta^j}{\partial x^i}\bigg|_{x(p)} \cdot \frac{\partial}{\partial x^j}\bigg|_p$$

for the derivative of η in direction ξ.

Problem. This "definition" depends on the choice of chart x.

Example 2.3.1. Let $M = \mathbb{R}^2$. In polar coordinates (r,φ) we set

$$\xi = \eta = \frac{\partial}{\partial\varphi}.$$

Then the derivative of η in direction ξ equals 0 because the coefficient functions η^j are constant. On the other hand, in Cartesian coordinates (x^1, x^2) we get

$$\xi = \eta = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}.$$

For the derivative of η in direction ξ we would then find

$$\begin{aligned}
&\left(-x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}\right)(-x^2)\frac{\partial}{\partial x^1} + \left(-x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}\right)(x^1)\frac{\partial}{\partial x^2} \\
&= \ -x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} \ = \ -r\frac{\partial}{\partial r} \ \neq \ 0.
\end{aligned}$$

We see that the idea of simply differentiating the coefficient functions was to naive. Since we do not know how to come up with a better definition we follow an axiomatic approach similar to the concept of derivations, except that this time we differentiate vector fields rather than functions.

Notation 2.3.2. Let M be a differentiable manifold and let $k \in \mathbb{N} \cup \{\infty\}$. For any open subset $U \subset M$ we put

$$C^k(U, TM) := \{C^k\text{-vector fields, defined on } U\}.$$

For $p \in M$ we set

$$\Xi_p := \bigcup_{\substack{U \subset M \text{ open} \\ \text{with } p \in U}} C^\infty(U, TM).$$

Now we list the properties that the derivative of vector fields should have. Differentiation takes a tangent vector $\xi \in T_pM$ and a smooth vector field η defined near p and gives us a tangent vector in T_pM as a result. Hence it is a map $T_pM \times \Xi_p \to T_pM$.

Definition 2.3.3. Let (M, g) be a semi-Riemannian manifold and $p \in M$. A map

$$\nabla : T_pM \times \Xi_p \to T_pM$$

is called **Levi-Civita connection** (at p), if the following holds:

(i) *Locality*

For all $\xi \in T_pM$, for all $\eta \in C^\infty(U, TM)$ and for all $\tilde{U} \subset U$ with $p \in \tilde{U}$ we have:

$$\nabla_\xi \eta = \nabla_\xi (\eta|_{\tilde{U}}).$$

(ii) *Linearity in the first argument*

For all $\xi_1, \xi_2 \in T_pM$, for all $\alpha, \beta \in \mathbb{R}$ and for all $\eta \in \Xi_p$ we have:

$$\nabla_{\alpha\xi_1 + \beta\xi_2}\eta = \alpha\nabla_{\xi_1}\eta + \beta\nabla_{\xi_2}\eta.$$

(iii) *Additivity in the second argument*

For all $\xi \in T_pM$ and for all $\eta_1, \eta_2 \in \Xi_p$ we have:

$$\nabla_\xi(\eta_1 + \eta_2) = \nabla_\xi \eta_1 + \nabla_\xi \eta_2.$$

(iv) *Product rule I*

For all $f \in C_p^\infty$, for all $\eta \in \Xi_p$ and for all $\xi \in T_pM$ we have:

$$\nabla_\xi(f \cdot \eta) = \partial_\xi f \cdot \eta|_p + f(p) \cdot \nabla_\xi \eta.$$

(v) *Product rule II*

For all $\xi \in T_pM$ and for all $\eta_1, \eta_2 \in \Xi_p$ we have:

$$\partial_\xi g(\eta_1, \eta_2) = g|_p(\nabla_\xi \eta_1, \eta_2|_p) + g|_p(\eta_1|_p, \nabla_\xi \eta_2).$$

(vi) *Torsion-freeness*

For all charts $x : U \to V$ of M with $p \in U$ we have:

$$\nabla_{\frac{\partial}{\partial x^i}\big|_p} \frac{\partial}{\partial x^j} = \nabla_{\frac{\partial}{\partial x^j}\big|_p} \frac{\partial}{\partial x^i}$$

for all i and j.

Remark 2.3.4. (1) From (iii) and (iv) we get the $\mathbb{R}$-linearity in the second argument. Let $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}
\nabla_\xi(\alpha\eta_1 + \beta\eta_2) &\overset{\text{(iii)}}{=} \nabla_\xi(\alpha\eta_1) + \nabla_\xi(\beta\eta_2) \\
&\overset{\text{(iv)}}{=} \underbrace{\partial_\xi \alpha}_{=0} \cdot \eta_1|_p + \alpha\nabla_\xi(\eta_1) + \underbrace{\partial_\xi \beta}_{=0} \cdot \eta_2|_p + \beta\nabla_\xi(\eta_2) \\
&= \alpha\nabla_\xi(\eta_1) + \beta\nabla_\xi(\eta_2).
\end{aligned}$$

(2) If (vi) holds in a chart x, then it also holds in every other chart y containing p.

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial y^i}\big|_p} \frac{\partial}{\partial y^j} &= \nabla_{\frac{\partial}{\partial y^i}\big|_p} \left(\sum_{k=1}^n \frac{\partial x^k}{\partial y^j}\bigg|_p \frac{\partial}{\partial x^k} \right) \\
&\overset{\substack{\text{(iii)}\\\text{(iv)}}}{=} \sum_{k=1}^n \left(\frac{\partial^2 x^k}{\partial y^i \partial y^j}\bigg|_{y(p)} \cdot \frac{\partial}{\partial x^k} + \frac{\partial x^k}{\partial y^j}\bigg|_{y(p)} \nabla_{\frac{\partial}{\partial y^i}\big|_p} \frac{\partial}{\partial x^k} \right) \\
&\overset{\text{(ii)}}{=} \sum_{k=1}^n \frac{\partial^2 x^k}{\partial y^i \partial y^j}\bigg|_{y(p)} \cdot \frac{\partial}{\partial x^k} + \sum_{k,l=1}^n \frac{\partial x^k}{\partial y^j}\bigg|_{y(p)} \cdot \frac{\partial x^l}{\partial y^i}\bigg|_{y(p)} \nabla_{\frac{\partial}{\partial x^l}\big|_p} \frac{\partial}{\partial x^k}
\end{aligned}$$

The first summand is symmetric in i and j due to Schwarz' Theorem. Concerning the second summand we have:

$$\begin{aligned}\sum_{k,l=1}^{n} \frac{\partial x^k}{\partial y^j}\Big|_{y(p)} \cdot \frac{\partial x^l}{\partial y^i}\Big|_{y(p)} \nabla_{\frac{\partial}{\partial x^l}\big|_p} \frac{\partial}{\partial x^k} &\overset{\text{(vi)}}{=} \sum_{k,l=1}^{n} \frac{\partial x^k}{\partial y^j}\Big|_{y(p)} \cdot \frac{\partial x^l}{\partial y^i}\Big|_{y(p)} \nabla_{\frac{\partial}{\partial x^k}\big|_p} \frac{\partial}{\partial x^l} \\ \text{change of indices} \qquad &= \sum_{l,k=1}^{n} \frac{\partial x^l}{\partial y^j}\Big|_{y(p)} \cdot \frac{\partial x^k}{\partial y^i}\Big|_{y(p)} \nabla_{\frac{\partial}{\partial x^l}\big|_p} \frac{\partial}{\partial x^k}\end{aligned}$$

Hence the second summand is also symmetric in i and j.

(3) In general, for non-coordinate fields ξ and η we have

$$\nabla_\xi \eta \neq \nabla_\eta \xi.$$

As an example we can choose $\xi = \frac{\partial}{\partial x^1}$ and $\eta = f \cdot \frac{\partial}{\partial x^1}$ with $\partial_\xi f \neq 0$.

Definition 2.3.5. Let $x : U \to V$ be a chart. Write

$$\boxed{\nabla_{\frac{\partial}{\partial x^i}\big|_p} \frac{\partial}{\partial x^j} = \sum_{k=1}^{n} \Gamma_{ij}^k(x(p)) \cdot \frac{\partial}{\partial x^k}\Big|_p} \tag{2.4}$$

The Γ_{ij}^k are called **Christoffel symbols**.

Remark 2.3.6. The Christoffel symbols determine ∇. Namely, let $\xi = \sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}\Big|_p \in T_pM$ and $\eta = \sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j} \in \Xi_p$. Then we compute:

$$\begin{aligned}\nabla_{\sum_{i=1}^n \xi^i \frac{\partial}{\partial x^i}\big|_p} \left(\sum_{j=1}^{n} \eta^j \frac{\partial}{\partial x^j} \right) &\overset{\substack{\text{(ii)}\\\text{(iii)}}}{=} \sum_{i,j=1}^{n} \xi^i \nabla_{\frac{\partial}{\partial x^i}\big|_p} \left(\eta^j \frac{\partial}{\partial x^j} \right) \\ &\overset{\text{(iv)}}{=} \sum_{i,j=1}^{n} \xi^i \left(\frac{\partial \eta^j}{\partial x^i}\Big|_{x(p)} \cdot \frac{\partial}{\partial x^j}\Big|_p + \eta^j|_{x(p)} \cdot \nabla_{\frac{\partial}{\partial x^i}\big|_p} \frac{\partial}{\partial x^j} \right) \\ &= \sum_{i,j=1}^{n} \xi^i \left(\frac{\partial \eta^j}{\partial x^i}\Big|_{x(p)} \cdot \frac{\partial}{\partial x^j}\Big|_p + \eta^j|_{x(p)} \cdot \sum_{k=1}^{n} \Gamma_{ij}^k(x(p)) \cdot \frac{\partial}{\partial x^k}\Big|_p \right) \\ &= \sum_{i,k=1}^{n} \xi^i \left(\frac{\partial \eta^k}{\partial x^i}\Big|_{x(p)} + \sum_{j=1}^{n} \eta^j|_{x(p)} \cdot \Gamma_{ij}^k(x(p)) \right) \frac{\partial}{\partial x^k}\Big|_p\end{aligned} \tag{2.5}$$

Remark 2.3.7. Torsion freeness is equivalent to the Christoffel symbols being symmetric in the two lower indices:

$$\Gamma_{ij}^k = \Gamma_{ji}^k \qquad \text{for all } i, j, k.$$

Theorem 2.3.8. *Let (M,g) be a semi-Riemannian manifold and let $p \in M$. Then there is exactly one Levi-Civita connection at p.*

Proof. *Uniqueness:* Let $x : U \to V$ be a chart of M with $p \in U$. We compute, using the Einstein summation convention:

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial x^k} &= \frac{\partial}{\partial x^k} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&\overset{(v)}{=} g\left(\nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^j}\right) \\
&= g\left(\Gamma^l_{ki}\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\right) + g\left(\frac{\partial}{\partial x^i}, \Gamma^l_{kj}\frac{\partial}{\partial x^j}\right) \\
&= \Gamma^l_{ki} \cdot g\left(\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^j}\right) + \Gamma^l_{kj} \cdot g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \Gamma^l_{ki} \cdot g_{lj} + \Gamma^l_{kj} \cdot g_{il}.
\end{aligned}$$

Renaming the indices we get the equations:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^l_{ki} \cdot g_{lj} + \Gamma^l_{kj} \cdot g_{il} \tag{2.6}$$

$$\begin{smallmatrix} i\to i \\ j\to k \\ k\to j \end{smallmatrix} \quad \frac{\partial g_{ik}}{\partial x^j} = \Gamma^l_{ji} \cdot g_{lk} + \Gamma^l_{jk} \cdot g_{il} \tag{2.7}$$

$$\begin{smallmatrix} i\to k \\ j\to j \\ k\to i \end{smallmatrix} \quad \frac{\partial g_{kj}}{\partial x^i} = \Gamma^l_{ik} \cdot g_{lj} + \Gamma^l_{ij} \cdot g_{kl} \tag{2.8}$$

Equation (2.6) $-$ (2.7) $+$ (2.8) together with the symmetry of the Christoffel symbols in the lower indices yields:

$$\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} = 2\Gamma^l_{ki} \cdot g_{lj}.$$

Let $(g^{ij})_{i,j=1,\dots,n}$ be the inverse matrix of $(g_{ij})_{i,j=1,\dots,n}$. This matrix exists because $g|_p$ is non-degenerate. In other words, we have:

$$g^{ij} \cdot g_{jk} = \delta^i_k.$$

Therefore

$$\left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i}\right) g^{jm} = 2\Gamma^l_{ki} \cdot g_{lj} \cdot g^{jm} = 2\Gamma^l_{ki} \cdot \delta^m_l = 2\Gamma^m_{ki}$$

and hence

$$\Gamma^m_{ki} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i}\right) g^{jm}.$$

Renaming indices ($k \to j, m \to k, j \to m$) we obtain:

$$\boxed{\Gamma^k_{ij} = \frac{1}{2}\sum_{m=1}^{n} g^{mk}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)} \tag{2.9}$$

Consequently, the Christoffel symbols are uniquely determined and hence ∇ is uniquely determined by the components of the semi-Riemannian metric and its first derivatives.
Existence: Define Γ_{ij}^k by equation (2.9) and ∇ by equation (2.5). Then conditions (i), (ii), (iii), and (vi) of the Levi-Civita connection are obvious. For the first product (iv) rule we have:

$$\begin{aligned}
\nabla_\xi(f\eta) &= \xi^i\left(\frac{\partial(f\cdot\eta^k)}{\partial x^i}+f\eta^j\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}\\
&= f\cdot\xi^i\left(\frac{\partial\eta^k}{\partial x^i}+\eta^j\Gamma_{ij}^k\right)\frac{\partial}{\partial x^k}+\xi^i\frac{\partial f}{\partial x^i}\eta^k\frac{\partial}{\partial x^k}\\
&= f\cdot\nabla_\xi\eta+\partial_\xi f\cdot\eta
\end{aligned}$$

We check the second product rule (v), using the Einstein summation convention and occasional renaming of indices:

$$\begin{aligned}
&\partial_\zeta g(\xi,\eta)-g(\nabla_\zeta\xi,\eta)-g(\xi,\nabla_\zeta\eta)\\
&\quad=\zeta^k\frac{\partial}{\partial x^k}(g_{ij}\xi^i\eta^j)-g_{ij}\zeta^k\left(\frac{\partial\xi^i}{\partial x^k}+\xi^l\Gamma_{lk}^i\right)\eta^j-g_{ij}\xi^i\zeta^k\left(\frac{\partial\eta^j}{\partial x^k}+\eta^l\Gamma_{lk}^j\right)\\
&\quad=\zeta^k\frac{g_{ij}}{\partial x^k}\xi^i\eta^j-g_{ij}\zeta^k\xi^l\Gamma_{lk}^i\eta^j-g_{ij}\xi^i\zeta^k\eta^l\Gamma_{lk}^j\\
&\quad=\xi^i\eta^j\zeta^k\left(\frac{g_{ij}}{\partial x^k}-g_{lj}\Gamma_{ik}^l-g_{il}\Gamma_{jk}^l\right)\\
&\quad\overset{(2.9)}{=}\xi^i\eta^j\zeta^k\left(\frac{g_{ij}}{\partial x^k}-\frac{1}{2}\underbrace{g_{lj}g^{ml}}_{=\delta_j^m}\left(\frac{\partial g_{im}}{\partial x^k}+\frac{\partial g_{km}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^m}\right)\right.\\
&\qquad\left.-\frac{1}{2}\underbrace{g_{il}g^{ml}}_{=\delta_i^m}\left(\frac{\partial g_{jm}}{\partial x^k}+\frac{\partial g_{km}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^m}\right)\right)\\
&\quad=\xi^i\eta^j\zeta^k\left(\frac{g_{ij}}{\partial x^k}-\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k}+\frac{\partial g_{kj}}{\partial x^i}-\frac{\partial g_{ik}}{\partial x^j}\right)-\frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^k}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^i}\right)\right)\\
&\quad=0.
\end{aligned}$$

□

Remark 2.3.9. For any chart $x: U \to V$ on (M,g) the Christoffel symbols are smooth functions

$$\Gamma_{ij}^k=\frac{1}{2}\sum_{m=1}^n g^{mk}\cdot\left(\frac{\partial g_{im}}{\partial x^j}+\frac{\partial g_{jm}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^m}\right): V\to\mathbb{R}.$$

Remark 2.3.10. Our naive ansatz to differentiate vector fields by simply differentiating the coefficient functions corresponds to formula (2.5) with $\Gamma_{ij}^k=0$. The problem was that this depends on the choice of coordinates. When we use formula (2.5) with the correct definition (2.9) for the Christoffel symbols, then we get the uniquely determined Levi-Civita connection. In particular, this kind of differentiating vector fields is independent of the choice of chart.

Note however, that the Levi-Civita connection depends on the semi-Riemannian metric. This cannot only seen from (2.9) but also from the second product rule (v) in Definition 2.3.3 which involves the metric. There is nothing we can do about this; different semi-Riemannian metrics will in general lead to different Levi-Civita connections.
So the situation is somewhat curious: Differentiability and the derivative of a function are well defined on a differentiable manifold. Differentiability of a vector field is also well defined on a differentiable manifold. But in order to define the derivative of a vector field we need a semi-Riemannian metric.

Definition 2.3.11. Let (M,g) be a semi-Riemannian manifold and let ∇ be its Levi-Civita connection. Let $p \in M$, let $\xi \in T_pM$ and let $\eta \in \Xi_p$. Then

$$\nabla_\xi \eta \in T_pM$$

is also called the **covariant derivative** of η in direction ξ.

Example 2.3.12. Let $(M,g) = (\mathbb{R}^2, g_{\mathrm{eucl}})$ be the 2-dimensional Euclidean space. In Cartesian coordinates x^1, x^2 the $g_{ij} = \delta_{ij}$ are constant. Therefore $\Gamma_{ij}^k = 0$. In this case, covariant differentiation is indeed given by differentiation of the coordinate functions. For example,

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} &= \nabla_{-x^2\frac{\partial}{\partial x^1}+x^1\frac{\partial}{\partial x^2}} \left(-x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}\right) \\
&= \left(-x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}\right)(-x^2)\frac{\partial}{\partial x^1} + \left(-x^2\frac{\partial}{\partial x^1} + x^1\frac{\partial}{\partial x^2}\right)(x^1)\frac{\partial}{\partial x^2} \\
&= -x^1\frac{\partial}{\partial x^1} - x^2\frac{\partial}{\partial x^2} \quad = \quad -r\frac{\partial}{\partial r}
\end{aligned}$$

In polar coordinates r, φ we have

$$(g_{ij})(r,\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (g^{ij})(r,\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}.$$

The Christoffel symbols with respect to polar coordinates are given by

$$\Gamma_{11}^1 = \frac{1}{2}\big(1\cdot(0+0-0) + 0\cdot\ldots\big) = 0.$$

and similarly

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = 0.$$

Moreover:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2}\left(\frac{1}{r^2}\left(\frac{\partial g_{12}}{\partial\varphi} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{12}}{\partial\varphi}\right) + 0\cdot\ldots\right) = \frac{1}{r} \quad \text{and} \quad \Gamma_{22}^1 = -r.$$

Thus

$$\nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} = \Gamma_{22}^1\frac{\partial}{\partial r} + \Gamma_{22}^2\frac{\partial}{\partial\varphi} = -r\frac{\partial}{\partial r}.$$

Indeed, we obtained the same result for both computations, one in Cartesian and one in polar coordinates.

Remark 2.3.13. We defined ∇ pointwise, i.e., as a map $T_pM \times \Xi_p \to T_pM$. We may also consider ∇ as a map

$$\nabla : \Xi(M) \times \Xi(M) \to \Xi(M),$$

where $\Xi(M)$ denotes the set of all smooth vector fields defined on all of M. Namely, we put

$$(\nabla_\xi \eta)(p) := \nabla_{\xi(p)} \eta.$$

We know

$$\nabla_\xi(\alpha_1\eta_1 + \alpha_2\eta_2) = \alpha_1\nabla_\xi\eta_1 + \alpha_2\nabla_\xi\eta_2$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$ and

$$\nabla_{f_1\xi_1+f_2\xi_2}\eta = f_1\nabla_{\xi_1}\eta + f_2\nabla_{\xi_2}\eta$$

for $f_1, f_2 \in C^\infty(M)$. This means that $\nabla_\xi\eta$ is $C^\infty(M)$-linear in ξ but only $\mathbb{R}$-linear in η.

Remark 2.3.14. To compute $\nabla_\xi\eta$ with $\xi = \dot{c}(0)$ we only need to know η along the curve c. Namely,

$$\begin{aligned}
\nabla_{\dot{c}(0)}\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right) &= \nabla_{\sum_{i=1}^n \dot{c}^i(0)\frac{\partial}{\partial x^i}}\left(\sum_{j=1}^n \eta^j \frac{\partial}{\partial x^j}\right) \\
&= \sum_{i,j=1}^n \dot{c}^i(0)\nabla_{\frac{\partial}{\partial x^i}}\left(\eta^j\frac{\partial}{\partial x^j}\right) \\
&= \sum_{i,j=1}^n \dot{c}^i(0)\left(\frac{\partial\eta^j}{\partial x^i}\frac{\partial}{\partial x^j} + \sum_{k=1}^n \eta^j\Gamma_{ij}^k\frac{\partial}{\partial x^k}\right) \\
&= \sum_{j=1}^n \frac{d}{dt}(\eta^j\circ c)|_{t=0}\frac{\partial}{\partial x^j} + \sum_{i,j,k=1}^n \dot{c}^i(0)\,\eta^j(c(0))\Gamma_{ij}^k(x(c(0)))\frac{\partial}{\partial x^k}.
\end{aligned}$$

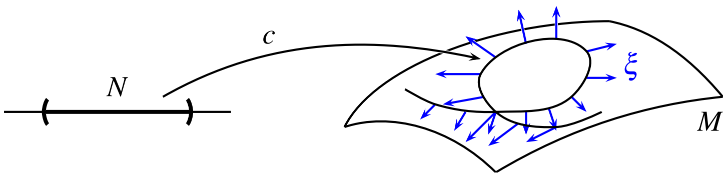
2.4 Vector fields along maps

Definition 2.4.1. Let M and N be differentiable manifolds and $\varphi : N \to M$ a map. Then a map $\xi : N \to TM$ is called a **vector field along φ**, if

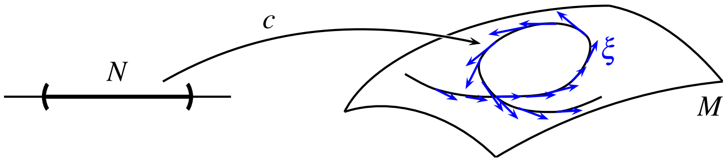
$$\pi_M \circ \xi = \varphi.$$

holds. Here $\pi_M : TM \to M$ is the "footpoint map".

Example 2.4.2. (1) *Vector fields along curves.* Let $N = I \subset \mathbb{R}$ be an open interval and $c = \varphi : N = I \to M$ be a curve.



An important special case is given by $\xi(t) = \dot{c}(t) := \dot{c}_t(0)$ where $c_t(s) := c(t+s)$. This is the *velocity field* of c.



(2) If $N = M$ and $\varphi = \mathrm{id}$ then a vector field along id is just a vector field in the usual sense.

(3) Let φ be constant, i.e., $\varphi(x) = p$ for all $x \in N$. Then a vector field along φ is a map $N \to T_pM$.

(4) Let φ be differentiable and let ξ be a vector field on N. Then

$$p \mapsto d\varphi|_p\big(\xi(p)\big) \in T_{\varphi(p)}M$$

is a vector field along φ.

(5) If ξ is a vector field on M then

$$p \mapsto \xi\big(\varphi(p)\big)$$

is a vector field along φ.

Definition 2.4.3. Let N be a differentiable manifold and (M,g) a semi-Riemannian manifold. Let $\varphi : N \to M$ be a differentiable map and $\eta : N \to TM$ a differentiable vector field along φ. For $p \in N$ and $\xi \in T_pN$ we define the **covariant derivative** $\nabla_\xi \eta \in T_{\varphi(p)}M$ as follows: Choose a chart $x : U \to V$ of M with $\varphi(p) \in U$ and write

$$\eta(q) = \sum_{j=1}^{n} \eta^j(q) \cdot \frac{\partial}{\partial x^j}\bigg|_{\varphi(q)}$$

with differentiable functions $\eta^1, \dots, \eta^n$ defined on $\varphi^{-1}(U)$. In addition, choose a curve $c : (-\varepsilon, \varepsilon) \to N$ with $\dot{c}(0) = \xi$ and set

$$\begin{aligned}
\nabla_\xi \eta \quad &:= \quad \sum_{k=1}^{n} \left(\frac{d}{dt}(\eta^k \circ c)\big|_{t=0} + \sum_{i,j=1}^{n} \eta^j(p)\, \frac{d}{dt}(\varphi^i \circ c)\big|_{t=0} \Gamma_{ij}^k\big(x(\varphi(p))\big) \right) \frac{\partial}{\partial x^k}\bigg|_{\varphi(p)} \\
&= \quad \sum_{k=1}^{n} \left(\partial_\xi \eta^k + \sum_{i,j=1}^{n} \eta^j(p)\, d\varphi(\xi)^i\, \Gamma_{ij}^k\big(x(\varphi(p))\big) \right) \frac{\partial}{\partial x^k}\bigg|_{\varphi(p)}.
\end{aligned}$$

Proposition 2.4.4. *Let N be a differentiable manifold, (M,g) a semi-Riemannian manifold and $\varphi : N \to M$ a differentiable map. Let η, η_1, η_2 be differentiable vector fields along φ. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $f : N \to \mathbb{R}$ be a differentiable function. Furthermore, let $p \in N$ and $\xi, \xi_1, \xi_2 \in T_pN$.*
Then the covariant derivative $\nabla_\xi \eta$ is defined independently of the choice of chart x and the choice of curve c with $\dot{c}(0) = \xi$ and we have:

(i) *If η is the form $\eta = \zeta \circ \varphi$ where ζ is a differentiable vector field on M, then we have*

$$\nabla_\xi \eta = \nabla_{d\varphi|_p(\xi)} \zeta.$$

(ii) Locality: *If η_1 and η_2 coincide on a neighborhood of p, then $\nabla_\xi \eta_1 = \nabla_\xi \eta_2$.*

(iii) Linearity in the first argument:

$$\nabla_{\alpha_1\xi_1+\alpha_2\xi_2}\eta = \alpha_1 \nabla_{\xi_1}\eta + \alpha_2 \nabla_{\xi_2}\eta.$$

(iv) Linearity in the second argument:

$$\nabla_\xi(\alpha_1\eta_1 + \alpha_2\eta_2) = \alpha_1 \nabla_\xi \eta_1 + \alpha_2 \nabla_\xi \eta_2.$$

(v) Product rule I:

$$\nabla_\xi(f \cdot \eta) = \partial_\xi f \cdot \eta(p) + f(p)\nabla_\xi \eta.$$

(vi) Product rule II:

$$\partial_\xi g(\eta_1, \eta_2) = g|_{\varphi(p)}\big(\nabla_\xi \eta_1, \eta_2(p)\big) + g|_{\varphi(p)}\big(\eta_1(p), \nabla_\xi \eta_2\big).$$

(vii) Torsion freeness: *For all charts y of N and all $i, j = 1, \ldots, \dim(N)$ we have:*

$$\nabla_{\frac{\partial}{\partial y^i}} d\varphi\left(\frac{\partial}{\partial y^j}\right) = \nabla_{\frac{\partial}{\partial y^j}} d\varphi\left(\frac{\partial}{\partial y^i}\right).$$

Proof. The assertions follow directly from the definition and the corresponding statements for the Levi-Civita connection. □

Notation 2.4.5. For local coordinates y on N we write

$$\frac{\nabla \eta}{\partial y^l}(p) := \nabla_{\frac{\partial}{\partial y^l}\big|_p} \eta = \sum_{k=1}^{n} \left(\frac{\partial \eta^k}{\partial y^l}\bigg|_{y(p)} + \sum_{i,j}^{n} \frac{\partial \varphi^i}{\partial y^l}(p) \cdot \eta^j\big(y(p)\big) \cdot \Gamma_{ij}^k\big(x(\varphi(p))\big) \right) \frac{\partial}{\partial x^k}(p).$$

If N is one-dimensional, we also write

$$\frac{\nabla \eta}{\partial t} =: \frac{\nabla \eta}{dt}.$$

Remark 2.4.6. For a vector field along a curve $c: I \to M$ we have the following formula in local coordinates on M:

$$\frac{\nabla \eta}{dt}(t) = \sum_{k=1}^{n} \left(\dot{\eta}^k(t) + \sum_{i,j}^{n} \dot{c}^i(t) \cdot \eta^j(t) \cdot \Gamma_{ij}^k\big(x(c(t))\big) \right) \frac{\partial}{\partial x^k}\bigg|_{c(t)}.$$

In particular, for the velocity field we get

$$\frac{\nabla \dot{c}}{dt}(t) = \sum_{k=1}^{n} \left(\ddot{c}^k(t) + \sum_{i,j}^{n} \dot{c}^i(t) \cdot \dot{c}^j(t) \cdot \Gamma_{ij}^k\big(x(c(t))\big) \right) \frac{\partial}{\partial x^k}\bigg|_{c(t)}.$$

Example 2.4.7. Let $(M,g) = (\mathbb{R}^n, g_{\mathrm{eucl}})$ or $(M,g) = (\mathbb{R}^n, g_{\mathrm{Mink}})$. Then the g_{ij} are constant in Cartesian coordinates. Consequently, the Christoffel symbols with respect to Cartesian coordinates vanish, $\Gamma_{ij}^k = 0$.
For a C^1-curve $c: I \to M$ and a C^1-vector field ξ along c with $\xi(t) = \sum_{j=1}^n \xi^j(t) \frac{\partial}{\partial x^j}|_{c(t)}$ we have:

$$\frac{\nabla}{dt}\xi(t) = \sum_{j=1}^{n} \dot{\xi}^j(t) \, \frac{\partial}{\partial x^j}\bigg|_{c(t)}.$$

Hence, in this case, covariant differentiation just consists of differentiation of the coefficient functions. Note however, that this is no longer true in other coordinate systems such as polar coordinates.

Example 2.4.8. In the Euclidean plane $(M,g) = (\mathbb{R}^2, g_{\mathrm{eucl}})$ we consider the circle line $c(t) = (\cos(t), \sin(t))$ and its velocity field

$$\xi(t) = \dot{c}(t) = -\sin(t) \, \frac{\partial}{\partial x^1}\bigg|_{c(t)} + \cos(t) \, \frac{\partial}{\partial x^2}\bigg|_{c(t)}.$$

In Cartesian coordinates we get by the previous example

$$\frac{\nabla}{dt}\xi(t) = \frac{\nabla}{dt}\dot{c}(t) = -\cos(t) \, \frac{\partial}{\partial x^1}\bigg|_{c(t)} - \sin(t) \, \frac{\partial}{\partial x^2}\bigg|_{c(t)} = -\frac{\partial}{\partial r}\bigg|_{c(t)}.$$

For the fun of it, let us also carry out the calculation in polar coordinates (r, φ). Now $c^1(t) = r(t) = 1$, $c^2(t) = \varphi(t) = t$ and $\xi(t) = \frac{\partial}{\partial \varphi}\Big|_{c(t)}$, i.e., $\xi^1(t) = 0$ and $\xi^2(t) = 1$. This time there are no derivatives of the coefficients of ξ but we have to take the Christoffel symbols into account.

Recall from Example 2.3.12 that there are three non-vanishing Christoffel symbols for polar coordinates,

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r.$$

Therefore we get

$$\begin{aligned}
\frac{\nabla}{dt}\xi(t) &= \sum_{ij=1}^{2} \dot{c}^i(t)\,\xi^j(t)\,\Gamma_{ij}^1\Big(r(t),\varphi(t)\Big)\,\frac{\partial}{\partial r}\Big|_{c(t)} + \sum_{ij=1}^{2} \dot{c}^i(t)\,\xi^j(t)\,\Gamma_{ij}^2\Big(r(t),\varphi(t)\Big)\,\frac{\partial}{\partial \varphi}\Big|_{c(t)} \\
&= \dot{c}^2(t)\,\xi^2(t)\,\big(-r(t)\big)\,\frac{\partial}{\partial r}\Big|_{c(t)} + \left(\dot{c}^1(t)\,\xi^2(t)\,\frac{1}{r(t)} + \dot{c}^2(t)\,\xi^1(t)\,\frac{1}{r(t)}\right)\frac{\partial}{\partial \varphi}\Big|_{c(t)} \\
&= 1\cdot 1\cdot(-1)\,\frac{\partial}{\partial r}\Big|_{c(t)} + (0\cdot 1\cdot 1 + 1\cdot 0\cdot 1)\,\frac{\partial}{\partial \varphi}\Big|_{c(t)} \\
&= -\frac{\partial}{\partial r}\Big|_{c(t)}.
\end{aligned}$$

So indeed, we have obtained the same result.

2.5 Parallel transport

Definition 2.5.1. Let (M,g) be a semi-Riemannian manifold and $c : I \to M$ be a C^1-curve. A C^1-vector field ξ along c is called **parallel**, if

$$\frac{\nabla}{dt}\xi \equiv 0.$$

Example 2.5.2. Let $(M,g) = (\mathbb{R}^n, g_{\mathrm{eucl}})$ or $(\mathbb{R}^n, g_{\mathrm{Mink}})$. In Cartesian coordinates, a vector field $\xi(t) = \sum_{j=1}^n \xi^j(t)\,\frac{\partial}{\partial x^j}\Big|_{c(t)}$ along a curve c is parallel if and only if $\dot{\xi}^j(t) = 0$ for all $t \in I$, i.e., if and only if the ξ^j are constant.

Example 2.5.3. Let $(M,g) = (\mathbb{R}^2, g_{\mathrm{eucl}})$. Recall from Example 2.3.12 that the Christoffel symbols in polar coordinates (r,φ) are given by:

$$\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{22}^2 = 0, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \quad \Gamma_{22}^1 = -r.$$

Thus $\xi = \xi^1 \frac{\partial}{\partial r} + \xi^2 \frac{\partial}{\partial \varphi}$ is parallel along a curve c if and only if

$$\begin{aligned} 0 &= \frac{\nabla}{dt}\xi \\ &= \dot\xi^1 \frac{\partial}{\partial r} + \xi^1 \nabla_{\dot c^1 \frac{\partial}{\partial r} + \dot c^2 \frac{\partial}{\partial \varphi}} \frac{\partial}{\partial r} + \dot\xi^2 \frac{\partial}{\partial \varphi} + \xi^2 \nabla_{\dot c^1 \frac{\partial}{\partial r} + \dot c^2 \frac{\partial}{\partial \varphi}} \frac{\partial}{\partial \varphi} \\ &= \dot\xi^1 \frac{\partial}{\partial r} + \xi^1 \left(\dot c^1 \cdot 0 + \dot c^2 \cdot \frac{1}{c^1} \frac{\partial}{\partial \varphi} \right) + \dot\xi^2 \frac{\partial}{\partial \varphi} + \xi^2 \left(\dot c^1 \frac{1}{c^1} \frac{\partial}{\partial \varphi} + \dot c^2 (-c^1) \frac{\partial}{\partial r} \right) \\ &= \left(\dot\xi^1 - c^1 \dot c^2 \xi^2 \right) \frac{\partial}{\partial r} + \left(\dot\xi^2 + \frac{\dot c^2}{c^1} \xi^1 + \frac{\dot c^1}{c^1} \xi^2 \right) \frac{\partial}{\partial \varphi}. \end{aligned}$$

This is equivalent to:

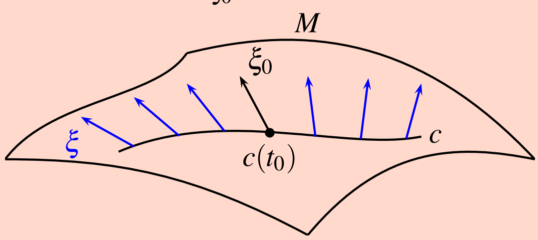
$$\dot\xi^1 - c^1 \dot c^2 \xi^2 = 0, \quad \dot\xi^2 + \frac{\dot c^2}{c^1} \xi^1 + \frac{\dot c^1}{c^1} \xi^2 = 0,$$

that is

$$\begin{pmatrix} \dot\xi^1 \\ \dot\xi^2 \end{pmatrix} = \begin{pmatrix} 0 & c^1 \dot c^2 \\ -\frac{\dot c^2}{c^1} & -\frac{\dot c^1}{c^1} \end{pmatrix} \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}.$$

This is a system of linear first order ordinary differential equations for (ξ^1, ξ^2).

Proposition 2.5.4. *Let (M,g) be a semi-Riemannian manifold and $c : I \to M$ be a C^1-curve and $t_0 \in I$. For any $\xi_0 \in T_{c(t_0)}M$ there exists exactly one parallel vector field ξ along c with $\xi(t_0) = \xi_0$.*



Proof. **Case 1:** Let $c(I)$ be contained in one chart and let $x : U \to V$ be such a chart. Then the condition $\frac{\nabla}{dt}\xi = 0$ is equivalent to

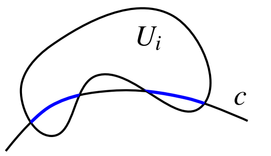
$$\dot\xi^k = - \sum_{i,j=1}^{n} \left(\Gamma_{ij}^k \circ x \circ c \right) \dot c^i \cdot \xi^j,$$

which is a system of linear ordinary equations of first order. Hence there exists a unique solution with initial condition

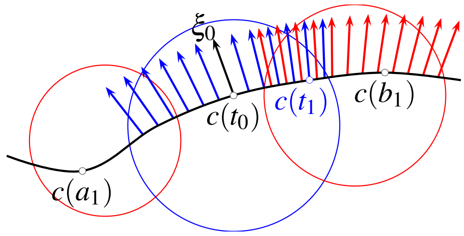
$$\left(\xi^1(t_0), \dots, \xi^n(t_0) \right) = (\xi_0^1, \dots, \xi_0^n).$$

Since the system is linear, the solution is defined on all of I.

Case 2: Suppose $c(I)$ is not contained in one chart.
Existence: The interval I can be open, closed or half-open. We restrict ourselves to open intervals, the other cases being slightly simpler. Write $I = (a,b)$ where $-\infty \leq a < b \leq \infty$. Choose $a < a_i < t_0 < b_i < b$ with $a_i \to a$ and $b_i \to b$ monotonically. Then $c([a_i,b_i])$ is compact and can be covered by finitely many charts $x_1 : U_1 \to V_1, \dots, x_N : U_N \to V_N$. W.l.o.g. we assume that $U_i \cap c([a_1,b_1])$ is connected.



Not something like this!



W.l.o.g. let $c(t_0) \in U_1$, otherwise renumber the charts. We solve the equation $\dfrac{\nabla}{dt}\xi = 0$ as in Case 1 with $\xi(t_0) = \xi_0$ in U_1.
If the solution is not defined on the whole of $[a_1,b_1]$, we choose $t_1 \in (a_1,b_1)$ with $c(t_1) \in U_1 \cap U_2$. Then we solve the equation in the chart x_2 with the initial condition $\xi(t_1)$, given by the previous solution.
Due to uniqueness in Case 1 both parallel vector fields coincide on $U_1 \cap U_2$. After finitely many steps we get a parallel vector field which is defined on $[a_1,b_1]$.
The same holds true for the next compact subinterval $[a_2,b_2]$ and we obtain a parallel vector field on $[a_2,b_2]$ which extends the one on $[a_1,b_1]$. By induction, we then find a parallel vector field on every $[a_i,b_i]$ extending the one on the smaller interval$[a_{i-1},b_{i-1}]$. Since $\bigcup_{i=1}^N [a_i,b_i] = (a,b)$ we obtain a parallel vector field ξ on (a,b) with $\xi(t_0) = \xi_0$.

Uniqueness: Let ξ and $\tilde{\xi}$ be two parallel vector fields along c with $\xi(t_0) = \tilde{\xi}(t_0) = \xi_0$. Write $I = I_{\text{good}} \sqcup I_{\text{bad}}$ where

$$\begin{aligned} I_{\text{good}} &= \{t \in I \mid \xi(t) = \tilde{\xi}(t)\} \\ I_{\text{bad}} &= \{t \in I \mid \xi(t) \neq \tilde{\xi}(t)\} \end{aligned}$$

Since ξ and $\tilde{\xi}$ are continuous, I_{good} is closed in I. For $t_1 \in I_{\text{good}}$ choose a chart $x : U \to V$ which contains $c(t_1)$. By uniqueness in Case 1 we then have $\xi(t) = \tilde{\xi}(t)$ for all $t \in I$ with $c(t) \in U$. Therefore a neighborhood of t_1 is contained in I_{good}. Hence I_{good} is open in I.
We have seen that I_{good} is open and closed in I. It is also non-empty because $t_0 \in I_{\text{good}}$. Since I is connected, we have $I = I_{\text{good}}$ and therefore $\xi(t) = \tilde{\xi}(t)$ for all $t \in I$. $\square$

Definition 2.5.5. Let M be a semi-Riemannian manifold and let $c : I \to M$ be a C^1-curve. Let $t_0, t_1 \in I$. The map

$$\begin{array}{rcl} P_{c,t_0,t_1} : T_{c(t_0)}M & \to & T_{c(t_1)}M, \\ \xi_0 & \mapsto & \xi(t_1), \end{array}$$

is called **parallel transport** along c. Here $\xi(t)$ is the parallel vector field along c with $\xi(t_0) = \xi_0$.

Proposition 2.5.6. *Let M, c, t_0, and t_1 as in Definition 2.5.5 and let $t_2 \in I$. Then we have:*

(a) $P_{c,t_0,t_1} : (T_{c(t_0)}M, g|_{c(t_0)}) \to (T_{c(t_1)}M, g|_{c(t_1)})$ *is a linear isometry;*

(b) $P_{c,t_0,t_2} = P_{c,t_1,t_2} \circ P_{c,t_0,t_1}$.

Proof. (a) Let $\xi_0, \eta_0 \in T_{c(t_0)}M$. Let ξ, η the corresponding parallel vector fields along c. Then

$$\frac{d}{dt} g(\xi, \eta) = g\Big(\underbrace{\frac{\nabla}{dt}\xi}_{=0}, \eta\Big) + g\Big(\xi, \underbrace{\frac{\nabla}{dt}\eta}_{=0}\Big) = 0.$$

Therefore $g(\xi, \eta)$ is constant, hence

$$\begin{array}{rcl} g\big(P_{c,t_0,t_1}(\xi_0), P_{c,t_0,t_1}(\eta_0)\big) & = & g\big(\xi(t_1), \eta(t_1)\big) \\ & = & g\big(\xi(t_0), \eta(t_0)\big) \\ & = & g(\xi_0, \eta_0). \end{array}$$

This proves that parallel transport is a linear isometry.

(b) is obvious. □

Remark 2.5.7. For $\xi_0 \in T_{c(t_0)}M$ the parallel vector field ξ with $\xi(t_0) = \xi_0$ is given by

$$\xi(t) = P_{c,t_0,t}(\xi_0).$$

We can reconstruct the Levi-Civita connection ∇ from parallel transport:

Proposition 2.5.8. *Let (M,g) be a semi-Riemannian manifold, let $c : I \to M$ be a C^1-curve, and let $t_0 \in I$. Then for every C^1-vector field ξ along c we get:*

$$\frac{\nabla}{dt}\xi\bigg|_{t_0} = \lim_{t\to t_0} \frac{P_{c,t,t_0}(\xi(t)) - \xi(t_0)}{t - t_0}.$$

Proof. Let $e_1(t_0), \ldots, e_n(t_0)$ be a basis of $T_{c(t_0)}M$. Let $e_1(t), \ldots, e_n(t)$ be the corresponding parallel vector fields along c.
By Proposition 2.5.6 (a), we know that $e_1(t), \ldots, e_n(t)$ form a basis of $T_{c(t)}M$ for every $t \in I$. Write $\xi(t) = \sum_{j=1}^n \xi^j(t) e_j(t)$. Then

$$\begin{aligned}
\frac{P_{c,t,t_0}(\xi(t)) - \xi(t_0)}{t - t_0} &= \frac{\sum_{j=1}^n \xi^j(t) \overbrace{P_{c,t,t_0}(e_j(t))}^{=e_j(t_0)} - \sum_{j=1}^n \xi^j(t_0) e_j(t_0)}{t - t_0} \\
&= \sum_{j=1}^n \frac{\xi^j(t) - \xi^j(t_0)}{t - t_0} e_j(t_0) \\
&\xrightarrow{t\to t_0} \sum_{j=1}^n \dot{\xi}^j(t_0) e_j(t_0).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\frac{\nabla}{dt}\xi|_{t_0} &= \frac{\nabla}{dt}\left(\sum_{j=1}^n \xi^j e_j\right)\bigg|_{t=0} \\
&= \sum_{j=1}^n \left(\dot{\xi}^j(t_0) e_j(t_0) + \xi^j(t_0) \underbrace{\frac{\nabla}{dt} e_j|_{t_0}}_{=0}\right) \\
&= \sum_{j=1}^n \dot{\xi}^j(t_0) e_j(t_0).
\end{aligned}$$

□

We have the following scheme of geometric structures:

semi-Riemannian metric $\rightsquigarrow$ covariant derivative ∇ $\leftrightsquigarrow$ parallel transport P

Remark 2.5.9. If $\psi : M \to \tilde{M}$ is a local isometry and if $c : I \to M$ is a C^1-curve, consider the image curve $\tilde{c} := \psi \circ c$. Then we have for every C^1-vector field ξ along c:

$$\xi \text{ parallel along } c \quad \Longleftrightarrow \quad \tilde{\xi} := d\psi \circ \xi \text{ parallel along } \tilde{c}.$$

In particular, the following diagram commutes:

$$\begin{tikzcd} T_{c(t_0)}M \arrow[r, "P_{c,t_0,t_1}"] \arrow[d, "d\psi|_{c(t_0)}"'] & T_{c(t_1)}M \arrow[d, "d\psi|_{c(t_1)}"] \\ T_{\tilde{c}(t_0)}\tilde{M} \arrow[r, "P_{\tilde{c},t_0,t_1}"] & T_{\tilde{c}(t_1)}\tilde{M} \end{tikzcd}$$

Remark 2.5.10. In general, parallel transport depends on the curve joining two given points. This means, in general we have $P_{c,t_0,t_1} \neq P_{\hat{c},s_0,s_1}$ if c and $\hat{c}$ are two curves in M with $c(t_0) = \hat{c}(s_0)$ and $c(t_1) = \hat{c}(s_1)$. In this respect, Euclidean space is not typical.

2.6 Geodesics

Definition 2.6.1. Let (M,g) be a semi-Riemannian manifold and $c : [a,b] \to M$ a C^1-curve. Then we call

$$E[c] := \frac{1}{2}\int_a^b g\big(\dot{c}(t),\dot{c}(t)\big)\,dt$$

the **energy** of c.

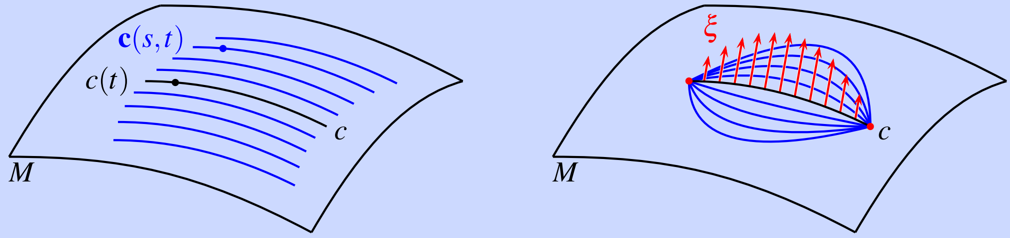
Remark 2.6.2. If (M,g) is Riemannian, then $g(\dot{c},\dot{c}) \geq 0$ and therefore $E[c] \geq 0$ (and equal to 0 if and only if c is constant).

Question. Are there curves with minimal energy joining two given endpoints? More generally, are there curves with "stationary energy"?

Definition 2.6.3. Let M be a differentiable manifold and $c : [a,b] \to M$ a smooth curve. A **variation** of c is a smooth map

$$\mathbf{c} : (-\varepsilon,\varepsilon) \times [a,b] \to M$$

with $\mathbf{c}(0,t) = c(t)$ for all $t \in [a,b]$. If $\mathbf{c}(s,a) = c(a)$ and $\mathbf{c}(s,b) = c(b)$ for all $s \in (-\varepsilon,\varepsilon)$ then we call $c(s,t)$ a **variation with fixed endpoints**.



The vector field $\xi(t) := \frac{\partial \mathbf{c}}{\partial s}(0,t)$ is called the **variational vector field**.

Remark 2.6.4. The variational vector field ξ of a variation with fixed endpoints satisfies

$$\xi(a) = 0 \quad \text{and} \quad \xi(b) = 0.$$

Theorem 2.6.5 (First variation of the energy). *Let (M,g) be a semi-Riemannian manifold, let $c : [a,b] \to M$ be a smooth curve and let $\mathbf{c} : (-\varepsilon,\varepsilon) \times [a,b] \to M$ be a variation of this curve. Let ξ be the variational vector field. Write $c_s(t) = c(s,t)$. Then*

$$\frac{d}{ds} E[c_s]\big|_{s=0} = -\int_a^b g\left(\xi(t), \frac{\nabla}{dt}\dot{c}(t)\right) dt + g\big(\xi(b),\dot{c}(b)\big) - g\big(\xi(a),\dot{c}(a)\big).$$

Proof. We compute:

$$\begin{aligned}
\frac{d}{ds} E[c_s]\big|_{s=0} &= \frac{1}{2}\,\frac{d}{ds}\Big|_{s=0} \int_a^b g\big(\dot{c}_s(t),\dot{c}_s(t)\big)\,dt \\
&= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\Big|_{s=0} g\left(\frac{\partial \mathbf{c}}{\partial t}(s,t), \frac{\partial \mathbf{c}}{\partial t}(s,t)\right) dt \\
&= \frac{1}{2}\int_a^b \left[g\left(\frac{\nabla}{\partial s}\frac{\partial \mathbf{c}}{\partial t}(0,t), \frac{\partial \mathbf{c}}{\partial t}(0,t)\right) + g\left(\frac{\partial \mathbf{c}}{\partial t}(0,t), \frac{\nabla}{\partial s}\frac{\partial \mathbf{c}}{\partial t}(0,t)\right)\right] dt \\
&= \int_a^b g\left(\frac{\nabla}{\partial s}\frac{\partial \mathbf{c}}{\partial t}(0,t), \frac{\partial \mathbf{c}}{\partial t}(0,t)\right) dt \\
&\overset{(*)}{=} \int_a^b g\left(\frac{\nabla}{\partial t}\frac{\partial \mathbf{c}}{\partial s}(0,t), \frac{\partial \mathbf{c}}{\partial t}(0,t)\right) dt
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b g\left(\frac{\nabla}{dt}\xi(t), \dot c(t)\right) dt \\
&= \int_a^b \left[\frac{d}{dt} g(\xi(t), \dot c(t)) - g\left(\xi(t), \frac{\nabla}{dt}\dot c(t)\right)\right] dt \\
&= g\big(\xi(b), \dot c(b)\big) - g\big(\xi(a), \dot c(a)\big) - \int_a^b g\left(\xi(t), \frac{\nabla}{dt}\dot c(t)\right) dt.
\end{aligned}$$

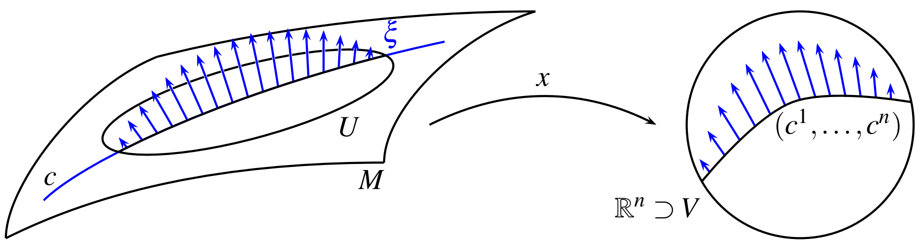
Equality $(*)$ holds because of torsion-freeness of the Levi-Civita connection. □

Corollary 2.6.6. *If the variation has fixed endpoints then*

$$\frac{d}{ds} E[c_s]\big|_{s=0} = -\int_a^b g\left(\xi(t), \frac{\nabla}{dt}\dot c(t)\right) dt.$$

Lemma 2.6.7. *Let $c : [a,b] \to M$ be a smooth curve and ξ a smooth vector field along c. Then there exists a variation $\mathbf{c}$ of c with variational vector field ξ. If $\xi(a) = 0$ and $\xi(b) = 0$, then we can choose the variation with fixed endpoints.*

Proof. a) We first consider the case that $\mathrm{supp}(\xi)$ is contained in a chart $x : U \to V$, i.e., $c(t) \in U$ whenever $\xi(t) \neq 0$.



We write $\xi(t) = \sum_{j=1}^n \xi^j(t) \left.\frac{\partial}{\partial x^j}\right|_{c(t)}$ and we set

$$\mathbf{c}(s,t) := \begin{cases} x^{-1}\Big(\big(c^1(t), \dots, c^n(t)\big) + s\big(\xi^1(t), \dots, \xi^n(t)\big)\Big), & c(t) \in U \\ c(t), & c(t) \notin U \end{cases}$$

Then we have for the corresponding variational vector field:

$$\begin{aligned}\left(\frac{\partial \mathbf{c}}{\partial s}(0,t)\right)^j &= dx^j\left(\frac{\partial \mathbf{c}}{\partial s}(0,t)\right)\\ &= \frac{\partial(x^j\circ \mathbf{c})}{\partial s}(0,t)\\ &= \left.\frac{\partial\left(c^j(t)+s\xi^j(t)\right)}{\partial s}\right|_{s=0}\\ &= \xi^j(t).\end{aligned}$$

Hence the variation $\mathbf{c}$ has the variational vector field ξ. Moreover, if ξ vanishes at the endpoints, then $\mathbf{c}$ has fixed endpoints.

b) In the general case, cover the compact set $c([a,b])$ with finitely many charts and construct the variation piecewise. □

Remark 2.6.8. Later, when we have the Riemannian exponential map at our disposal, we will be able to directly write down a suitable variation without usage of charts.

Notation 2.6.9. Let M be a differentiable manifold and $p,q\in M$. Then we set

$$\Omega_{p,q}(M) := \left\{\text{smooth curves } c:[a,b]\to M \text{ with } c(a)=p \text{ and } c(b)=q\right\}.$$

Corollary 2.6.10. *Let (M,g) be a semi-Riemannian manifold and $c\in\Omega_{p,q}(M)$. Then the following are equivalent:*

(i) The curve c is a "critical point" of the energy functional, i.e.,

$$\frac{d}{ds}E[c_s]\big|_{s=0}=0$$

for all variations c_s of c with fixed endpoints;

(ii) For all t we have

$$\frac{\nabla}{dt}\dot c(t)=0.$$

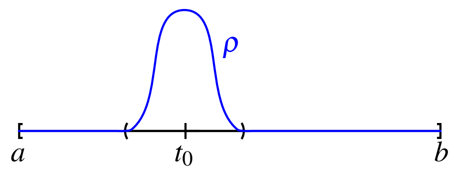
Proof. The implication "(ii)⇒(i)" is directly clear by Corollary 2.6.6. We show "(i)⇒(ii)". Let $[a,b]$ be the parameter interval of c. Assume there exists a $t_0\in(a,b)$ with $\frac{\nabla}{dt}\dot c(t_0)\neq 0$. Then there exists a $\xi_0 \in T_{c(t_0)}M$ with

$$g\left(\xi_0, \frac{\nabla}{dt}\dot{c}(t_0)\right) > 0$$

because g is non-degenerate. Let $\tilde{\xi}$ be the parallel vector field along c with $\tilde{\xi}(t_0) = \xi_0$. By continuity there exists an $\varepsilon > 0$ such that $(t_0 - \varepsilon, t_0 + \varepsilon) \subset (a,b)$ and

$$g\left(\tilde{\xi}(t), \frac{\nabla}{dt}\dot{c}(t)\right) > 0$$

holds for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. We choose a smooth function $\rho : [a,b] \to \mathbb{R}$ with $\rho(t) > 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\rho(t) = 0$ otherwise.



Set $\xi(t) := \rho(t) \cdot \tilde{\xi}(t)$. Then we have:

$$g\left(\xi(t), \frac{\nabla}{dt}\dot{c}(t)\right) = \rho(t) \cdot g\left(\tilde{\xi}(t), \frac{\nabla}{dt}\dot{c}(t)\right) \begin{cases} > 0 & \text{for } t \in (t_0 - \varepsilon, t_0 + \varepsilon) \\ = 0 & \text{otherwise} \end{cases}.$$

By Lemma 2.6.7 we can choose a variation of c with fixed endpoints and variational vector field ξ. Then we have for this variation

$$\frac{d}{ds}E[c_s]\Big|_{s=0} = -\int_a^b g\left(\xi(t), \frac{\nabla}{dt}\dot{c}(t)\right) dt < 0$$

which contradicts the assumption. Hence we have $\frac{\nabla}{dt}\dot{c} = 0$ on (a,b) and by continuity also on the whole of $[a,b]$. □

Definition 2.6.11. A smooth curve c with $\frac{\nabla}{dt}\dot{c} = 0$ is called a **geodesic**.

Example 2.6.12. Let $(M,g) = (\mathbb{R}^n, g_{\mathrm{eucl}})$ or $(\mathbb{R}^n, g_{\mathrm{Mink}})$. In Cartesian coordinates $x^1, \ldots, x^n$ we have:

$$\begin{aligned} \frac{\nabla}{dt}\dot{c} = 0 \quad &\Longleftrightarrow \quad \ddot{c}^1 = 0, \ldots, \ddot{c}^n = 0 \\ &\Longleftrightarrow \quad c^j(t) = p^j + tv^j \\ &\Longleftrightarrow \quad c(t) = p + tv. \end{aligned}$$

Hence geodesics are straight lines, parametrized with constant speed.

Lemma 2.6.13. *For any geodesic c the quantity $g\big(\dot c, \dot c\big)$ is constant.*

Proof. We compute

$$\tfrac{d}{dt}\, g\big(\dot c, \dot c\big) = 2 \cdot g\Big(\underbrace{\tfrac{\nabla}{dt}\dot c}_{=0}, \dot c\Big) = 0.$$

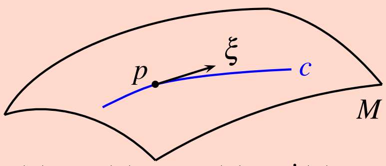
Λ

Definition 2.6.14. A smooth curve c is called

- **parametrized by arc-length** , if $g(\dot c, \dot c) \equiv 1$,
- **parametrized by proper time** , if $g(\dot c, \dot c) \equiv -1$
- **parametrized proportional to arc-length** , if $g(\dot c, \dot c) \equiv \alpha > 0$,
- **parametrized proportional proper time** , if $g(\dot c, \dot c) \equiv -\alpha < 0$ and
- a **null curve**, if $g(\dot c, \dot c) \equiv 0$.

Theorem 2.6.15 (Existence and uniqueness of geodesics). *Let (M, g) be a semi-Riemannian manifold.*

For any $p \in M$ and $\xi \in T_pM$ there exists an open interval I with $0 \in I$ and a geodesic $c : I \to M$ with $c(0) = p$ and $\dot c(0) = \xi$.



If $c : I \to M$ and $\tilde c : \tilde I \to M$ are two such geodesics with $c(0) = \tilde c(0)$ and $\dot c(0) = \dot{\tilde c}(0)$, then c and $\tilde c$ coincide on their common domain $I \cap \tilde I$.

Proof. In a chart $x : U \to V$ in p we consider the equation for a geodesic

$$\frac{\nabla}{dt}\dot c = 0 \quad \Longleftrightarrow \quad \ddot c^k + \sum_{i,j=1}^{n} \Gamma^k_{ij}\big(c^1, \dots, c^n\big) \cdot \dot c^i \cdot \dot c^j = 0$$

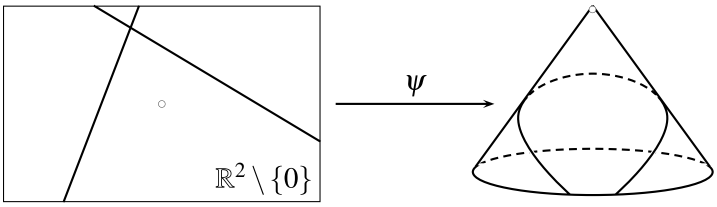
for $k = 1, \dots, n$ and $c^k = x^k \circ c$. This is a system of ordinary differential equations of second order. By the Theorem of Picard-Lindelöf the we get the assertion. $\square$

Remark 2.6.16. The system of differential equations is non-linear. Therefore we do not have a-priori control over the maximal domain of definition I of the geodesic.

Remark 2.6.17. If $\psi : M \to \tilde{M}$ is a local isometry, then

$$c : I \to M \text{ is a geodesic} \quad \Longleftrightarrow \quad \psi \circ c : I \to \tilde{M} \text{ is a geodesic}.$$

Example 2.6.18. Let $M = (\mathbb{R}^2 \setminus \{0\}, g_{\mathrm{eucl}})$ be the Euclidean plane with the origin removed and let $\tilde{M} = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = z^2/3,\, z < 0)\}$ be a cone with the cone tip removed and equipped with the first fundamental form $\tilde{g}$. Now $\psi : M \to \tilde{M}$, $\psi(u,v) = \frac{1}{2\sqrt{u^2+v^2}}(u^2 - v^2, 2uv, -\sqrt{3}(u^2 + v^2))$, can be checked to be a local isometry. Hence ψ maps straight lines in M onto geodesics in $\tilde{M}$.



Definition 2.6.19. Let $\psi : M \to M$ be a diffeomorphism. Then we call

$$\mathrm{Fix}(\psi) := \{p \in M \,|\, \psi(p) = p\}$$

the **fixed point set** of ψ.

Proposition 2.6.20. *Let (M,g) be a semi-Riemannian manifold and $\psi \in \mathrm{Isom}(M,g)$. Then for any $p \in \mathrm{Fix}(\psi)$ and any $\xi \in T_pM$ with $d\psi|_p(\xi) = \xi$ the geodesic $c : I \to M$ with*

$$c(0) = p \quad \text{and} \quad \dot{c}(0) = \xi$$

is entirely contained in $\mathrm{Fix}(\psi)$, i.e., for all $t \in I$ we have $c(t) \in \mathrm{Fix}(\psi)$.

Proof. Set $\tilde{c}(t) := \psi \circ c(t)$. Since ψ is an isometry, $\tilde{c}$ is also a geodesic. Furthermore, we have:

$$\begin{aligned} \tilde{c}(0) &= \psi\big(c(0)\big) = \psi(p) = p = c(0) \text{ and} \\ \dot{\tilde{c}}(0) &= d\psi|_{c(0)}\big(\dot{c}(0)\big) = d\psi|_p(\xi) = \xi = \dot{c}(0). \end{aligned}$$

Applying the uniqueness part of Theorem 2.6.15 we get for all $t \in I$:

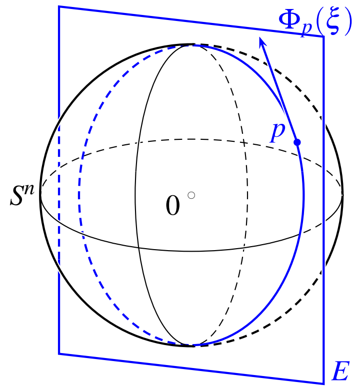
$$c(t) = \tilde{c}(t) = \psi\big(c(t)\big).$$

This means $c(t) \in \mathrm{Fix}(\psi)$ for all t. $\square$

Example 2.6.21. We use Proposition 2.6.20 to determine the geodesics of the sphere (S^n, g_{std}). Let $p \in S^n$ and $\xi \in T_pS^n$. Let $E \subset \mathbb{R}^{n+1}$ be the two-dimensional vector subspace spanned by p and $\Phi_p(\xi)$. Let $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection about E. Then $A \in \mathrm{O}(n+1)$. Hence

$$\psi := A|_{S^n} \in \mathrm{Isom}(S^n, g_{\mathrm{std}}).$$

Then $\mathrm{Fix}(A) = E$ and therefore $\mathrm{Fix}(\psi) = E \cap S^n$ is a great circle.



Proposition 2.6.20 implies that $c(t) \in E \cap S^n$ for all t. Since geodesics on a Riemannian manifold are parametrized proportional to arc-length we seek an arc-length parametrization of this great circle:

$$c(t) = p \cdot \cos(\alpha t) + \frac{\Phi_p(\xi)}{\|\Phi_p(\xi)\|} \cdot \sin(\alpha t).$$

We have to satisfy the initial conditions:

$$\begin{aligned} c(0) &= p \quad \text{is satisfied.} \\ \frac{d}{dt}c(0) &= \frac{\Phi_p(\xi)}{\|\Phi_p(\xi)\|} \cdot \alpha \quad \text{and therefore} \quad \alpha = \|\Phi_p(\xi)\| = \|\xi\|. \end{aligned}$$

Then we get $\frac{d}{dt}c(0) = \Phi_p(\xi)$, i.e., $\dot{c}(0) = \xi$. Thus the geodesic c with initial conditions $c(0) = p$ and $\dot{c}(0) = \xi$ is given by

$$c(t) = p \cdot \cos\left(\|\xi\| t\right) + \frac{\Phi_p(\xi)}{\|\xi\|} \cdot \sin\left(\|\xi\| t\right).$$

Remark 2.6.22. Let (M, g) be a semi-Riemannian manifold and $p \in M$. For $\xi \in T_pM$ let c_ξ be the geodesic with

$$c_\xi(0) = p \quad \text{and} \quad \dot{c}_\xi(0) = \xi.$$

For $\alpha \in \mathbb{R}$ set $\tilde{c}(t) := c_\xi(\alpha t)$. Then

$$\frac{\nabla}{dt}\dot{\tilde{c}}(t) = \frac{\nabla}{dt}\left(\alpha \cdot \dot{c}_\xi(\alpha t)\right) = \alpha^2 \left(\frac{\nabla}{dt}\dot{c}_\xi\right)(\alpha t) = 0.$$

Hence $\tilde{c}$ is also a geodesic. Since its initial conditions are

$$\tilde{c}(0) = c_\xi(0) = p,$$
$$\dot{\tilde{c}}(0) = \alpha \cdot \dot{c}_\xi(0) = \alpha\xi,$$

we conclude $\tilde{c} = c_{\alpha\xi}$. In particular, $c_\xi(\alpha) = c_{\alpha\xi}(1)$.

Definition 2.6.23. Let M be a semi-Riemannian manifold and $p \in M$. For $\xi \in T_pM$ set

$$\exp_p(\xi) := c_\xi(1)$$

if the maximal domain of the geodesic c_ξ contains 1. Furthermore, set

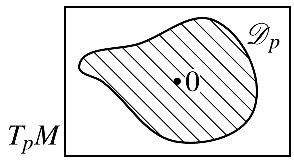
$$\mathscr{D}_p := \{\xi \in T_pM \,|\, 1 \text{ is contained in the maximal domain of } c_\xi\}.$$

Then we call $\exp_p : \mathscr{D}_p \to M$ the **Riemannian exponential map** (at the point p).

Remark 2.6.24. (1) By Remark 2.6.22 we know $\exp_p(t \cdot \xi) = c_{t\xi}(1) = c_\xi(t)$. Thus $t \mapsto \exp_p(t\xi)$ is *the* geodesic with initial values p and ξ.

(2) For any $p \in M$ we have $\exp_p(0) = p$ because c_0 is the constant curve $c_0(t) = p$.

(3) Let $\xi \in \mathscr{D}_p$. Then c_ξ is defined on $[0,1]$. Let $0 \leq \alpha \leq 1$. From $c_{\alpha\xi}(t) = c_\xi(\alpha t)$ we see that $c_{\alpha\xi}$ is defined on $\left[0, \frac{1}{\alpha}\right] \supset [0,1]$. Therefore $\alpha\xi \in \mathscr{D}_p$. This shows that $\mathscr{D}_p$ is star-shaped with respect to $0 \in T_pM$.



(4) Set $\mathscr{D} := \bigcup_{p \in M} \mathscr{D}_p \subset TM$ and $\exp : \mathscr{D} \to M$, $\exp(\xi) := \exp_{\pi(\xi)}(\xi)$. The theory of ordinary differential equations implies that $\mathscr{D}$ is open and that $\exp$ is a smooth map (smooth dependence of solutions of the initial values). In particular, $\mathscr{D}_p = \mathscr{D} \cap T_pM$ is open in T_pM.

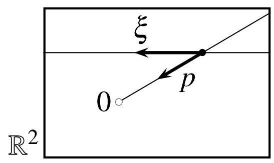
Example 2.6.25. (1) Let $(M,g) = (\mathbb{R}^n, g_{\mathrm{eucl}})$ or $(\mathbb{R}^n, g_{\mathrm{Mink}})$. Then we have:

$$\exp_p(\xi) = p + 1 \cdot \Phi_p(\xi) = p + \Phi_p(\xi).$$

Here $\mathscr{D}_p = T_p\mathbb{R}^n$.

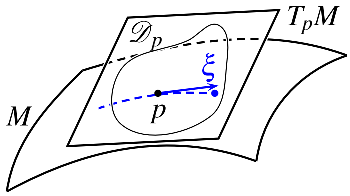
(2) Let $(M,g) = (\mathbb{R}^2 \setminus \{0\}, g_{\mathrm{eucl}})$. Then

$$\mathscr{D}_p = T_pM \setminus \left\{ -t \cdot {\Phi_p}^{-1}(p) \,|\, t \geq 1 \right\}.$$



(3) Let $(M,g) = (S^n, g_{\text{std}})$. Then we have $\mathscr{D}_p = T_pM$ and

$$\exp_p(\xi) = p\cdot\cos(\|\xi\|) + \frac{\Phi_p(\xi)}{\|\xi\|}\cdot\sin(\|\xi\|).$$



Lemma 2.6.26. *The differential of the map $\exp_p : \mathscr{D}_p \to M$ at 0 is given by the canonical isomorphism*

$$d\exp_p|_0 = \Phi_0 : T_0\mathscr{D}_p = T_0T_pM \to T_pM.$$

Proof. Let $\xi \in T_pM$. Then we have:

$$d\exp_p\big|_0(\Phi_0{}^{-1}(\xi)) = d\exp_p\big|_0\left(\frac{d}{dt}(t\xi)\big|_{t=0}\right) = \frac{d}{dt}\exp_p(t\xi)\big|_{t=0} = \xi. \qquad \square$$

In the literature Lemma 2.6.26 is sometimes formulated slightly imprecisely as follows

$$\mathrm{id}_{T_pM} = d\exp_p\big|_0 : T_pM \to T_pM.$$

Corollary 2.6.27. *For $p \in M$ there exists an open neighborhood $\mathscr{V}_p \subset \mathscr{D}_p \subset T_pM$ of 0, such that*

$$\exp_p|_{\mathscr{V}_p} : \mathscr{V}_p \to \exp_p(\mathscr{V}_p) =: U_p$$

is a diffeomorphism.

Proof. By Lemma 2.6.26 $d\exp_p|_0$ is invertible. The inverse function theorem yields the claim. $\square$

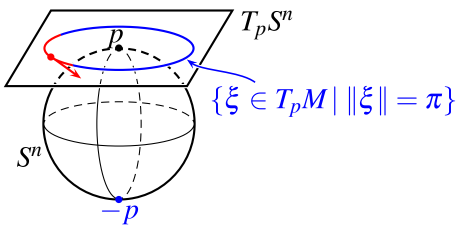
Remark 2.6.28. In general, $\exp_p : \mathscr{D}_p \to \exp_p(\mathscr{D}_p) \subset M$ is not a diffeomorphism because $\exp_p$ is not injective in general. Moreover, $d\exp_p|_\xi$ is not necessarily invertible for $\xi \neq 0$.

Example 2.6.29. Let $(M,g) = (S^n, g_{\text{std}})$. For $p \in S^n$ we have $\mathscr{D}_p = T_pM$ and

$$\exp_p(\xi) = p\cdot\cos\big(\|\xi\|\big) + \frac{\Phi_p(\xi)}{\|\xi\|}\cdot\sin\big(\|\xi\|\big).$$

In particular, for any $\xi \in T_pM$ with $\|\xi\| = \pi$ we have

$$\exp_p(\xi) = p\cdot\cos(\pi) = -p.$$



For $\xi \in T_pM$ with $\|\xi\| = \pi$ the differential $d\exp_p|_\xi$ has the $(n-1)$-dimensional kernel

$$\{\eta \in T_\xi T_p S^n \,|\, \Phi_\xi(\eta) \perp \xi\}.$$

Now we construct coordinates which are well adapted to the geometry and to this end we choose a generalized orthonormal basis $E_1, \dots, E_n$ of T_pM regarding $g|_p$, that is

$$g|_p(E_i, E_j) = \varepsilon_i \delta_{ij}, \quad \varepsilon_i \in \{\pm 1\}.$$

We get a linear isomorphism $A : \mathbb{R}^n \to T_pM$, $(\alpha^1, \dots, \alpha^n) \mapsto \sum\limits_{i=1}^{n} \alpha^i E_i$.

$$\begin{tikzcd}
T_pM \supset \mathscr{V}_p \arrow[r, "\exp_p", "\approx"'] & U_p \subset M \\
\mathbb{R}^n \supset V_p \arrow[u, "\cong"', "A"] \arrow[ur, "\approx"'] &
\end{tikzcd}$$

We put $V_p := A^{-1}(\mathscr{V}_p)$. Then $\exp_p \circ A : V_p \to U_p$ is a diffeomorphism. Set $x := (\exp_p \circ A)^{-1}$. Then $x : U_p \to V_p$ is a chart.

Definition 2.6.30. The coordinates we just defined are called **Riemannian normal coordinates** around the point p.

In which sense are these coordinates well adapted to the geometry?

Proposition 2.6.31. *Let (M, g) be a semi-Riemannian manifold and $p \in M$. Let $g_{ij} : V_p \to \mathbb{R}$ be the metric coefficients and $\Gamma_{ij}^k : V_p \to \mathbb{R}$ be the Christoffel symbols in Riemannian normal coordinates around p. Then we have:*

$$x(p) = 0, \quad g_{ij}(0) = \varepsilon_i \delta_{ij}, \quad \Gamma_{ij}^k(0) = 0.$$

Proof. a) Clearly, we have $x(p) = A^{-1}\big(\exp_p{}^{-1}(p)\big) = A^{-1}(0) = 0$.

b) Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$. Then

$$\begin{aligned}
g_{ij}(0) &= g|_p\big(dx^{-1}|_0(e_i), dx^{-1}|_p(e_j)\big) \\
&= g|_p\big(d(\exp_p \circ A)|_0(e_i), d(\exp_p \circ A)|_0(e_i)\big) \\
&= g|_p\big(d\exp_p|_0(E_i), d\exp_p|_0(E_j)\big) \\
&\overset{\text{L. 2.6.26}}{=} g|_p(E_i, E_j) \\
&= \varepsilon_i \delta_{ij}
\end{aligned}$$

c) Let $v = (v^1, \dots, v^n) \in \mathbb{R}^n$. Then $c(t) = x^{-1}(tv) = \exp_p(tAv)$ is a geodesic with $c(0) = p$ and $\dot{c}(0) = Av$. In Riemannian normal coordinates the equation for a geodesic is in this case

$$0 = \ddot{c}^k(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k\big(c^1(t), \dots, c^n(t)\big) \cdot \dot{c}^i(t) \cdot \dot{c}^j(t).$$

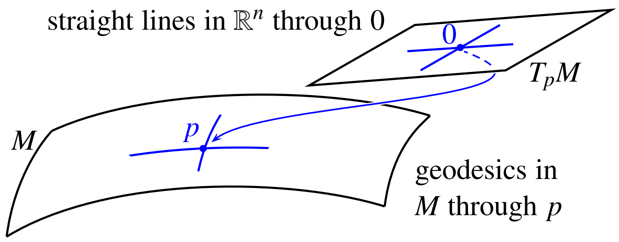
Here $c^k(t) = x^k(c(t)) = tv^k$, $\dot{c}^k(t) = v^k$ and $\ddot{c}^k(t) = 0$. For $t = 0$ we get

$$0 = 0 + \sum_{i,j=1}^{n} \Gamma_{ij}^k(0, \dots, 0) \cdot v^i \cdot v^j.$$

For each k we define a bilinear form β^k on $\mathbb{R}^n$ by $\beta^k(y,z) := \sum_{i,j=1}^n \Gamma_{i,j}^k(0)\, y^i z^j$. These bilinear forms are symmetric because:

$$\beta^k(z,y) = \sum_{i,j=1}^{n} \Gamma_{ij}^k(0)\, z^i y^j \underset{\substack{\text{Exchanging}\\ \text{indices}}}{=} \sum_{j,i=1}^{n} \Gamma_{ji}^k(0)\, z^j y^i \underset{\substack{\nabla \text{ free of}\\ \text{torsion}}}{=} \sum_{i,j=1}^{n} \Gamma_{ij}^k(0)\, y^i z^j = \beta^k(y,z).$$

Since we know that $\beta^k(v,v) = 0$ for all $v \in \mathbb{R}^n$, polarization yields $\beta^k(y,z) = 0$ for all $y,z \in \mathbb{R}^n$. This means $\Gamma_{ij}^k(0) = 0$ for all i,j,k.



□

Example 2.6.32. Let $(M,g) = (\mathbb{R}^n, g_{\text{eucl}})$ or $(\mathbb{R}^n, g_{\text{Mink}})$ and $p \in M$. Choose

$$A = \Phi_p = \text{ canonical isomorphism } \mathbb{R}^n \to T_p\mathbb{R}^n.$$

Then we have $\exp_p(Av) = p + v$, thus Riemannian normal coordinates around p are given by

$$x : \mathbb{R}^n \to \mathbb{R}^n, \quad x(q) = q - p.$$

Up to translation by $-p$, Riemannian normal coordinates coincide with Cartesian coordinates.

Corollary 2.6.33. *In Riemannian normal coordinates we have for the Taylor expansion of $g_{ij} : V_p \to \mathbb{R}$ around 0:*

$$g_{ij}(x) = \varepsilon_i \delta_{ij} + O(\|x\|^2).$$

Proof. Expanding g_{ij} into a Taylor series at 0 yields

$$g_{ij}(x) = g_{ij}(0) + \sum_{k=1}^{n} \frac{\partial g_{ij}}{\partial x^k}(0) \cdot x^k + O\big(\|x\|^2\big).$$

In the proof of Theorem 2.3.8 we found

$$\frac{\partial g_{ij}}{\partial x^k}(0) = \sum_{l=1}^{n} \Big(\Gamma^l_{ki}(0)\, g_{lj}(0) + \Gamma^l_{kj}(0)\, g_{il}(0)\Big)$$

which is zero in our situation because the Christoffel symbols vanish at 0. ◻

3 Curvature

We now come to one of the central concepts of differential geometry, that of curvature. We will see that there are various inequivalent notions of curvature. We start with the most basic one.

3.1 The Riemannian curvature tensor

Definition 3.1.1. Let (M,g) be a semi-Riemannian manifold and $p \in M$.
Let $\xi \in T_pM$ and $\eta,\zeta \in \Xi_p(M)$. Then we have $\nabla_\eta \zeta \in \Xi_p(M)$ and

$$\nabla^2_{\xi,\eta}\zeta := \nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta}\zeta \in T_pM$$

is called the **second covariant derivative** of ζ in the direction ξ and η.

Lemma 3.1.2. *The second covariant derivative $\nabla^2_{\xi,\eta}\zeta$ depends on η only via $\eta|_p$, i.e., if $\eta,\tilde{\eta} \in \Xi_p(M)$ with $\eta|_p = \tilde{\eta}|_p$ then*

$$\nabla^2_{\xi,\eta}\zeta = \nabla^2_{\xi,\tilde{\eta}}\zeta.$$

Proof. We choose Riemannian normal coordinates x around p. In these coordinates we write (using the Einstein summation convention) the vector fields locally as:

$$\xi = \xi^i \left.\frac{\partial}{\partial x^i}\right|_p, \quad \eta = \eta^j \frac{\partial}{\partial x^j}, \quad \zeta = \zeta^k \frac{\partial}{\partial x^k}.$$

Since the Christoffel symbols vanish at 0 we get

$$\nabla_\xi \eta = \nabla_{\xi^i \left.\frac{\partial}{\partial x^i}\right|_p}\left(\eta^j \frac{\partial}{\partial x^j}\right) \;=\; \xi^i \frac{\partial \eta^j}{\partial x^i}(0) \left.\frac{\partial}{\partial x^j}\right|_p$$

and therefore

$$\nabla_{\nabla_\xi \eta}\zeta = \xi^i \frac{\partial \eta^j}{\partial x^i}(0) \nabla_{\left.\frac{\partial}{\partial x^j}\right|_p}\left(\zeta^k \frac{\partial}{\partial x^k}\right) = \xi^i \frac{\partial \eta^j}{\partial x^i}(0) \frac{\partial \zeta^k}{\partial x^j}(0) \left.\frac{\partial}{\partial x^k}\right|_p. \tag{3.1}$$

Moreover,

$$\nabla_\eta \zeta = \nabla_{\eta^j \frac{\partial}{\partial x^j}}\left(\zeta^k \frac{\partial}{\partial x^k}\right) = \eta^j \left(\frac{\partial \zeta^k}{\partial x^j}\frac{\partial}{\partial x^k} + \zeta^k \Gamma^m_{jk} \frac{\partial}{\partial x^m}\right)$$

and hence (again using that the Christoffel symbols vanish)

$$\begin{aligned}\nabla_\xi \nabla_\eta \zeta &= \nabla_{\xi^i \frac{\partial}{\partial x^i}\big|_p} \left(\eta^j \frac{\partial \zeta^k}{\partial x^j} \frac{\partial}{\partial x^k} + \eta^j \zeta^k \Gamma^m_{jk} \frac{\partial}{\partial x^m} \right) \\ &= \xi^i \frac{\partial \eta^j}{\partial x^i}(0) \frac{\partial \zeta^k}{\partial x^j}(0) \left.\frac{\partial}{\partial x^k}\right|_p + \xi^i \eta^j(0) \frac{\partial^2 \zeta^k}{\partial x^j \partial x^i}(0) \left.\frac{\partial}{\partial x^k}\right|_p + \xi^i \eta^j(0) \zeta^k \frac{\partial \Gamma^m_{jk}}{\partial x^i}(0) \left.\frac{\partial}{\partial x^m}\right|_p .\end{aligned} \tag{3.2}$$

Subtracting (3.1) from (3.2) we see that the terms containing a derivative of the η^j cancel and we are left with

$$\nabla^2_{\xi,\eta} \zeta = \left[\xi^i \eta^j(0) \frac{\partial^2 \zeta^k}{\partial x^i \partial x^j}(0) + \xi^i \eta^j(0) \zeta^m(0) \frac{\partial \Gamma^k_{im}}{\partial x^i}(0) \right] \left.\frac{\partial}{\partial x^k}\right|_p . \tag{3.3}$$

This expression depends on η only via the $\eta^j(0)$ which are the coefficients of $\eta|_p$. □

Consequence. *The expression $\nabla^2_{\xi,\eta}\zeta$ is well defined for $\xi, \eta \in T_pM$ and $\zeta \in \Xi_p$.*

Lemma 3.1.4. *For $\xi, \eta \in T_pM$ and $\zeta \in \Xi_p(M)$*

$$R(\xi,\eta)\zeta := \nabla^2_{\xi,\eta}\zeta - \nabla^2_{\eta,\xi}\zeta$$

depends only on ζ via $\zeta|_p$. Thus $R(\xi,\eta)\zeta \in T_pM$ is well defined for $\xi, \eta, \zeta \in T_pM$.

Proof. Again we choose Riemann normal coordinates around p and recall (3.3):

$$\nabla^2_{\xi,\eta}\zeta = \xi^i \eta^j \left(\frac{\partial^2 \zeta^k}{\partial x^i \partial x^j}(0) + \zeta^m(0) \frac{\partial \Gamma^k_{jm}}{\partial x^i}(0) \right) \left.\frac{\partial}{\partial x^k}\right|_p .$$

Relabeling summation indices and using the Schwarz theorem we get

$$\begin{aligned} R(\xi,\eta)\zeta &= \left(\xi^i \eta^j - \xi^j \eta^i \right) \left(\frac{\partial^2 \zeta^k}{\partial x^i \partial x^j}(0) + \zeta^m(0) \frac{\partial \Gamma^k_{jm}}{\partial x^i}(0) \right) \left.\frac{\partial}{\partial x^k}\right|_p \\ &= \xi^i \eta^j \left(\frac{\partial^2 \zeta^k}{\partial x^i \partial x^j}(0) - \frac{\partial^2 \zeta^k}{\partial x^j \partial x^i}(0) + \zeta^m(0) \frac{\partial \Gamma^k_{jm}}{\partial x^i}(0) - \zeta^m(0) \frac{\partial \Gamma^k_{im}}{\partial x^j}(0) \right) \left.\frac{\partial}{\partial x^k}\right|_p \\ &= \xi^i \eta^j \zeta^m(0) \left(\frac{\partial \Gamma^k_{jm}}{\partial x^i}(0) - \frac{\partial \Gamma^k_{im}}{\partial x^j}(0) \right) \left.\frac{\partial}{\partial x^k}\right|_p . \end{aligned}$$

□

Definition 3.1.5. The map

$$\begin{aligned} R : T_pM \times T_pM \times T_pM &\to T_pM \\ (\xi,\eta,\zeta) &\mapsto R(\xi,\eta)\zeta \end{aligned}$$

is called the **Riemann curvature tensor** at the point p.

Representation in local coordinates.
Let $x : U \to V$ be a chart on M. Then R is determined on U by smooth functions $R^l_{kij} : V \to \mathbb{R}$, defined by

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \sum_{l=1}^n R^l_{kij}\frac{\partial}{\partial x^l}. \tag{3.4}$$

As we have already seen, we have in Riemann normal coordinates:

$$\boxed{R^l_{kij}(0) = \frac{\partial \Gamma^l_{jk}}{\partial x^i}(0) - \frac{\partial \Gamma^l_{ik}}{\partial x^j}(0)}$$

Remark 3.1.6. One can check (not difficult but tedious) that we have in *arbitrary* coordinates

$$\boxed{R^l_{kij} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_{m=1}^n (\Gamma^m_{kj}\Gamma^l_{mi} - \Gamma^m_{ki}\Gamma^l_{mj})}$$

In particular, if the curvature tensor $R : T_pM \times T_pM \times T_pM \to T_pM$ does not vanish at the point p, then there does not exist a chart containing p for which $\Gamma^k_{ij} \equiv 0$.

Proposition 3.1.7 (Symmetries of the curvature tensor). *Let (M,g) be a semi-Riemannian manifold, $p \in M$ and $\xi,\eta,\zeta,\nu \in T_pM$. Then we have:*

(1) $R : T_pM \times T_pM \times T_pM \to T_pM$ is trilinear;

(2) $R(\xi,\eta)\zeta = -R(\eta,\xi)\zeta$;

(3) $g|_p(R(\xi,\eta)\zeta,\nu) = -g|_p(R(\xi,\eta)\nu,\zeta)$;

(4) First Bianchi identity:
$R(\xi,\eta)\zeta + R(\eta,\zeta)\xi + R(\zeta,\xi)\eta = 0$;

(5) $g|_p(R(\xi,\eta)\zeta,\nu) = g|_p(R(\zeta,\nu)\xi,\eta)$.

Proof. (1) is obvious because $\nabla^2_{\xi,\eta}\zeta$ is already $\mathbb{R}$-linear in ξ, η and ζ.

(2) is also clear by definition.

(3) We choose Riemannian normal coordinates around p and consider the special case

$$\xi = \frac{\partial}{\partial x^i}\bigg|_p, \quad \eta = \frac{\partial}{\partial x^j}\bigg|_p, \quad \zeta = \frac{\partial}{\partial x^k}\bigg|_p, \quad \nu = \frac{\partial}{\partial x^l}\bigg|_p.$$

Then we find

$$\begin{aligned} g|_p\Big(R(\xi,\eta)\zeta,\nu\Big) &= g|_p\left(R\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p\right)\frac{\partial}{\partial x^k}\bigg|_p, \frac{\partial}{\partial x^l}\bigg|_p\right) \\ &= g|_p\left(\sum_{m=1}^n R^m_{kij}(0)\,\frac{\partial}{\partial x^m}\bigg|_p, \frac{\partial}{\partial x^l}\bigg|_p\right) \\ &= \sum_{m=1}^n R^m_{kij}(0)\cdot g|_p\left(\frac{\partial}{\partial x^m}\bigg|_p, \frac{\partial}{\partial x^l}\bigg|_p\right) \\ &= \sum_{m=1}^n g_{ml}(0)\cdot R^m_{kij}(0). \end{aligned}$$

From the proof of Theorem 2.3.8 we recall

$$\frac{\partial g_{ij}}{\partial x^k} = \sum_{m=1}^n (g_{mj}\Gamma^m_{ki} + g_{mi}\Gamma^m_{kj})$$

and thus, in Riemannian normal coordinates,

$$\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0) = \sum_{m=1}^n \left(g_{mj}(0)\frac{\partial \Gamma^m_{ki}}{\partial x^l}(0) + g_{mi}(0)\frac{\partial \Gamma^m_{kj}}{\partial x^l}(0)\right).$$

Thus

$$\begin{aligned} 0 &= \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l}(0) - \frac{\partial^2 g_{ij}}{\partial x^l \partial x^k}(0) \\ &= \sum_{m=1}^n \left(g_{mj}(0)\frac{\partial \Gamma^m_{ki}}{\partial x^l}(0) + g_{mi}(0)\frac{\partial \Gamma^m_{kj}}{\partial x^l}(0) - g_{mj}(0)\frac{\partial \Gamma^m_{li}}{\partial x^k}(0) - g_{mi}(0)\frac{\partial \Gamma^m_{lj}}{\partial x^k}(0)\right) \\ &= \sum_{m=1}^n \Big(g_{mj}(0)R^m_{ilk}(0) + g_{mi}(0)R^m_{jlk}(0)\Big) \end{aligned}$$

Renaming the indices via $l \mapsto i$, $k \mapsto j$, $i \mapsto k$, $j \mapsto l$ leads to

$$0 = \sum_{m=1}^n \Big(g_{ml}(0)R^m_{kij}(0) + g_{mk}(0)R^m_{lij}(0)\Big)$$

and therefore

$$\sum_{m=1}^n g_{ml}(0)R^m_{kij}(0) = -\sum_{m=1}^n g_{mk}(0)R^m_{lij}(0).$$

This proves the assertion for coordinate fields ξ, η, ζ, ν of Riemannian normal coordinates. By multilinearity the assertion follows for general ξ, η, ζ and ν.

(4) The first Bianchi identity is equivalent to

$$R^l_{kij} + R^l_{ijk} + R^l_{jki} = 0.$$

We check this in Riemann normal coordinates:

$$\begin{aligned}
&R^l_{kij}(0) + R^l_{ijk}(0) + R^l_{jki}(0) \\
&= \cancel{\frac{\partial \Gamma^l_{jk}}{\partial x^i}}(0) - \cancel{\frac{\partial \Gamma^l_{ik}}{\partial x^j}}(0) + \cancel{\frac{\partial \Gamma^l_{ki}}{\partial x^j}}(0) - \cancel{\frac{\partial \Gamma^l_{ji}}{\partial x^k}}(0) + \cancel{\frac{\partial \Gamma^l_{ij}}{\partial x^k}}(0) - \cancel{\frac{\partial \Gamma^l_{kj}}{\partial x^i}}(0) \\
&= 0.
\end{aligned}$$

(5) Proof by an explicit calculation:

$$\begin{aligned}
0 &\overset{(4)}{=} \cancel{g|_p(R(\eta,\zeta)\xi,\nu)} + \cancel{g|_p(R(\zeta,\xi)\eta,\nu)} + \underline{g|_p(R(\xi,\eta)\zeta,\nu)} \\
&\quad + \cancel{g|_p(R(\zeta,\xi)\nu,\eta)} + \cancel{g|_p(R(\xi,\nu)\zeta,\eta)} + \underline{\underline{g|_p(R(\nu,\zeta)\xi,\eta)}} \\
&\quad + \cancel{g|_p(R(\xi,\nu)\eta,\zeta)} + \cancel{g|_p(R(\nu,\eta)\xi,\zeta)} + \underline{g|_p(R(\eta,\xi)\nu,\zeta)} \\
&\quad + \cancel{g|_p(R(\nu,\eta)\zeta,\xi)} + \cancel{g|_p(R(\eta,\zeta)\nu,\xi)} + \underline{\underline{g|_p(R(\zeta,\nu)\eta,\xi)}} \\
&\overset{(2),(3)}{=} 2g|_p(R(\xi,\eta)\zeta,\nu) + 2g|_p(R(\zeta,\nu)\eta,\xi) \\
&= 2(g|_p(R(\xi,\eta)\zeta,\nu) - g|_p(R(\zeta,\nu)\xi,\eta)).
\end{aligned}$$

□

Example 3.1.8. Let $(M,g) = (\mathbb{R}^n, g_{\mathrm{eucl}})$ or $(\mathbb{R}^n, g_{\mathrm{Mink}})$. In Cartesian coordinates we have $\Gamma^k_{ij} = 0$. Thus we get $R^l_{kij} = 0$ for all i,j,k,l and therefore $R \equiv 0$.

Definition 3.1.9. A semi-Riemannian manifold with $R \equiv 0$ is called **flat**.

Warning. In the literature there are two different sign conventions for R: For example, our R is the negative of the curvature tensor as defined in [ON83].

Lemma 3.1.11. *Let (M,g) and $(\tilde{M},\tilde{g})$ be semi-Riemannian manifolds and $\psi : M \to \tilde{M}$ a local isometry. Let $p \in M$. Then the curvature tensors R of M at p and $\tilde{R}$ of $\tilde{M}$ at $\psi(p)$ are related by:*

$$d\psi|_p\big(R(\xi,\eta)\zeta\big) = \tilde{R}\big(d\psi|_p(\xi), d\psi|_p(\eta)\big)d\psi|_p(\zeta)$$

for all $\xi,\eta,\zeta \in T_pM$.

Proof. Let $x : U \to V$ be a chart on M with $p \in U$. By making U smaller if necessary we can assume that $\psi : U \to \tilde{U} := \psi(U)$ is a diffeomorphism. Then $\tilde{x} := x \circ \psi^{-1} : \tilde{U} \to V$ is a chart on $\tilde{M}$.

Since ψ is a local isometry, it follows that $g_{ij} = \tilde{g}_{ij} : V \to \mathbb{R}$, where the g_{ij} are the components of g w.r.t. x and the $\tilde{g}_{ij}$ are the components of $\tilde{g}$ w.r.t. $\tilde{x}$. Therefore the Christoffel symbols coincide, $\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k$, hence so do the components of the curvature tensors, $R_{kij}^l = \tilde{R}_{kij}^l$. From (3.4) we conclude

$$\begin{aligned}
d\psi\left(R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\right) &= \sum_{l=1}^n R_{kij}^l d\psi\Big(\frac{\partial}{\partial x^l}\Big) \\
&= \sum_{l=1}^n R_{kij}^l \frac{\partial}{\partial \tilde{x}^l} \\
&= \sum_{l=1}^n \tilde{R}_{kij}^l \frac{\partial}{\partial \tilde{x}^l} \\
&= \tilde{R}\left(\frac{\partial}{\partial \tilde{x}^i},\frac{\partial}{\partial \tilde{x}^j}\right)\frac{\partial}{\partial \tilde{x}^k} \\
&= \tilde{R}\left(d\psi\Big(\frac{\partial}{\partial x^i}\Big), d\psi\Big(\frac{\partial}{\partial x^j}\Big)\right) d\psi\Big(\frac{\partial}{\partial x^k}\Big).
\end{aligned}$$

This proves the lemma for the coordinate basis tangent vectors $\frac{\partial}{\partial x^i}$. By trilinearity of R it follows for all tangent vectors. $\square$

Alternatively one can define the curvature tensor as a multilinear map $R : T_pM \times T_pM \times T_pM \times T_pM \to \mathbb{R}$ by

$$\mathbf{R}(\xi,\eta,\zeta,\nu) = g\big(R(\xi,\eta)\zeta,\nu\big).$$

In this version, $\mathbf{R}$ is known as the *Riemannian* $(4,0)$*-curvature tensor*. In local coordinates $x : U \to V$ around p, we define $R_{ijkl} : V \to \mathbb{R}$ by

$$R_{ijkl}(x(p)) := \mathbf{R}\left(\frac{\partial}{\partial x^i}\bigg|_p, \frac{\partial}{\partial x^j}\bigg|_p, \frac{\partial}{\partial x^k}\bigg|_p, \frac{\partial}{\partial x^l}\bigg|_p\right).$$

Then we have

$$\begin{aligned}
R_{ijkl} &\overset{\substack{\text{Prop.}\\ 3.1.7(5)}}{=} R_{klij} \\
&= g\left(R\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right)\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
&= g\left(\sum_{m=1}^n R_{ikl}^m \frac{\partial}{\partial x^m},\frac{\partial}{\partial x^j}\right) \\
&= \sum_{m=1}^n R_{ikl}^m\, g\left(\frac{\partial}{\partial x^m},\frac{\partial}{\partial x^j}\right) \\
&= \sum_{m=1}^n g_{mj} R_{ikl}^m.
\end{aligned}$$

Hence we have

$$\boxed{R_{ijkl} \;=\; \sum_{m=1}^{n} g_{mj} R^m_{ikl}}$$

We have lowered the upper index. On the other hand we have

$$R^l_{kij} = \sum_{m=1}^{n} \delta^l_m R^m_{kij} = \sum_{a,m=1}^{n} g^{al} g_{ma} R^m_{kij},$$

hence

$$\boxed{R^l_{kij} \;=\; \sum_{a=1}^{n} g^{al} R_{kaij}}$$

In this case we have raised the index.

Proposition 3.1.12. *Let (M,g) be a semi-Riemannian manifold. In Riemannian normal coordinates we have:*

$$g_{ij}(x) = \varepsilon_i \delta_{ij} + \frac{1}{3} \sum_{k,l=1}^{n} R_{ikjl}(0) x^k x^l + O\big(\|x\|^3\big).$$

Proof. We already know that $g_{ij}(x) = \varepsilon_i \delta_{ij} + O(\|x\|^2)$ by Corollary 2.6.33. In the following we will use the Einstein summation convention and the following abbreviations

$$f_{,k} := \frac{\partial f}{\partial x^k} \quad \text{and} \quad f_{,k\ell} = \frac{\partial^2 f}{\partial x^k \partial x^\ell}$$

for the first and the second partial derivatives. In the proof of Theorem 2.3.8 we have seen that

$$g_{ij,k} = \Gamma^m_{ki} g_{mj} + \Gamma^m_{kj} g_{mi} \,.$$

We differentiate this equation with respect to x^ℓ, evaluate at 0 and use that the Christoffel symbols vanish at 0:

$$g_{ij,k\ell}(0) = \Gamma^m_{ki,\ell}(0) \cdot g_{mj}(0) + \Gamma^m_{kj,\ell}(0)\, g_{mi}(0) \,. \tag{3.5}$$

Claim:

$$\Gamma^k_{ij,\ell}(0) + \Gamma^k_{\ell i,j}(0) + \Gamma^k_{j\ell,i}(0) \;=\; 0\,. \tag{3.6}$$

Proof of the claim: In normal coordinates the straight lines $t \mapsto t \cdot x$ give geodesics. The equation for geodesics then looks like:

$$0 = \Gamma^k_{ij}(t \cdot x) x^i x^j \,.$$

We differentiate this with respect to t and evaluate at $t = 0$:

$$0 = \left.\frac{d}{dt}\right|_{t=0} \Gamma^k_{ij}(tx) x^i x^j = \Gamma^k_{ij,\ell}(0) x^\ell x^i x^j \,.$$

Thus we have for every k a polynomial of third degree in x, namely $P^k(x) := \Gamma^k_{ij,\ell}(0)x^i x^j x^\ell$, which vanished identically. Thus for every monomial $x^\alpha x^\beta x^\gamma$ the sum of coefficients $\Gamma^k_{ij,\ell}(0)$ with $x^i x^j x^\ell = x^\alpha x^\beta x^\gamma$ has to vanish. The six permutations of the three lower indices yield

$$\Gamma^k_{ij,\ell}(0) + \Gamma^k_{\ell i,j}(0) + \Gamma^k_{j\ell,i}(0) + \Gamma^k_{ji,\ell}(0) + \Gamma^k_{i\ell,j}(0) + \Gamma^k_{\ell j,i}(0) = 0.$$

The symmetry of the Christoffel symbols in their two lower indices implies the claim. ✓
From $R^\ell_{kij}(0) = \Gamma^\ell_{jk,i}(0) - \Gamma^\ell_{ik,j}(0)$ we conclude:

$$\begin{aligned} R_{k\ell ij}(0) &= \left(\Gamma^m_{jk,i}(0) - \Gamma^m_{ik,j}(0)\right) g_{m\ell}(0) \\ &\overset{(3.6)}{=} -\left(\Gamma^m_{ij,k}(0) + \Gamma^m_{ki,j}(0) + \Gamma^m_{ik,j}(0)\right) g_{m\ell}(0) \\ &= -\left(\Gamma^m_{ij,k}(0) + 2\Gamma^m_{ki,j}(0)\right) g_{m\ell}(0)\,. \end{aligned} \tag{3.7}$$

Thus we have:

$$\begin{aligned} 2R_{ikj\ell}(0)x^k x^\ell &\overset{Prop.\ 3.1.7}{=} \left(-R_{kij\ell}(0) - R_{\ell jik}(0)\right) x^k x^\ell \\ &\overset{(3.7)}{=} \left(\Gamma^m_{j\ell,k}(0) + 2\Gamma^m_{kj,\ell}(0)\right) g_{mi}(0)x^k x^\ell \\ &\quad + \left(\Gamma^m_{ik,\ell}(0) + 2\Gamma^m_{\ell i,k}(0)\right) g_{mj}(0)x^k x^\ell \\ &\overset{(*)}{=} \left(\Gamma^m_{j\ell,k}(0) + 2\Gamma^m_{kj,\ell}(0)\right) g_{mi}(0)x^k x^\ell \\ &\quad + \left(\Gamma^m_{i\ell,k}(0) + 2\Gamma^m_{ki,\ell}(0)\right) g_{mj}(0)x^\ell x^k \\ &\overset{(3.5)}{=} \left(g_{ij,\ell k}(0) + 2g_{ij,k\ell}(0)\right) \cdot x^k x^\ell \\ &= 3g_{ij,k\ell}(0) \cdot x^k x^\ell\,. \end{aligned}$$

At equality $(*)$ we renamed the summation parameter k to ℓ and vice versa. Thus we get for the second term in the Taylor expansion

$$\frac{1}{2} g_{ij,k\ell}(0)x^k x^\ell = \frac{1}{3} R_{ikj\ell}(0) \cdot x^k x^\ell\,. \qquad \square$$

3.2 Sectional curvature

The Riemannian curvature tensor contains the full curvature information of a Riemannian manifold but for many applications other curvature entities are more suitable. We will introduce the sectional curvature, Ricci curvature and scalar curvature in this and the following sections.
We start with some linear algebra. Let V be a finite dimensional real vector space with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Later we will apply this to $V = T_pM$ and $\langle\cdot,\cdot\rangle = g|_p(\cdot,\cdot)$.

Definition 3.2.1. A subvector space $U \subset V$ is called **non-degenerate**, if $\langle\cdot,\cdot\rangle|_{U\times U} : U \times U \to \mathbb{R}$ is a non-degenerate bilinear form on U. We define:

$$G_k\big(V, \langle\cdot,\cdot\rangle\big) := \big\{k\text{-dimensional, non-degenerate subvector spaces of } V\big\}.$$

Note that every subvector space is non-degenerate if $\langle\cdot,\cdot\rangle$ is definite. We set

$$Q : V \times V \to \mathbb{R}, \quad Q(\xi,\eta) := \langle\xi,\xi\rangle\langle\eta,\eta\rangle - \langle\xi,\eta\rangle^2.$$

Lemma 3.2.2. *For two-dimensional subvector spaces $E \subset V$ the following assertions are equivalent:*

(i) $E \in G_2(V, \langle\cdot,\cdot\rangle)$*;*

(ii) there exists a basis ξ,η of E with $Q(\xi,\eta) \neq 0$;

(iii) for all basis ξ,η of E we have $Q(\xi,\eta) \neq 0$.

Proof. With respect to any basis ξ,η of E, the bilinear form $\langle\cdot,\cdot\rangle|_{E\times E}$ is represented by the matrix

$$A_{\xi,\eta} := \begin{pmatrix} \langle\xi,\xi\rangle & \langle\eta,\xi\rangle \\ \langle\xi,\eta\rangle & \langle\eta,\eta\rangle \end{pmatrix}.$$

Then we have $Q(\xi,\eta) = \det A_{\xi,\eta}$ which proves the lemma. □

Remark 3.2.3. (a) If $\langle\cdot,\cdot\rangle$ is positive definite, then

$$\sqrt{Q(\xi,\eta)} = \text{area of the parallelogram spanned by } \xi \text{ and } \eta.$$

(b) The two-dimensional subvector space $E \subset V$ is degenerate if and only if there exists a basis ξ,η of E with $\langle\xi,\xi\rangle = \langle\xi,\eta\rangle = 0$. Namely,

"$\Leftarrow$": $Q(\xi,\eta) = \underbrace{\langle\xi,\xi\rangle}_{=0}\langle\eta,\eta\rangle - \underbrace{\langle\xi,\eta\rangle}_{=0}^2 = 0.$

"$\Rightarrow$": Let E be degenerate, i.e., $\langle\cdot,\cdot\rangle|_{E\times E}$ is degenerate. Then there exists a $\xi \in E\setminus\{0\}$ with $\langle\xi,\zeta\rangle = 0$ for all $\zeta \in E$. Now complete ξ by some η to a basis of E. ✓

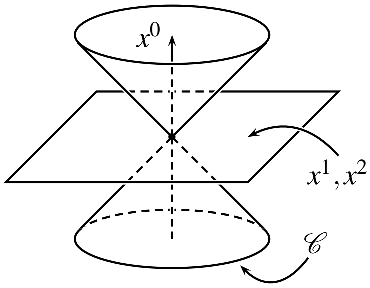
Example 3.2.4. Let $V = \mathbb{R}^3$ with the Minkowski product

$$\langle\langle \xi, \eta \rangle\rangle = -\xi^0\eta^0 + \xi^1\eta^1 + \xi^2\eta^2.$$

Consider the lightcone

$$\mathscr{C} := \{\xi \in \mathbb{R}^3 \setminus \{0\} \mid \langle\langle \xi, \xi \rangle\rangle = 0\}.$$

Then the plane $E \subset \mathbb{R}^3$ is degenerate if and only if $E = T_p\mathscr{C}$ for some $p \in \mathscr{C}$.



Namely, assume $c : (-\varepsilon, \varepsilon) \to \mathscr{C}$ is a smooth curve with $c(0) = p$ and $\dot{c}(0) = \xi$. Then we have:

$$\langle\langle c(t), c(t) \rangle\rangle = 0 \quad \forall\, t \in (-\varepsilon, \varepsilon)$$

$$\Rightarrow \quad 0 = \left.\frac{d}{dt}\right|_{t=0} \langle\langle c(t), c(t) \rangle\rangle = 2\,\langle\langle \dot{c}(0), c(0) \rangle\rangle = 2\,\langle\langle \xi, p \rangle\rangle$$

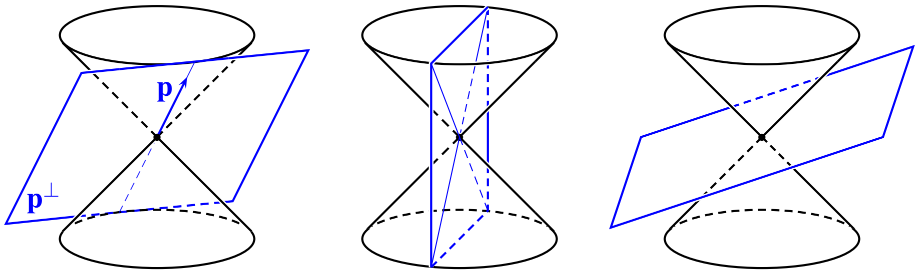
$\Rightarrow$ $T_p\mathscr{C} \subset p^\perp$, where both are two-dimensional subvector spaces of $\mathbb{R}^3$

$\Rightarrow$ $T_p\mathscr{C} = p^\perp$

$\Rightarrow$ for $\xi = p$ and any $\eta \in T_p\mathscr{C}$ which is not a multiple of ξ we obtain a basis of $T_p\mathscr{C}$ with $\langle\langle \xi, \xi \rangle\rangle = \langle\langle \xi, \eta \rangle\rangle = 0$

$\Rightarrow$ $T_p\mathscr{C}$ is degenerate.

Conversely, if E is degenerate, then we choose a basis ξ, η of E such that $\langle\langle \xi, \xi \rangle\rangle = \langle\langle \xi, \eta \rangle\rangle = 0$. We put $p := \xi$. Clearly $p \in \mathscr{C}$. Now we have $E \subset p^\perp = T_p\mathscr{C}$. Since both E and $T_p\mathscr{C}$ are two-dimensional we conclude $E = T_p\mathscr{C}$. ✓



degenerate | non-degenerate (indefinite) | non-degenerate (definite)

Lemma 3.2.5. *Let V be a finite-dimensional real vector space with non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let $\mathbf{R}: V \times V \times V \times V \to \mathbb{R}$ be multilinear with*

$$\mathbf{R}(\xi,\eta,\zeta,\nu) = -\mathbf{R}(\eta,\xi,\zeta,\nu) = -\mathbf{R}(\xi,\eta,\nu,\zeta)$$

for all $\xi,\eta,\zeta,\nu \in V$. Then for $E \in G_2(V,\langle\cdot,\cdot\rangle)$ and any basis ξ,η of E the expression

$$K(E) := \frac{\mathbf{R}(\xi,\eta,\eta,\xi)}{Q(\xi,\eta)}$$

does not depend on the choice of the basis ξ,η of E, but only on E itself.

Proof. Let μ,ν be another basis of E with $\mu = a\xi + b\eta$ and $\nu = c\xi + d\eta$. Then we have:

$$\begin{aligned}
\mathbf{R}(\mu,\nu,\nu,\mu) &= \mathbf{R}(a\xi+b\eta, c\xi+d\eta, c\xi+d\eta, a\xi+b\eta) \\
&= adcb\cdot\mathbf{R}(\xi,\eta,\xi,\eta) + adda\cdot\mathbf{R}(\xi,\eta,\eta,\xi) + bccb\cdot\mathbf{R}(\eta,\xi,\xi,\eta) \\
&\quad\ + bcda\cdot\mathbf{R}(\eta,\xi,\eta,\xi) \\
&= \big(-abcd + a^2d^2 + b^2c^2 - abcd\big)\cdot\mathbf{R}(\xi,\eta,\eta,\xi) \\
&= (ad-bc)^2\cdot\mathbf{R}(\xi,\eta,\eta,\xi) \qquad (3.8)
\end{aligned}$$

The map $\mathbf{R}_1 : V \times V \times V \times V \to \mathbb{R}$, defined by

$$\mathbf{R}_1(\xi,\eta,\zeta,\nu) := \langle\xi,\nu\rangle\langle\eta,\zeta\rangle - \langle\xi,\zeta\rangle\langle\eta,\nu\rangle$$

has all the symmetries of the curvature tensor as in Proposition 3.1.7. Hence we get

$$\underbrace{\mathbf{R}_1(\mu,\nu,\nu,\mu)}_{=Q(\mu,\nu)} = (ad-bc)^2\underbrace{\mathbf{R}_1(\xi,\eta,\eta,\xi)}_{=Q(\xi,\eta)}. \qquad (3.9)$$

Dividing (3.8) by (3.9) proves the lemma. □

Set $G_2(M,g) := \bigcup_{p\in M} G_2(T_pM, g|_p)$.

Definition 3.2.6. The function $K : G_2(M,g) \to \mathbb{R}$, defined by

$$K(E) := \frac{\mathbf{R}(\xi,\eta,\eta,\xi)}{Q(\xi,\eta)},$$

where ξ,η is a basis of E, is called **sectional curvature** of (M,g). Here $\mathbf{R}$ is the Riemannian $(4,0)$-curvature tensor.

Remark 3.2.7. The sectional curvature is only defined for manifolds of dimension at least 2. If $\dim(M) = 1$, then $\mathbf{R}(\xi,\eta,\zeta,\nu) = 0$ for all $\xi,\eta,\zeta,\nu \in T_pM$ due to the skew-symmetry in ξ and η.

Definition 3.2.8. If (M,g) is a two-dimensional semi-Riemannian manifold, then we call

$$K : M \to \mathbb{R}, \quad K(p) := K(T_pM)$$

the **Gauß curvature** of M.

Remark 3.2.9. The sectional curvature determines the curvature tensor, as can be seen by

$$\begin{aligned}
6R(\xi,\eta,\zeta,\nu) \;=\; & K(\xi+\nu,\eta+\zeta)Q(\xi+\nu,\eta+\zeta) - K(\eta+\nu,\xi+\zeta)Q(\eta+\nu,\xi+\zeta) \\
& - K(\xi,\eta+\zeta)Q(\xi,\eta+\zeta) - K(\eta,\xi+\nu)Q(\eta,\xi+\nu) \\
& - K(\zeta,\xi+\nu)Q(\zeta,\xi+\nu) - K(\nu,\eta+\zeta)Q((\nu,\eta+\zeta) \\
& + K(\xi,\eta+\nu)Q(\xi,\eta+\nu) + K(\eta,\zeta+\xi)Q(\eta,\zeta+\xi) \\
& + K(\zeta,\eta+\nu)Q(\zeta,\eta+\nu) + K(\nu,\xi+\zeta)Q(\nu,\xi+\zeta) \\
& + K(\xi,\zeta)Q(\xi,\zeta) + K(\eta,\nu)Q(\eta,\nu) - K(\xi,\nu)Q(\xi,\nu) - K(\eta,\zeta)Q(\eta,\zeta)
\end{aligned}$$

for all $\xi,\eta,\zeta,\nu \in T_pM$, for which the corresponding sectional curvatures are defined. The set of quadruples (ξ,η,ζ,ν), that satisfies this, is open and dense in $T_pM \times T_pM \times T_pM \times T_pM$. By continuity this determines R on all of $T_pM \times T_pM \times T_pM \times T_pM$.

Special case: If $K(E)$ only depends on p but not on the particular plane $E \subset T_pM$ (satisfied automatically if $\dim(M) = 2$, but not in general if $\dim(M) \geq 3$), then:

$$R(\xi,\eta,\zeta,\nu) = K(p)\big(\langle\eta,\zeta\rangle\langle\xi,\nu\rangle - \langle\xi,\zeta\rangle\langle\eta,\nu\rangle\big).$$

Moreover, we always have: $K = 0 \;\Leftrightarrow\; R = 0$.

3.3 Ricci- and scalar curvature

The Riemann curvature tensor and sectional curvature can be computed from one another. They contain the same amount of information. Both are rather complicated objects. In this section we

introduce two simplified curvature concepts which however contain less information than the full curvature tensor.
Let (M,g) be a semi-Riemannian manifold and $p \in M$. The Riemann curvature tensor at the point $p \in M$ is a multilinear map

$$R : T_pM \times T_pM \times T_pM \to T_pM.$$

For fixed $\xi, \eta \in T_pM$ we get a linear map

$$R(\xi,\cdot)\eta : T_pM \to T_pM, \quad \zeta \mapsto R(\xi,\zeta)\eta.$$

Definition 3.3.1. The map $\mathrm{ric} : T_pM \times T_pM \to \mathbb{R}$,

$$\mathrm{ric}(\xi,\eta) := -\mathrm{tr}(R(\xi,\cdot)\eta) = \mathrm{tr}(R(\cdot,\xi)\eta)$$

is called the **Ricci curvature** at the point p.

Remark 3.3.2. Let V be a n-dimensional $\mathbb{R}$-vector space with non-degenerate symmetric bilinear form g and $E_1,\dots,E_n$ be a generalized orthonormal basis of (V,g), that is $g(E_i,E_j) = \varepsilon_i\delta_{i,j}$ with $\varepsilon_i = \pm 1$. Then for every endomorphism $A : V \to V$ we have

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \varepsilon_i \cdot g(A(E_i),E_i). \tag{3.10}$$

Why? If we define $\omega_i : V \to \mathbb{R}$ by $\omega_i(\xi) := \varepsilon_i \cdot g(\xi,E_i)$ then $\omega_1,\dots,\omega_n$ is the dual basis of V^* to $E_1,\dots,E_n$. Hence

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \omega_i(A(E_i)) = \sum_{i=1}^{n} \varepsilon_i \cdot g(A(E_i),E_i).$$

The local description of Ricci curvature is similar to that of the semi-Riemannian metric itself: For any chart $x : U \to V$ of M we define the functions

$$\mathrm{ric}_{ij} : V \to \mathbb{R}, \quad \mathrm{ric}_{ij}(x(p)) := \mathrm{ric}\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right).$$

Lemma 3.3.3 (Properties of the Ricci curvature). *(i) The map* ric *is bilinear and symmetric on T_pM.*

(ii) For any generalized orthonormal basis $E_1,\dots,E_n$ of $(T_pM, g|_p)$ we have:

$$\mathrm{ric}(\xi,\eta) = \sum_{i=1}^{n} \varepsilon_i \cdot g(R(\xi,E_i)E_i,\eta).$$

(iii) We have:

$$\mathrm{ric}_{ij} = \sum_{k=1}^{n} R^k_{ikj}.$$

Proof. (i) Bilinearity of ric follows directly from trilinearity of R. We show symmetry of ric:

$$
\begin{aligned}
\mathrm{ric}(\eta,\xi) &= \sum_{i=1}^{n} \varepsilon_i \cdot g\big(R(\eta,E_i)E_i,\xi\big) \\
&\overset{\substack{\text{Prop.}\\ 3.1.7(5)}}{=} \sum_{i=1}^{n} \varepsilon_i \cdot g\big(R(E_i,\xi)\eta,E_i\big) \\
&\overset{\substack{\text{Prop.}\\ 3.1.7\\ (2),(3)}}{=} \sum_{i=1}^{n} \varepsilon_i \cdot g\big(R(\xi,E_i)E_i,\eta\big) \\
&= \mathrm{ric}(\xi,\eta).
\end{aligned}
$$

(ii) is clear from (3.10).

(iii) We fix i and j and we have $\mathrm{ric}_{ij} = \mathrm{ric}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) = \mathrm{tr}\left(\zeta \mapsto -R\left(\frac{\partial}{\partial x^i},\zeta\right)\frac{\partial}{\partial x^j}\right)$. W.r.t. the basis $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}$ the endomorphism $\zeta \mapsto -R\left(\frac{\partial}{\partial x^i},\zeta\right)\frac{\partial}{\partial x^j}$ has the matrix representation

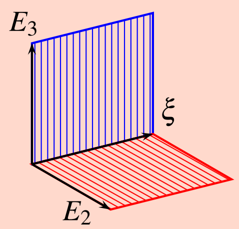
$$(-R^l_{jik})_{kl} = (R^l_{jki})_{kl}.$$

Thus we get that $\mathrm{ric}_{ij} = \sum_{k=1}^{n} R^k_{jki}$ and because of (i) we have $\mathrm{ric}_{ij} = \mathrm{ric}_{ji}$, which yields the assertion. □

We defined Ricci curvature using the Riemann curvature tensor. Since the curvature tensor and sectional curvature contain the same information, Ricci curvature should also be computable in terms of sectional curvature. Indeed, Ricci curvature can can be computed by averaging the sectional curvature of certain planes.

Lemma 3.3.4. *Let (M,g) be a semi-Riemannian manifold and $p \in M$. If $\xi \in T_pM$ with $g(\xi,\xi) \neq 0$ and if $E_2,\ldots,E_n$ is a generalized orthonormal basis of $\xi^\perp$, then*

$$\mathrm{ric}(\xi,\xi) = g(\xi,\xi)\cdot \underbrace{\sum_{j=2}^{n} K(\mathrm{span}\{\xi,E_j\})}_{\substack{\text{This is essentially the}\\ \text{mean value of } K \text{ on all}\\ \text{planes containing } \xi.}}$$



Proof. W.l.o.g. let $g(\xi,\xi) = \pm 1$. Write $\xi =: E_1$. Then $E_1,\ldots,E_n$ forms a generalized orthonormal basis of T_pM. Therefore

$$\begin{aligned}
\mathrm{ric}(\xi,\xi) &= \sum_{i=1}^{n} g(E_i,E_i)\cdot g\big(R(\xi,E_i)E_i,\xi\big) \\
&= \sum_{i=2}^{n} g(E_i,E_i)\cdot g\big(R(\xi,E_i)E_i,\xi\big) \\
&= \sum_{i=2}^{n} g(E_i,E_i)\cdot K\big(\mathrm{span}\{\xi,E_i\}\big)\cdot\big(g(\xi,\xi)g(E_i,E_i)-\underbrace{g(\xi,E_i)^2}_{=0}\big) \\
&= g(\xi,\xi)\cdot\sum_{i=2}^{n} K\big(\mathrm{span}\{\xi,E_i\}\big).
\end{aligned}$$

□

Remark 3.3.5. Lemma 3.3.4 expresses $\mathrm{ric}(\xi,\xi)$ in terms of sectional curvatures provided $g(\xi,\xi)\neq 0$. Since g is non-degenerate the set of vectors $\xi\in T_pM$ with $g(\xi,\xi)\neq 0$ is dense in T_pM. By continuity, $\mathrm{ric}(\xi,\xi)$ is determined for all $\xi\in T_pM$. By polarization, this determines the values of $\mathrm{ric}(\xi,\eta)$ for all $\xi,\eta\in T_pM$ via

$$\mathrm{ric}(\xi,\eta)=\tfrac{1}{2}\big(\mathrm{ric}(\xi+\eta,\xi+\eta)-\mathrm{ric}(\xi,\xi)-\mathrm{ric}(\eta,\eta)\big).$$

Remark 3.3.6. Both maps $\mathrm{ric}: T_pM\times T_pM\to\mathbb{R}$ and $g: T_pM\times T_pM\to\mathbb{R}$ are bilinear and symmetric. The second map g is in addition non-degenerate. Thus there exists a unique endomorphism $\mathrm{Ric}: T_pM\to T_pM$ such that

$$\mathrm{ric}(\xi,\eta)=g\big(\mathrm{Ric}(\xi),\eta\big)$$

for all $\xi,\eta\in T_pM$.

In local coordinates: For any chart $x: U\to V$ we get functions $\mathrm{Ric}_i^j: V\to\mathbb{R}$ by:

$$\mathrm{Ric}\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)=\sum_{j=1}^{n}\mathrm{Ric}_i^j(x(p))\left.\frac{\partial}{\partial x^j}\right|_p$$

We compute:

$$\begin{aligned}
\mathrm{ric}_{ij} &= \mathrm{ric}\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right) \\
&= g\left(\mathrm{Ric}\left(\frac{\partial}{\partial x^i}\right),\frac{\partial}{\partial x^j}\right) \\
&= g\left(\sum_{k=1}^{n}\mathrm{Ric}_i^k\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^j}\right) \\
&= \sum_{k=1}^{n}\mathrm{Ric}_i^k\cdot g\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^j}\right).
\end{aligned}$$

We have shown:

$$\boxed{\mathrm{ric}_{ij} \;=\; \sum_{k=1}^{n} \mathrm{Ric}_i^k \cdot g_{kj}}$$

The functions ric_{ij} are obtained from the functions Ric_i^k by lowering the upper index. Similarly, the Ric_i^k can be obtained from the ric_{ij} by raising one index.

Definition 3.3.7. The map $\mathrm{scal} : M \to \mathbb{R}$ defined by

$$\mathrm{scal}(p) := \mathrm{tr}(\mathrm{Ric}|_p)$$

is called the **scalar curvature** of M.

Lemma 3.3.8. *(i) In local coordinates we have*

$$\mathrm{scal}(p) = \sum_{i=1}^{n} \mathrm{Ric}_i^i\big(x(p)\big) = \sum_{i,j=1}^{n} \mathrm{ric}_{ij}\big(x(p)\big) \cdot g^{ij}\big(x(p)\big).$$

(ii) For a generalized orthonormal basis $E_1, \dots, E_n$ of T_pM we have

$$\mathrm{scal}(p) = \sum_{i=1}^{n} \varepsilon_i \cdot \mathrm{ric}(E_i, E_i).$$

Proof. Clear. □

Remark 3.3.9. Let us consider the special case when $\dim(M) = 2$. Let K be the Gauß curvature, i.e., $K(p) = K(T_pM)$. Then the curvature tensor is given by

$$R(\xi,\eta,\zeta,\nu) = K(p)\big(g(\eta,\zeta)g(\xi,\nu) - g(\xi,\zeta)g(\eta,\nu)\big).$$

Thus we get for the Ricci curvature

$$\begin{aligned}
\mathrm{ric}(\xi,\eta) &= \sum_{i=1}^{2} \varepsilon_i \cdot R(\xi, E_i, E_i, \eta) \\
&= K(p) \sum_{i=1}^{2} \varepsilon_i \big(g(E_i,E_i)g(\xi,\eta) - g(\xi,E_i)g(E_i,\eta)\big) \\
&= K(p)\big(2g(\xi,\eta) - g(\xi,\eta)\big) \\
&= K(p) \cdot g(\xi,\eta).
\end{aligned}$$

This shows

$$\mathrm{ric} = K \cdot g$$

and

$$\mathrm{scal} = 2K.$$

In the case of surfaces the Riemann curvature tensor, sectional curvature (Gauß curvature), Ricci curvature and scalar curvature all determine each other. In higher dimensions this is no longer so.

Remark 3.3.10. The following table shows how the different notions of curvature depend on each other:

$\dim M$	2	3	≥ 4
R			
	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
K			
	$\Updownarrow$	$\Updownarrow$	$\Downarrow$
ric			
	$\Updownarrow$	$\Downarrow$	$\Downarrow$
scal			

Remark 3.3.11. In the physics literature the following notation in local coordinates is often used:

- for R and $\mathbf{R}$ ones writes: R^l_{ijk} and R_{ijkl} (as here),
- for Ric and ric one write: $\mathrm{ric}_{ij} = R_{ij}$ and $\mathrm{Ric}^j_i = R^j_i$,
- for scal one write: $\mathrm{scal} = R$.

3.4 Jacobi fields

In order to better understand the behavior of geodesics we will linearize the geodesic equations. This leads to the Jacobi fields and relates geodesics and curvature.

Definition 3.4.1. Let M be a semi-Riemannian manifold. A variation of curves $c : (-\varepsilon, \varepsilon) \times I \to M$ is called a **geodesic variation** if for every $s \in (-\varepsilon, \varepsilon)$ the curve

$$t \mapsto c_s(t) := c(s,t)$$

is a geodesic.

Let $\xi(t) := \frac{\partial}{\partial s}c(0,t)$ be the corresponding variational vector field. Then we have:

$$\begin{aligned}
\left(\frac{\nabla}{dt}\right)^2 \xi(t) &= \frac{\nabla}{\partial t}\frac{\nabla}{\partial t}\frac{\partial}{\partial s}c(s,t)|_{s=0} \\
&= \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\frac{\partial}{\partial t}c(s,t)|_{s=0} \\
&= \frac{\nabla}{\partial s}\underbrace{\frac{\nabla}{\partial t}\frac{\partial}{\partial t}c(s,t)}_{\equiv 0 \text{ since } c_s \text{ geodesic}}|_{s=0} + R\left(\frac{\partial c}{\partial t}(0,t), \frac{\partial c}{\partial s}(0,t)\right)\frac{\partial c}{\partial t}(0,t) \\
&= R(\dot{c}_0(t), \xi(t))\dot{c}_0(t)
\end{aligned}$$

Definition 3.4.2. The equation for vector fields ξ along a geodesic c_0

$$\left(\frac{\nabla}{dt}\right)^2 \xi = R(\dot{c}_0, \xi)\dot{c}_0$$

is called the **Jacobi equation**. Its solutions are called **Jacobi fields**.

The above computation shows that the variational vector field of a geodesic variation is a Jacobi field.

Proposition 3.4.3. *Let M be a n-dimensional semi-Riemannian manifold, $c : I \to M$ a geodesic and $t_0 \in I$.*
For all $\xi, \eta \in T_{c(t_0)}M$ there exists a unique Jacobi field J along c with

$$J(t_0) = \xi \quad \textit{and} \quad \frac{\nabla}{dt}J(t_0) = \eta.$$

The set of Jacobi fields along c forms a $2n$-dimensional vector space.

Proof. Let $E_1(t_0), \dots, E_n(t_0)$ be a basis of $T_{c(t_0)}M$. By parallel transport along c we obtain a basis $E_1(t), \dots, E_n(t)$ of $T_{c(t)}M$ for all $t \in I$. Write $J(t) = \sum_{j=1}^n v^j(t)E_j(t)$. Then $\left(\frac{\nabla}{dt}\right)^2 J(t) = \sum_{j=1}^n \ddot{v}^j(t)E_j(t)$ and

$$R(\dot{c}(t), J(t))\dot{c}(t) = \sum_{j=1}^n v^j(t)R(\dot{c}(t), E_j(t))\dot{c}(t).$$

Write $R(\dot{c}(t), E_j(t))\dot{c}(t) = \sum_{k=1}^n a_j^k(t)E_k(t)$. Then J is a Jacobi field if and only if

$$\sum_{k=1}^n \ddot{v}^k E_k = \sum_{j,k=1}^n a_j^k v^j E_k,$$

hence if and only if

$$\boxed{\dot{v}^k = \sum_{j=1}^{n} a_j^k v^j} \qquad \text{for all } k = 1, \dots, n$$

This is a *linear* system of ordinary differential equations of second order. Thus solutions exist (on all of I) and are uniquely determined by the initial data $v^k(t_0)$ and $\dot{v}^k(t_0)$, i.e., by $J(t_0)$ and $\frac{\nabla}{dt}J(t_0)$.
The linearity of the Jacobi equation implies that its solution space forms a vector space. The map $\{\text{Jacobi fields}\} \to T_{c(t_0)}M \oplus T_{c(t_0)}M$, $J \mapsto (J(t_0), \frac{\nabla}{dt}J(t_0))$ is a vector space isomorphism. In particular, the dimension of the space of Jacobi fields along c equals $2n$. □

Example 3.4.4. If M is flat then the equation for Jacobi fields is simply given by

$$\left(\frac{\nabla}{dt}\right)^2 J \equiv 0.$$

Hence

$$\{\text{Jacobi fields}\} = \big\{\xi(t) + t\,\eta(t) \,|\, \xi, \eta \text{ parallel}\big\}.$$

Example 3.4.5. Let c be a geodesic in an arbitrary semi-Riemannian manifold. Then the vector field $J(t) := (a+bt)\dot{c}(t)$ is a Jacobi field for any $a, b \in \mathbb{R}$. Namely, we have:

$$\left(\frac{\nabla}{dt}\right)^2 J(t) = 0, \quad \text{and} \quad R(\dot{c}, J)\dot{c} = (a+bt)R(\dot{c}, \dot{c})\dot{c} = 0.$$

Such a J is the variational vector field of the geodesic variation

$$c(s,t) = c(t + s(a+bt)) = c\big((1+sb)t + sa\big).$$

This is a variation of c which is obtained by simply reparametrizing the geodesic. It contains no geometric information. Therefore such a Jacobi field is *uninteresting*. Thus there is a two-dimensional space of uninteresting Jacobi fields.

Remark 3.4.6. If a Jacobi field $J : I \to TM$ satisfies:

$$J(t_0) \perp \dot{c}(t_0) \quad \text{and} \quad \frac{\nabla}{dt}J(t_0) \perp \dot{c}(t_0) \quad \text{for a } t_0 \in I,$$

then we have

$$J(t) \perp \dot{c}(t) \quad \text{and} \quad \frac{\nabla}{dt}J(t) \perp \dot{c}(t) \quad \text{for all } t \in I.$$

Namely,

$$\frac{d}{dt}\left\langle \frac{\nabla}{dt}J, \dot{c} \right\rangle = \left\langle \left(\frac{\nabla}{dt}\right)^2 J, \dot{c} \right\rangle + \left\langle \frac{\nabla}{dt}J, \underbrace{\frac{\nabla}{dt}\dot{c}}_{=0} \right\rangle = \langle R(\dot{c}, J)\dot{c}, \dot{c} \rangle = 0$$

implies $\left\langle \frac{\nabla}{dt} J, \dot{c} \right\rangle \equiv 0$ and from

$$\frac{d}{dt}\langle J, \dot{c}\rangle = \left\langle \frac{\nabla}{dt} J, \dot{c} \right\rangle \equiv 0$$

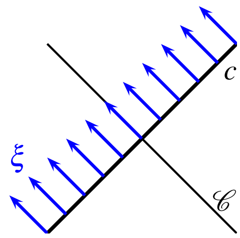
we see that $\langle J, \dot{c}\rangle \equiv 0$.

Consequence. *Let c be* non *light-like. In this case we have $T_{c(t)}M = \mathbb{R}\dot{c}(t) \oplus \dot{c}(t)^{\perp}$. Then*

$$\{\textit{Jacobi fields along } c\} = \underbrace{\mathbb{R}\cdot\dot{c} \oplus \mathbb{R}\cdot t\,\dot{c}}_{\substack{\textit{uninteresting}\\ \textit{Jacobi fields}}} \oplus \underbrace{\left\{\textit{Jacobi fields } J \textit{ along } c \,|\, J \perp \dot{c}, \tfrac{\nabla}{dt}J \perp \dot{c}\right\}}_{\textit{interesting Jacobi fields}}.$$

Remark 3.4.8. For light-like geodesics c this is not true because $\dot{c} \perp \dot{c}$.

Example 3.4.9. Let $(M,g) = (\mathbb{R}^2, g_{\mathrm{Mink}})$, let c be a light-like geodesic and let ξ be a light-like parallel vector field along c which is linearly independent of $\dot{c}$.
Since ξ is parallel and $R = 0$, the vector field ξ is also a Jacobi field and we have:



$$\begin{aligned}\{\text{Jacobi field along } c\} &= \underbrace{\mathbb{R}\cdot\dot{c} \oplus \mathbb{R}(t\dot{c})} \oplus \mathbb{R}\xi \oplus \mathbb{R}(t\xi)\\ &\ = \left\{\text{Jacobi field } J \text{ along } c \,|\, J \perp \dot{c}, \tfrac{\nabla}{dt}J \perp \dot{c}\right\}\end{aligned}$$

Definition 3.4.10. For $\kappa \in \mathbb{R}$ the **generalized sine** and **cosine function** $\mathfrak{s}_\kappa, \mathfrak{c}_\kappa : \mathbb{R} \to \mathbb{R}$ are defined by

$$\mathfrak{s}_\kappa(r) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}\cdot r), & \kappa > 0\\ r, & \kappa = 0\\ \frac{1}{\sqrt{|\kappa|}}\sinh(\sqrt{|\kappa|}\cdot r), & \kappa < 0\end{cases} \quad \text{and} \quad \mathfrak{c}_\kappa(r) := \begin{cases} \cos(\sqrt{\kappa}\cdot r), & \kappa > 0\\ 1, & \kappa = 0\\ \cosh(\sqrt{|\kappa|}\cdot r), & \kappa < 0\end{cases}$$

respectively.

It is easy to check that

$$\begin{aligned}\kappa\,\mathfrak{s}_\kappa^2+\mathfrak{c}_\kappa^2&=1,\\ \mathfrak{s}_\kappa'&=\mathfrak{c}_\kappa \quad&&\text{and}\quad \mathfrak{s}_\kappa(0)=0,\\ \mathfrak{c}_\kappa'&=-\kappa\,\mathfrak{s}_\kappa \quad&&\text{and}\quad \mathfrak{c}_\kappa(0)=1.\end{aligned}$$

Example 3.4.11. Let (M,g) be a Riemannian manifold with constant sectional curvature $K\equiv\kappa$. Let c be a geodesic, parametrized by arc-length. Let ξ be a parallel vector field along c with $\xi\perp\dot c$. Set

$$J(t):=\big(a\,\mathfrak{s}_\kappa(t)+b\,\mathfrak{c}_\kappa(t)\big)\,\xi(t)\qquad\text{with } a,b\in\mathbb{R}.$$

Then

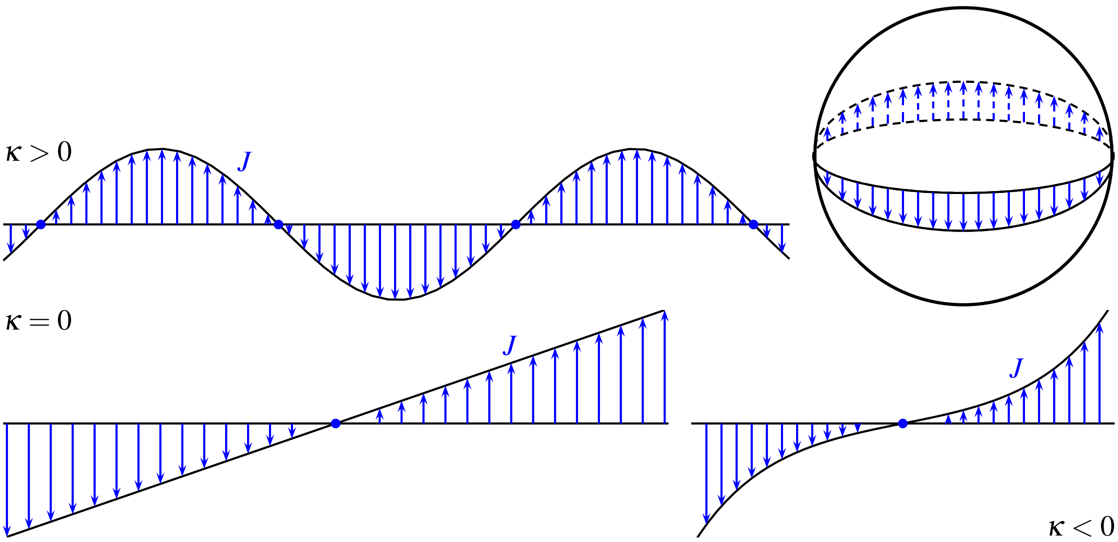
$$\left(\frac{\nabla}{dt}\right)^2 J=(a\ddot{\mathfrak{s}}_\kappa+b\ddot{\mathfrak{c}}_\kappa)\xi=-\kappa(a\,\mathfrak{s}_\kappa+b\,\mathfrak{c}_\kappa)\xi=-\kappa J.$$

For the curvature tensor we here have $R(\xi,\eta)\zeta=\kappa(\langle\eta,\zeta\rangle\,\xi-\langle\xi,\zeta\rangle\,\eta)$. Thus

$$R(\dot c,J)\dot c=(a\,\mathfrak{s}_\kappa+b\,\mathfrak{c}_\kappa)\cdot\kappa\big(\underbrace{\langle\xi,\dot c\rangle}_{=0}\,\dot c-\underbrace{\langle\dot c,\dot c\rangle}_{=1}\,\xi\big)=-\kappa\big(a\,\mathfrak{s}_\kappa+b\,\mathfrak{c}_\kappa\big)\xi=-\kappa J.$$

Hence J is a Jacobi field and

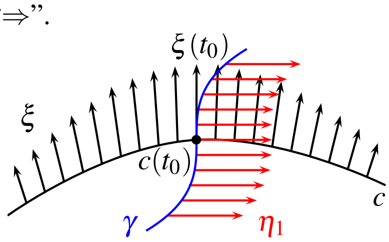
$$\begin{aligned}&\Big\{\text{Jacobi fields along } c\,\big|\,J\perp\dot c,\ \frac{\nabla}{dt}J\perp\dot c\Big\}\\ &\qquad\qquad=\big\{(a\,\mathfrak{s}_\kappa+b\,\mathfrak{c}_\kappa)\,\xi\,\big|\,a,b\in\mathbb{R},\ \xi\text{ parallel along } c,\xi\perp\dot c\big\}.\end{aligned}$$



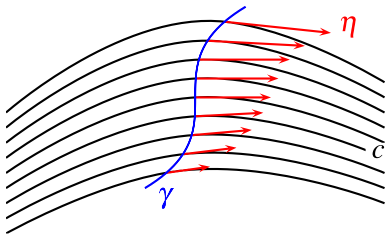
Proposition 3.4.12. *Let M be a semi-Riemannian manifold and $c : [a,b] \to M$ a geodesic. Let ξ be a smooth vector field along c. Then*

$$\xi \text{ is a Jacobi field } \Longleftrightarrow \xi \text{ is the variational field of a geodesic variation.}$$

Proof. The implication "$\Leftarrow$" is already known. We show "$\Rightarrow$".



Let ξ be a Jacobi field along c. Choose a $t_0 \in [a,b]$ and choose a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = c(t_0)$ and $\dot\gamma(0) = \xi(t_0)$. Let η_1 be the parallel vector field along γ with $\eta_1(0) = \dot c(t_0)$. Let η_2 be the parallel vector field along γ with $\eta_2(0) = \frac{\nabla}{dt}\xi(t_0)$.



Set $\eta(s) := \eta_1(s) + s\eta_2(s)$ and

$$c(s,t) := \exp_{\gamma(s)}\big((t-t_0)\eta(s)\big).$$

Since the domain of definition of exp is open, $c(s,t)$ is defined for $|s|$ sufficiently small and for all $t \in [a,b]$. Then we have

$$c(0,t) = \exp_{\gamma(0)}\big((t-t_0)\eta(0)\big) = \exp_{c(t_0)}\big((t-t_0)\dot c(t_0)\big) = c(t)$$

Hence $c(s,t)$ is a geodesic variation of $c(t)$. Let $J(t) := \frac{\partial c}{\partial s}(0,t)$ be the corresponding variational field. Then J is a Jacobi field. We show:

$$\xi(t_0) = J(t_0) \quad \text{and} \quad \frac{\nabla}{dt}\xi(t_0) = \frac{\nabla}{dt}J(t_0).$$

Then we get $\xi = J$ because Jacobi fields are uniquely determined by their initial data and hence ξ is the variational field of the geodesic variation $c(s,t)$.
We calculate

$$J(t_0) = \frac{\partial c}{\partial s}(0,t_0) = \frac{d}{ds}\Big|_{s=0} \exp_{\gamma(s)}(0) = \frac{d}{ds}\Big|_{s=0} \gamma(s) = \dot\gamma(0) = \xi(t_0)$$

and

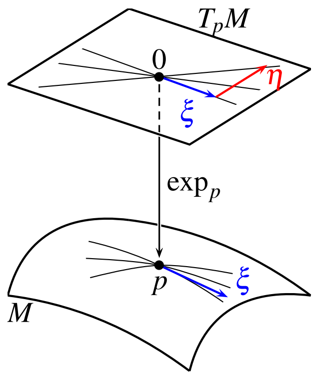
$$\frac{\nabla}{dt}J(t_0) = \frac{\nabla}{\partial t}\frac{\partial c}{\partial s}(0,t_0) = \frac{\nabla}{\partial s}\frac{\partial c}{\partial t}(0,t_0) = \frac{\nabla}{ds}\eta(0) = \eta_2(0) = \frac{\nabla}{dt}\xi(t_0). \qquad \square$$

We are now able to generalize Lemma 2.6.26 and can identify the differential of the exponential at arbitrary points in its domain.

Proposition 3.4.13. *Let M be a semi-Riemannian manifold, $p \in M$ and $\xi \in T_pM$. We assume that the geodesic $\gamma(t) := \exp_p(t\xi)$ is defined on $[0,1]$, i.e., ξ lies in the domain of $\exp_p$.*
For $\eta \in T_pM (\cong T_{t\xi}T_pM)$ let J be the Jacobi field along γ with $J(0) = 0$ and $\frac{\nabla}{dt}J(0) = \eta$. Then we have for all $t \in (0,1]$:

$$d\exp_p|_{t\xi}(\eta) = \frac{J(t)}{t}.$$

Proof. Consider the geodesic variation $c(s,t) := \exp_p\big(t(\xi + s\eta)\big)$.



Let $\zeta := \frac{\partial c}{\partial s}|_{s=0}$ be the corresponding variational Jacobi field. Then we have

$$\zeta(0) = \frac{\partial c}{\partial s}(0,0) = \frac{d}{ds}\exp_p|_{s=0}(0) = 0 = J(0)$$

and

$$\frac{\nabla}{dt}\zeta(0) = \frac{\nabla}{dt}\frac{\partial c}{\partial s}(0,0) = \frac{\nabla}{\partial s}\frac{\partial c}{\partial t}(0,0) = \frac{\nabla}{ds}(\xi + s\eta)|_{s=0} = \eta = \frac{\nabla}{dt}J(0)$$

Hence $\zeta = J$. Now we compute for fixed $t \in (0,1]$:

$$d\exp_p|_{t\xi}(\eta) = \frac{\partial}{\partial s}\exp_p(t\xi + s\eta)|_{s=0} = \frac{\partial}{\partial s}\exp_p\left(t\left(\xi + \frac{s}{t}\eta\right)\right)\Big|_{s=0} = \frac{1}{t}\zeta(t) = \frac{1}{t}J(t). \qquad \square$$

Corollary 3.4.14. *Let M be a semi-Riemannian manifold, let $p \in M$ and let ξ be in the domain of $\exp_p$. Then*

$$\ker(d\exp_p|_\xi) \cong \big\{\textit{Jacobi field along } \gamma(t) = \exp_p(t\xi)\,|\,J(0) = 0, J(1) = 0\big\}.$$

Definition 3.4.15. Let M be a semi-Riemannian manifold and $\gamma : I \to M$ a geodesic. Then $t_1, t_2 \in I$, $t_1 \neq t_2$ are called **conjugate points** along γ, if there exists a non-trivial Jacobi field J along γ with $J(t_1) = 0$ and $J(t_2) = 0$.

Consequence. *$d\exp_p|_\xi$ is non-invertible if and only if 0 and 1 are conjugate points along $\gamma(t) = \exp_p(t\xi)$.*

Example 3.4.17. Let M be a Riemannian manifold with constant sectional curvature $K \equiv \kappa$.

Case 1: $\kappa \leq 0$. Every Jacobi field has at most one zero.

$\Rightarrow$ There are no conjugate points.

$\Rightarrow$ $d\exp_p|_\xi$ is invertible for all $\xi \in \mathscr{D}_p$.

$\Rightarrow$ The map $\exp_p : \mathscr{D}_p \to M$ is a local diffeomorphism.

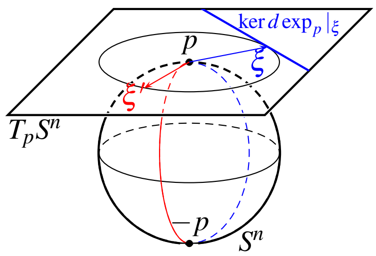
Case 2: $\kappa > 0$.

For a geodesic parametrized by arc-length, the conjugate points belonging to t_0 are the points $t_0 + m\frac{\pi}{\sqrt{\kappa}}$ for $m \in \mathbb{Z} \setminus \{0\}$. Considering the case $m = 1$ we have

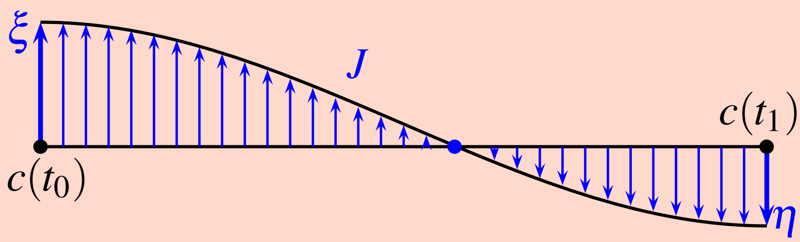
$$\exp_p\left(\{\xi \in T_pS^n \mid \|\xi\| = \pi\}\right) = \{-p\}.$$

For $\|\xi\| = \pi$ we obtain

$$\ker d\exp_p|_\xi = \xi^\perp.$$



Proposition 3.4.18. *Let M be a semi-Riemannian manifold and let $c : [t_0, t_1] \to M$ be a geodesic. Let t_0 and t_1 be* not *conjugate with each other along c. Then for $\xi \in T_{c(t_0)}M$ and $\eta \in T_{c(t_1)}M$ there exists exactly one Jacobi field J along c with $J(t_0) = \xi$ and $J(t_1) = \eta$.*



Proof. The linear map

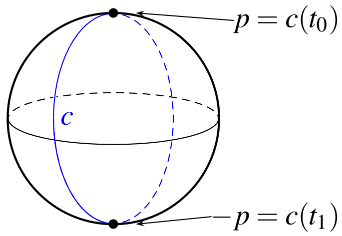
$$\begin{array}{rcl} \overbrace{\{\text{Jacobi field along } c\}}^{2n\text{-dimensional}} & \to & \overbrace{T_{c(t_0)}M \oplus T_{c(t_1)}M}^{(n+n)\text{-dimensional}}, \\ J & \mapsto & \big(J(t_0), J(t_1)\big), \end{array}$$

is injective since t_0 and t_1 are not conjugate to each other along c. For dimensional reasons, this map is an isomorphism. □

Proposition 3.4.18 means that in the non-conjugate case we can also characterize Jacobi fields by the boundary values $J(t_0)$ and $J(t_1)$ instead of the initial values $J(t_0)$ and $\frac{\nabla}{dt}J(t_0)$. In the conjugate case this is certainly wrong.

Example 3.4.19. Let c be a geodesic emanating from $p \in S^n$ which is parametrized by arc-length. The set of $\eta \in T_{-p}S^n$ for which exists a Jacobi field J along c with $J(0) = 0$ and $J(\pi) = \eta$ is given by

$$\{\eta = \alpha \cdot \dot{c}(\pi) \,|\, \alpha \in \mathbb{R}\}.$$

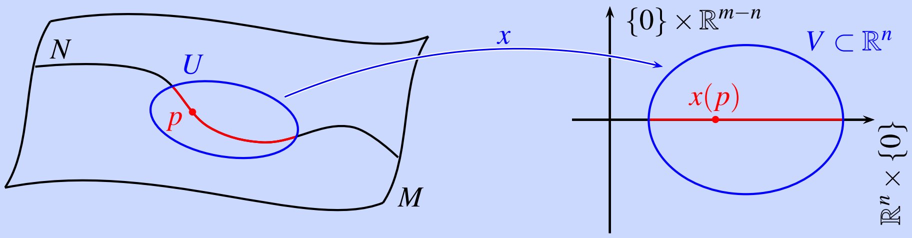


4 Submanifolds

4.1 Submanifold of differentiable manifolds

Definition 4.1.1. Let M be an m-dimensional differentiable manifold. A subset $N \subset M$ is called an ***n*-dimensional submanifold** if for every $p \in N$ there exists a chart $x : U \to V$ of M with $p \in U$ such that

$$x(N \cap U) = V \cap (\mathbb{R}^n \times \{0\}).$$



Such a chart is called **submanifold chart** of N. The number $m - n$ is called the **codimension** of N in M.

Example 4.1.2. (1) Codimension $n = 0$: A subset $N \subset M$ is a submanifold of codimension 0 if and only if N is open subset of M.

(2) Dimension $n = 0$: A subset $N \subset M$ is a submanifold of dimension 0 if and only if N is a discrete subset of M.

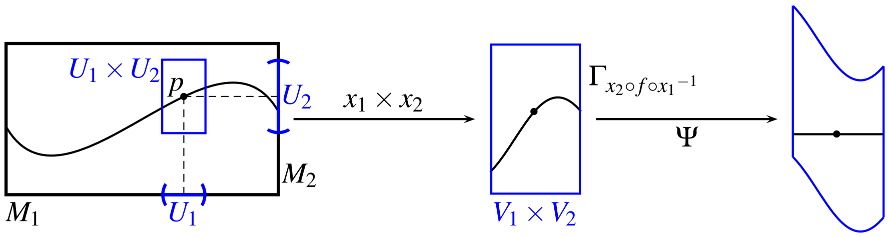
(3) Affine subspaces: Let $N \subset M = \mathbb{R}^m$ be an affine subspace, i.e., N is of the form $N = N' + p$, where $N' \subset \mathbb{R}^m$ is an n-dimensional vector subspace and $p \in \mathbb{R}^m$ fixed. Choose $A \in GL(m)$ with $AN' = \mathbb{R}^n \times \{0\}$. Then $x : U = \mathbb{R}^m \to V = \mathbb{R}^m$, given by

$$x(q) := A(q - p),$$

is a submanifold chart.

(4) Graphs: Let M_1 and M_2 be differentiable manifolds and let $f : M_1 \to M_2$ be a smooth map. Set $M = M_1 \times M_2$ and

$$N = \Gamma_f = \big\{(\xi, \eta) \in M_1 \times M_2 \,|\, \eta = f(\xi)\big\}.$$



Choose charts $x_i : U_i \to V_i$ of M_i with $p \in U_1 \times U_2$. W.l.o.g. let $f(U_1) \subset U_2$. For $w \in V_1$ and $z \in V_2$ set

$$\psi(w,z) := \big(w, z - \big(x_2 \circ f \circ x_1{}^{-1}\big)(w)\big).$$

Then $x := \psi \circ (x_1 \times x_2)$ is a submanifold chart, defined on $U_1 \times U_2$.

Theorem 4.1.3. *Let M be an m-dimensional differentiable manifold. Let $N \subset M$ be a subset. Then the following assertions are equivalent:*

(i) N is an n-dimensional submanifold.

(ii) For every $p \in N$ there exists an open neighborhood U of p and smooth functions $f_1, \ldots, f_{m-n} : U \to \mathbb{R}$ such that

(a) $N \cap U = \{q \in U \mid f_1(q) = \ldots = f_{m-n}(q) = 0\}$;

(b) The differentials $df_1|_p, \ldots, df_{m-n}|_p \in T_p^*M$ are linearly independent.

(iii) For every $p \in N$ there exists an open neighborhood U of p, an $(m-n)$-dimensional differentiable manifold R and a smooth map $f : U \to R$ with

(a) $N \cap U = f^{-1}(q)$ where $q = f(p)$;

(b) $df|_p : T_pM \to T_qR$ has maximal rank.

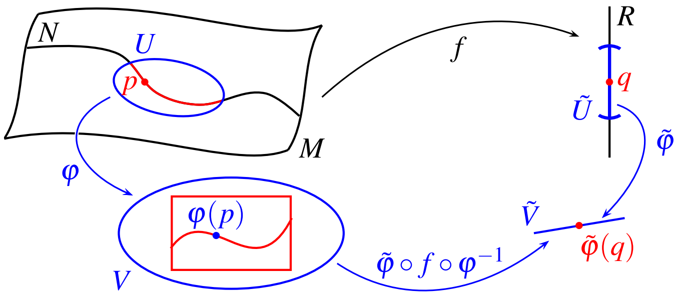
Proof. "(i)⇒(ii)": Let $p \in N$ and let $x : U \to V$ be a submanifold chart for N with $p \in U$. W.l.o.g. let

(1) $x(p) = 0 \in \mathbb{R}^m$ (otherwise compose x with a suitable translation);

(2) $V = V_1 \times V_2$ where $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^{m-n}$ (otherwise make U smaller).

Now $f_j : U \to \mathbb{R}$, $f_j := x^{n+j}$, do the job ($j = 1, \ldots, m-n$).
"(ii)⇒(iii)" is obvious. Simply set $R := \mathbb{R}^{m-n}$ and $f := (f_1, \ldots, f_{m-n})$.

"(iii)⇒(i)":



Choose a chart $\varphi : U \to V$ of M around p and a chart $\tilde\varphi : \tilde U \to \tilde V$ of R around $q := f(p)$. W.l.o.g. we assume that $f(U) \subset \tilde U$. Since φ and $\tilde\varphi$ are diffeomorphisms, we have

$$\operatorname{rank} D(\tilde\varphi \circ f \circ \varphi^{-1})|_{\varphi(p)} = \operatorname{rank} df|_p = m-n.$$

The implicit function theorem yields: One can shrink V and U to smaller neighborhoods of q and p, respectively, such that $V = V_1 \times V_2$ and one can find a smooth map $g : V_1 \to V_2$ such that

$$(\tilde\varphi \circ f \circ \varphi^{-1})^{-1}(\tilde\varphi(q)) = (f \circ \varphi^{-1})^{-1}(q) = \Gamma_g.$$

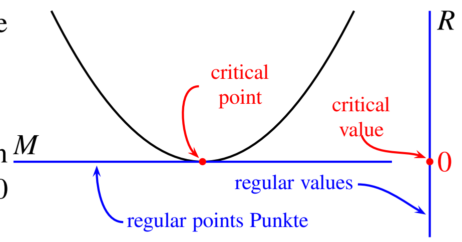
If we compose φ with a submanifold chart for graphs as in the Example 4.1.2 (4) then we get a submanifold chart for N in M around p. □

Definition 4.1.4. Let M and R be differentiable manifolds and let $f : M \to R$ be smooth. A point $p \in M$ is called a **regular point** of f if $df|_p$ has maximal rank. Otherwise p is called a **critical point** of f.
A point $q \in R$ is called a **regular value** of f if all $p \in f^{-1}(q)$ are regular points. Otherwise q is called a **critical value** of f.

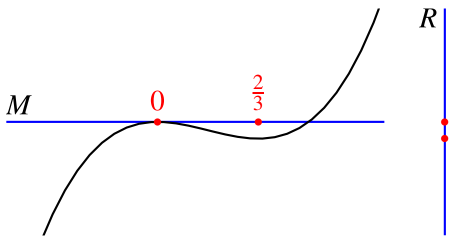
Example 4.1.5. Let $M = R = \mathbb{R}$ and $f(t) = t^2$. We have

$$df|_t(\xi) = f'(t)\cdot\xi.$$

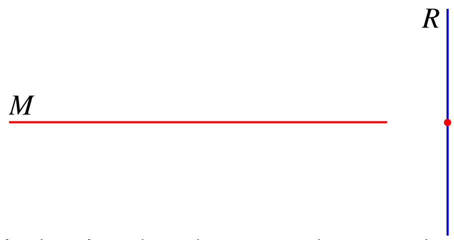
Hence t is a critical point of f if and only if $f'(t) = 0$. In this example $t = 0$ is the only critical point and $f(0) = 0$ is the only critical value.



Example 4.1.6. Let $M = R = \mathbb{R}$ and $f(t) = t^2(t-1)$. Here f has the critical points $t = 0$ and $t = \frac{2}{3}$. The critical values are $f(0) = 0$ and $f(\frac{2}{3}) = -\frac{4}{27}$.



Example 4.1.7. Let $M = R = \mathbb{R}$ and $f(t) = 0$. In this case all $t \in \mathbb{R}$ are critical points but 0 is the only critical value.



The examples indicate that there may be many critical points but there are always only few critical values. This is true in general:

Theorem 4.1.8 (Sard). *Let M and R be differentiable manifolds and let $f : M \to R$ be smooth. Then the set of critical values of f is a null set in R. In other words, for every chart $x : U \to V$ of R the set $x(U \cap \{\text{critical values of } f\}) \subset V$ is a null set (in the sense of Lebesgue measure theory).*

For a proof see [M65, Chapter 3].

Corollary 4.1.9. *If $f : M \to R$ is smooth and if $q \in R$ is a regular value of f, then $N = f^{-1}(q)$ is empty or a submanifold of M with* $\mathrm{codim}(N) = \dim(R)$.

Proof. This follows directly from Criterion (iii) in Theorem 4.1.3. □

Example 4.1.10. Let $M = \mathbb{R}^{n+1}$ and $R = \mathbb{R}$. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be given by $f(x) = \|x\|^2 = (x^0)^2 + \ldots + (x^n)^2$. Then $S^n = f^{-1}(1)$ and for any $x \in \mathbb{R}^{n+1}$ we have

$$Df|_x = (2x^0, \ldots, 2x^n).$$

$$\Rightarrow \quad \mathrm{rank}(Df|_x) = \begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}$$

$\Rightarrow$ For all $x \in f^{-1}(1)$ we have $\mathrm{rank}(Df|_x) = 1$.

$\Rightarrow$ 1 is a regular value of f.

$\overset{4.1.9}{\Rightarrow}$ $S^n \subset \mathbb{R}^{n+1}$ is a submanifold of codimension 1.

Remark 4.1.11. In this example all $q \in \mathbb{R} \setminus \{0\}$ are regular values. We have

$$f^{-1}(q) = \begin{cases} S^n(\sqrt{q}), & q > 0 \\ \emptyset & , q < 0 \end{cases}$$

For the critical value $q = 0$ we have that $f^{-1}(0) = \{0\}$ is also (by coincidence) a submanifold, but of the wrong codimension $n+1$. In general, the preimage of a critical value is not a submanifold.

Remark 4.1.12. Sometimes the set $f^{-1}(q)$ is a submanifold with codimension $\dim R$ even if q is a critical value.

Example 4.1.13. Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$, $g(x) = (\|x\|^2 - 1)^2$. Then 0 is critical value of g but $S^n = g^{-1}(0)$ is a submanifold of codimension 1.

Remark 4.1.14. Submanifolds of differentiable manifolds are themselves differentiable manifolds. Namely:
Let $N \subset M$ be a submanifold and $p \in N$ and $x : U \to V$ a submanifold chart with $x = (x^1, \dots, x^n, x^{n+1}, \dots, x^m)$, then

$$(x^1, \dots, x^n) : U \cap N \to V \cap \mathbb{R}^n$$

is a chart of N. The set of charts of N obtained in this manner by restricting submanifold charts to N is a C^∞-atlas for N.

Theorem 4.1.15. *Let $N \subset M$ be a submanifold. Let $\iota : N \hookrightarrow M$ be the inclusion map, $\iota(p) = p$. Then we have:*

(i) ι is smooth and $d\iota|_p : T_pN \to T_pM$ is injective.

(ii) If $f : M \to P$ is smooth then $f|_N : N \to P$ is also smooth.

(iii) If $g : Q \to M$ is smooth with $g(Q) \subset N$ then $g : Q \to N$ is also smooth.

Proof. (i) Let $x = (x^1, \dots, x^m)$ be a submanifold chart of N in M and $\tilde{x} = (x^1, \dots, x^n)$ the corresponding chart of N. The following diagram commutes:

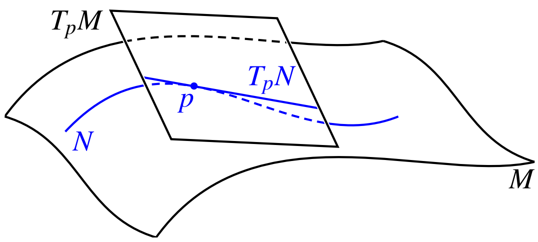
$$\begin{tikzcd} N \supset U \cap N \arrow[r, "\iota"] \arrow[d, "\tilde{x}"'] & U \subset M \arrow[d, "x"] \\ V \cap \mathbb{R}^n \arrow[r, "\xi \mapsto (\xi, 0)"] & V \end{tikzcd}$$

Obviously, $\xi \mapsto (\xi, 0)$ is smooth. Since this map is linear, it coincides with its differential, such that the differential is in particular injective.

(ii) The function $f|_N = f \circ \iota$ is the composition of two smooth maps and therefore again smooth.

(iii) Let $q \in Q$ and $x = (x^1, \dots, x^m)$ be a submanifold chart of M around $g(q)$. Since g is smooth the functions $g^i := x^i \circ g$ are also smooth. From $g(Q) \subset N$ we see that $(g^1, \dots, g^m) = (g^1, \dots, g^n, 0, \dots, 0)$. Now $(g^1, \dots, g^n)$ is smooth and thus also $g : Q \to N$. □

Remark 4.1.16. One identifies T_pN with $d\iota|_p(T_pN)$ and thinks of it as a vector subspace of T_pM.



Remark 4.1.17. If $M = \mathbb{R}^m$, i.e., $N \subset \mathbb{R}^m$, then one often considers T_pN as a vector subspace of $\mathbb{R}^m$ via $T_pN \subset T_p\mathbb{R}^m \xrightarrow[\text{canon. isom.}]{\cong} \mathbb{R}^m$.

Example 4.1.18. For $N = S^n \subset \mathbb{R}^{n+1}$ we have $T_pS^n = p^\perp$.

4.2 Semi-Riemannian submanifolds

Definition 4.2.1. Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold. A submanifold $M \subset \bar{M}$ is called a **semi-Riemannian submanifold**, if for all $p \in M$

$$(\bar{g}|_p)\big|_{T_pM \times T_pM} =: g|_p$$

is non-degenerate.

Example 4.2.2. If $(\bar{M}, \bar{g})$ is Riemannian, then every submanifold is a semi-Riemannian submanifold.

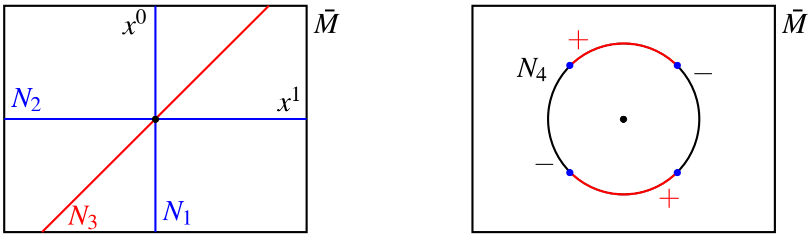
Example 4.2.3. Let $(\bar{M}, \bar{g}) = (\mathbb{R}^2, g_{\mathrm{Mink}})$ be 2-dimensional Minkowski space, i.e., $g_{\mathrm{Mink}} = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1$. Then

$N_1 = \{(x^0, 0) \,|\, x^0 \in \mathbb{R}\}$ is semi-Riemannian (with negative-definite metric).

$N_2 = \{(0, x^1) \,|\, x^1 \in \mathbb{R}\}$ is semi-Riemannian (with positive-definite metric).

$N_3 = \{(t, t) \,|\, t \in \mathbb{R}\}$ is not semi-Riemannian, since the restriction of g_{Mink} on T_pN_3 vanishes.

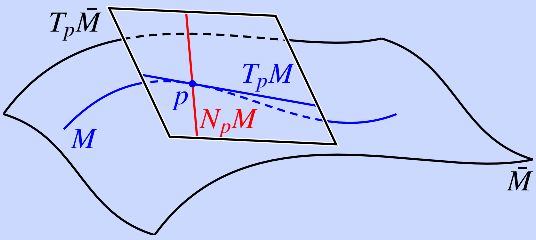
$N_4 = S^1$ has 4 points at which the restriction of g_{Mink} degenerates.



Definition 4.2.4. Let $M \subset \bar{M}$ be a semi-Riemannian submanifold. Then we call

$$N_pM := T_pM^\perp = \left\{\xi \in T_p\bar{M} \mid \bar{g}|_p(\xi,\eta) = 0 \,\forall \eta \in T_pM\right\}$$

the **normal space** of M at the point p.



Remark 4.2.5. We have $T_p\bar{M} = T_pM \oplus N_pM$ since

(1) $\dim N_pM \geq \dim T_p\bar{M} - \dim T_pM$.

(2) If there existed a $\xi \in T_pM \cap N_pM$ with $\xi \neq 0$, then we would have $\xi \in T_pM$ with $\bar{g}|_P(\xi,\eta) = 0$ for all $\eta \in T_pM$. Then $(\bar{g}|_p)|_{T_pM\times T_pM}$ would be degenerate, which is a contradiction.

Let $M \subset \bar{M}$ be a semi-Riemannian submanifold and $p \in M$. Let

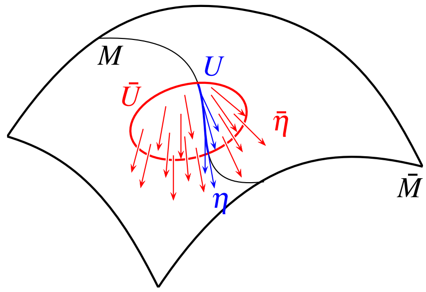
$$\begin{aligned} \tan : & T_p\bar{M} \to T_pM, \\ \mathrm{nor} : & T_p\bar{M} \to N_pM, \end{aligned}$$

be the orthogonal projections. Both M and also $\bar{M}$ have, when seen as semi-Riemannian manifolds in their own rights, a Levi-Civita connection ∇ and $\bar{\nabla}$, respectively. Now we want to investigate, how we can determine ∇ directly from $\bar{\nabla}$.

$$\begin{tikzcd} (\bar{M},\bar{g}) \arrow[r, dashed] \arrow[d, dashed] & \bar{\nabla} \text{ Levi-Civita connection} \arrow[d, dashed, "?"] \\ (M,g) \arrow[r, dashed] & \nabla \text{ Levi-Civita connection} \end{tikzcd}$$

Here $g|_p := (\bar{g}|_p)_{T_pM\times T_pM}$.

Let $p \in M$, $\xi \in T_pM$ and $\eta \in C^\infty(U,TM)$, where $U \subset M$ is an open neighborhood of p. Choose a smooth extension $\bar{\eta}$ of η to an open neighborhood $\bar{U}$ of p in $\bar{M}$. Then $\bar{\nabla}_\xi \bar{\eta} \in T_p\bar{M}$ does not depend on the choice of continuation $\bar{\eta}$.
Namely: the tangent vector $\xi \in T_pM$ is of the form $\xi = \dot{c}(0)$ with a curve $c : (-\varepsilon, \varepsilon) \to M$. Hence $\bar{\nabla}_\xi \bar{\eta}$ depends on $\bar{\eta}$ only along c, that is, only on η.



We can also write:

$$\bar{\nabla}_\xi \eta := \bar{\nabla}_\xi \bar{\eta}.$$

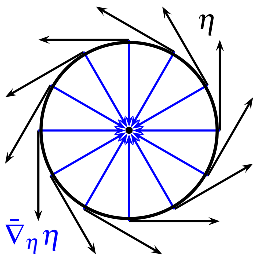
Example 4.2.6. Let $\bar{M} = (\mathbb{R}^2, g_{\mathrm{eucl}})$ and $M = S^1$. Set $\eta(x^1,x^2) = (-x^2,x^1)$. For $c : \mathbb{R} \to S^1$ with $c(t) = (\cos(t), \sin(t))$ we have

$$\dot{c}(t) = \eta(c(t)).$$

Then we get:

$$\bar{\nabla}_\eta \eta = \frac{\bar{\nabla}}{dt}\dot{c} = \ddot{c} = \big(-\cos(t), -\sin(t)\big)$$

which is *not* tangent to S^1.



We set $\nabla_\xi \eta := \tan(\bar{\nabla}_\xi \eta)$.

Theorem 4.2.7. *Let $(\bar{M}, \bar{g})$ be a semi-Riemannian manifold and $M \subset \bar{M}$ a semi-Riemannian submanifold with induced semi-Riemannian metric g. Let $\bar{\nabla}$ be the Levi-Civita connection of $(\bar{M}, \bar{g})$. Then*

$$\nabla_\xi \eta = \tan\left(\bar{\nabla}_\xi \eta\right)$$

is the Levi-Civita connection of (M,g).

Proof. We check that ∇ satisfies the axioms of the Levi-Civita connection for (M,g). By the uniqueness statement in Theorem 2.3.8, ∇ must then be the Levi-Civita connection of (M,g).

(i) *Locality* is clear because $\bar{\nabla}$ is local.

(ii) *Linearity in* ξ is clear because tan is linear and $\bar{\nabla}$ is linear in ξ.

(iii) *Linearity in* η is clear by a similar argument.

(iv) *Product rule I:* Let $f \in C^\infty(U)$ and $\eta \in C^\infty(U,TM)$, where $U \subset M$ is an open neighborhood of p and $\xi \in T_pM$. Let $\bar{\eta}$ and $\bar{f}$ be smooth extensions of η and f to an open neighborhood $\bar{U}$ of p in $\bar{M}$. Then

$$\begin{aligned}
\nabla_\xi(f\cdot\eta) &= \tan\left(\bar{\nabla}_\xi\left(\bar{f}\cdot\bar{\eta}\right)\right)\\
&= \tan\left(\partial_\xi \bar{f}\cdot\bar{\eta}|_p + \bar{f}(p)\cdot\bar{\nabla}_\xi\bar{\eta}\right)\\
&= \tan\left(\partial_\xi f\cdot\bar{\eta}|_p + f(p)\cdot\bar{\nabla}_\xi\bar{\eta}\right)\\
&= \partial_\xi f\cdot\tan\left(\bar{\eta}|_p\right) + f(p)\cdot\tan\left(\bar{\nabla}_\xi\bar{\eta}\right)\\
&= \partial_\xi f\cdot\eta|_p + f(p)\nabla_\xi\eta.
\end{aligned}$$

(v) *Product rule II:* Let $\xi \in T_pM$ and $\eta_1,\eta_2 \in C^\infty(U,TM)$. Choose smooth extensions $\bar{\eta}_1,\bar{\eta}_2 \in C^\infty(\bar{U},T\bar{M})$. Then

$$\begin{aligned}
\partial_\xi g(\eta_1,\eta_2) &= \partial_\xi \bar{g}(\bar{\eta}_1,\bar{\eta}_2)\\
&= \bar{g}|_p\big(\bar{\nabla}_\xi\bar{\eta}_1,\underbrace{\bar{\eta}_2|_p}_{=\eta_2|_p,\text{ in particular tangent to }M}\big) + \bar{g}|_p\left(\bar{\eta}_1|_p,\bar{\nabla}_\xi\bar{\eta}_2\right)\\
&= g|_p\left(\tan\left(\bar{\nabla}_\xi\bar{\eta}_1\right),\eta_2|_p\right) + g|_p\left(\eta_1|_p,\tan\left(\bar{\nabla}_\xi\bar{\eta}_2\right)\right)\\
&= g|_p\left(\nabla_\xi\eta_1,\eta_2|_p\right) + g|_p\left(\eta_1|_p,\nabla_\xi\eta_2\right).
\end{aligned}$$

(vi) *Freeness of torsion:* Let $x^1,\dots,x^m,x^{m+1},\dots,x^{\bar{m}}$ be submanifold coordinates on $\bar{M}$. Here $x^1,\dots,x^m$ are coordinates on M. For $1 \le i,j \le m$:

$$\nabla_{\frac{\partial}{\partial x^i}\big|_p}\frac{\partial}{\partial x^j} = \tan\left(\bar{\nabla}_{\frac{\partial}{\partial x^i}\big|_p}\frac{\partial}{\partial x^j}\right) = \tan\left(\bar{\nabla}_{\frac{\partial}{\partial x^j}\big|_p}\frac{\partial}{\partial x^i}\right) = \nabla_{\frac{\partial}{\partial x^j}\big|_p}\frac{\partial}{\partial x^i}. \qquad \square$$

Example 4.2.8. Let $M = S^1 \subset \bar{M} = \mathbb{R}^2$ with $\bar{g} = g_{\mathrm{eucl}}$. Set $\eta(c(t)) = \dot{c}(t)$ where $c(t) = (\cos(t),\sin(t))$. Then

$$\nabla_\eta\eta = \tan\left(\bar{\nabla}_\eta\eta\big|_p\right) = \tan(-p) = 0.$$

Hence c is a geodesic in S^1 (but not in $\mathbb{R}^2$).

Lemma 4.2.9. *Let $\xi \in T_pM$ and $\eta \in C^\infty(U,TM)$, where $U \subset M$ is an open neighborhood of p. Then $\mathrm{nor}(\bar\nabla_\xi \eta) \in N_pM$ only depends η via $\eta|_p$.*

Proof. Let $x^1,\dots,x^m,x^{m+1},\dots,x^{\bar m}$ be submanifold coordinates on $\bar M$ around p. Let $\Gamma^k_{ij} : U \to \mathbb{R}$ be the Christoffel symbols of ∇, $1 \le i,j,k \le m$, and $\bar\Gamma^k_{ij} : U \to \mathbb{R}$ be the Christoffel symbols of $\bar\nabla$, $1 \le i,j,k \le \bar m$. On U we write $\eta = \sum_{j=1}^m \eta^j \frac{\partial}{\partial x^j}$ and we define on $\bar U$:

$$\bar\eta^j(x^1,\dots,x^{\bar m}) := \begin{cases} \eta^j(x^1,\dots,x^m) & \text{for } j = 1,\dots,m \\ 0 & \text{for } j = m+1,\dots,\bar m \end{cases}.$$

Set $\bar\eta := \sum_{j=1}^{\bar m} \bar\eta^j \frac{\partial}{\partial x^j}$. Furthermore, write $\xi = \sum_{i=1}^m \xi^i \left.\frac{\partial}{\partial x^i}\right|_p$. Then we have:

$$\begin{aligned}
\mathrm{nor}(\bar\nabla_\xi \eta) &= \mathrm{nor}(\bar\nabla_\xi \bar\eta) \\
&= \bar\nabla_\xi \bar\eta - \nabla_\xi \eta \\
&= \sum_{i=1}^m \xi^i \sum_{k=1}^{\bar m} \left(\left.\frac{\partial \bar\eta^k}{\partial x^i}\right|_{x(p)} + \sum_{j=1}^{\bar m} \bar\Gamma^k_{ij}|_{x(p)} \cdot \bar\eta^j|_{x(p)} \right) \left.\frac{\partial}{\partial x^k}\right|_p \\
&\quad - \sum_{i=1}^m \xi^i \sum_{k=1}^m \left(\left.\frac{\partial \eta^k}{\partial x^i}\right|_{x(p)} + \sum_{j=1}^m \Gamma^k_{ij}|_{x(p)} \cdot \eta^j|_{x(p)} \right) \left.\frac{\partial}{\partial x^k}\right|_p \\
&= \sum_{i=1}^m \xi^i \sum_{j=1}^m \eta^j|_{x(p)} \left(\sum_{k=1}^{\bar m} \bar\Gamma^k_{ij}|_{x(p)} \left.\frac{\partial}{\partial x^k}\right|_p - \sum_{k=1}^m \Gamma^k_{ij}|_{x(p)} \left.\frac{\partial}{\partial x^k}\right|_p \right).
\end{aligned}$$

This only depends on $\eta^j|_{x(p)}$, i.e., only on $\eta|_p$. □

Definition 4.2.10. The map $II : T_pM \times T_pM \to N_pM$, given by

$$\mathrm{II}(\xi,\eta) = \mathrm{nor}\big(\bar\nabla_\xi \eta\big),$$

is called the **second fundamental form** of M in $\bar M$ (at the point $p \in M$).

Lemma 4.2.11. *The second fundamental form* II *is bilinear and symmetric.*

Proof. In the previous proof we have shown that

$$\mathrm{II}(\xi,\eta) = \sum_{i,j=1}^m \left(\sum_{k=1}^{\bar m} \bar\Gamma^k_{ij} \left.\frac{\partial}{\partial x^k}\right|_p - \sum_{k=1}^m \Gamma^k_{ij} \left.\frac{\partial}{\partial x^k}\right|_p \right) \xi^i \eta^j.$$

Clearly, II is bilinear. By the symmetry of the Christoffel symbols in the lower indices, II is also symmetric. □

Example 4.2.12. Let $M = S^1 \subset \bar{M} = \mathbb{R}^2$ and η as in Example 4.2.8. The second fundamental form is then given by $\mathrm{II}(\eta,\eta) = -p$.

Conclusion. The equation

$$\bar{\nabla}_\xi \eta = \nabla_\xi \eta + II(\xi, \eta|_p)$$

is the splitting of $\bar{\nabla}_\xi \eta$ into its tangential and normal parts.

Notation 4.2.13. For better readability we will from now on write $\langle \xi, \eta \rangle$ instead of $g(\xi,\eta)$ or $\bar{g}(\xi,\eta)$.

Since one can compute the Levi-Civita connection ∇ of the submanifold M from the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$, one should also be able to compute the curvature tensor R of M from that of $\bar{M}$. Indeed this is possible.

Theorem 4.2.14 (Gauß Formula). *Let $M \subset \bar{M}$ be a semi-Riemannian submanifold and $p \in M$. Let $\xi, \eta, \zeta, \nu \in T_pM$. Then we have*

$$\langle R(\zeta,\nu)\xi,\eta\rangle = \langle \bar{R}(\zeta,\nu)\xi,\eta\rangle + \langle \mathrm{II}(\nu,\xi), \mathrm{II}(\zeta,\eta)\rangle - \langle \mathrm{II}(\zeta,\xi), \mathrm{II}(\nu,\eta)\rangle .$$

Proof. Let $x^1,\dots,x^m$ be coordinates of M around p coming from a submanifold chart $x^1,\dots,x^m,x^{m+1},\dots,x^{\bar{m}}$. By multilinearity, it suffices to show the assertion for $\xi = \frac{\partial}{\partial x^i}\Big|_p$, $\eta = \frac{\partial}{\partial x^j}\Big|_p$, $\zeta = \frac{\partial}{\partial x^k}\Big|_p$ and $\nu = \frac{\partial}{\partial x^l}\Big|_p$. We have

$$\begin{aligned}
\langle \bar{R}(\zeta,\nu)\xi,\eta\rangle &= \left\langle \bar{\nabla}_\zeta \bar{\nabla}_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i} - \bar{\nabla}_{\bar{\nabla}_{\frac{\partial}{\partial x^k}|_p}\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i} - \bar{\nabla}_\nu \bar{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} + \bar{\nabla}_{\bar{\nabla}_{\frac{\partial}{\partial x^l}|_p}\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \eta \right\rangle \\
&\overset{\text{torsion freeness}}{=} \left\langle \bar{\nabla}_\zeta \bar{\nabla}_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i} - \bar{\nabla}_\nu \bar{\nabla}_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \eta \right\rangle \\
&= \left\langle \bar{\nabla}_\zeta \nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i} + \bar{\nabla}_\zeta \mathrm{II}\left(\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^i}\right) - \bar{\nabla}_\nu \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i} - \bar{\nabla}_\nu \mathrm{II}\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right), \eta \right\rangle \\
&= \left\langle \nabla_\zeta \nabla_{\frac{\partial}{\partial x^l}} \frac{\partial}{\partial x^i} - \nabla_\nu \nabla_{\frac{\partial}{\partial x^k}} \frac{\partial}{\partial x^i}, \eta \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\partial_\zeta \overbrace{\left\langle \mathrm{II}\left(\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^i}\right),\frac{\partial}{\partial x^j}\right\rangle}^{\equiv 0} - \left\langle \mathrm{II}\left(\frac{\partial}{\partial x^l},\frac{\partial}{\partial x^i}\right),\bar\nabla_\zeta \frac{\partial}{\partial x^j}\right\rangle \\
&-\partial_\nu \overbrace{\left\langle \mathrm{II}\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^i}\right),\frac{\partial}{\partial x^j}\right\rangle}^{\equiv 0} + \left\langle \mathrm{II}\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^i}\right),\bar\nabla_\nu \frac{\partial}{\partial x^j}\right\rangle \\
=\ &\langle R(\zeta,\nu)\xi,\eta\rangle + \langle \mathrm{II}(\zeta,\xi),\mathrm{II}(\nu,\eta)\rangle - \langle \mathrm{II}(\nu,\xi),\mathrm{II}(\zeta,\eta)\rangle .
\end{aligned}
$$
□

Corollary 4.2.15. *If $E \subset T_pM$ is a non-degenerate plane with basis ξ,η, then we have*

$$K(E) = \bar K(E) + \frac{\langle \mathrm{II}(\xi,\xi),\mathrm{II}(\eta,\eta)\rangle - \langle \mathrm{II}(\xi,\eta),\mathrm{II}(\xi,\eta)\rangle}{\langle\xi,\xi\rangle\langle\eta,\eta\rangle - \langle\xi,\eta\rangle^2}.$$

Proof. This follows directly from the definition of sectional curvature and the Gauß formula. □

Lemma 4.2.16. *Let $M \subset \bar M$ be a semi-Riemannian submanifold. Let $\varphi : \bar M \to \bar N$ be a local isometry. Set $\varphi(M) =: N$. For $\xi,\eta \in T_pM$ we have*

$$\mathrm{II}_N\big(d\varphi|_p(\xi), d\varphi|_p(\eta)\big) = d\varphi|_p\big(\mathrm{II}_M(\xi,\eta)\big).$$

Proof. Local isometries preserve ∇ and $\bar\nabla$. Since II is the difference of ∇ and $\bar\nabla$ we get the assertion. □

4.3 Totally geodesic submanifolds

Let $M \subset \bar M$ be a semi-Riemannian submanifold and $c : I \to M$ a smooth curve. Let ξ be a smooth vector field at M along c. Then the splitting in tangential and normal parts of the covariant derivative is given by

$$\frac{\bar\nabla}{dt}\xi = \frac{\nabla}{dt}\xi + \mathrm{II}(\xi,\dot c).$$

In particular, we have for $\xi = \dot c$

$$\frac{\bar\nabla}{dt}\dot c = \frac{\nabla}{dt}\dot c + \mathrm{II}(\dot c,\dot c).$$

Therefore the curve c is a geodesic in M if and only if

$$\frac{\bar\nabla}{dt}\dot c = \mathrm{II}(\dot c,\dot c), \quad \text{i.e., if} \quad \frac{\bar\nabla}{dt}\dot c(t) \in N_{c(t)}M \text{ for all } t \in I.$$

Example 4.3.1. Let $M = S^n \subset \bar M = \mathbb{R}^{n+1}$ with Euclidean metric. Let $c : I \to S^n$ be a great circle,

$$c(t) = \cos(t)\cdot p + \sin(t)\cdot \xi.$$

with $p \in S^n$, $\xi \in T_pS^n \subset \mathbb{R}^{n+1}$ and $\|\xi\| = 1$. From this we get

$$\frac{\bar\nabla}{dt}\dot c(t) = \ddot c(t) = -\cos(t)\cdot p - \sin(t)\cdot\xi = -c(t) \in N_{c(t)}S^n.$$

Hence c is a geodesic in S^n.

Definition 4.3.2. A semi-Riemannian submanifold is called **totally geodesic** if $\mathrm{II} \equiv 0$.

Theorem 4.3.3. *For a semi-Riemannian submanifold $M \subset \bar M$ the following statements are equivalent:*

(i) M ist totally geodesic.

(ii) Every geodesic in M is also a geodesic in $\bar M$.

(iii) For any $p \in M$ and $\xi \in T_pM$ there exists an $\varepsilon > 0$ such that the $\bar M$-geodesic $c : (-\varepsilon,\varepsilon) \to \bar M$ with $c(0) = p$ and $\dot c(0) = \xi$ lies in M, i.e., $c(t) \in M$ for all $t \in (-\varepsilon,\varepsilon)$.

(iv) Let $c : I \to M$ be a smooth curve. Then the parallel transport along c w.r.t. M and w.r.t. $\bar M$ coincide (for tangent vectors of M).

Proof. **"(ii)⇒(iii)":** Let $p \in M$ and $\xi \in T_pM$. Let c be the $\bar M$-geodesic with $c(0) = p$ and $\dot c(0) = \xi$. Let $\tilde c$ be the M-geodesic with $\tilde c(0) = p$ and $\dot{\tilde c}(0) = \xi$. By (ii), $\tilde c$ is also geodesic in $\bar M$. Since we have $\tilde c(0) = c(0)$ and $\dot{\tilde c}(0) = \dot c(0)$, the two $\bar M$-geodesics must coincide, $c = \tilde c$ on $(-\varepsilon,\varepsilon)$ for a $\varepsilon > 0$. In particular, c lies in M.
"(iii)⇒(i)": Let $p \in M$ and $\xi \in T_pM$. Let c_ξ be the $\bar M$-geodesic with $c_\xi(0) = p$ and $\dot c_\xi(0) = \xi$. By (iii), c_ξ lies in M for $t \in (-\varepsilon,\varepsilon)$ with suitable $\varepsilon > 0$. On $(-\varepsilon,\varepsilon)$ we get:

$$0 = \frac{\bar\nabla}{dt}\dot c_\xi = \underbrace{\frac{\nabla}{dt}\dot c_\xi}_{\text{tangential}} + \underbrace{\mathrm{II}(\dot c_\xi,\dot c_\xi)}_{\text{normal}}$$

In particular, we have

$$\mathrm{II}\big(\dot c_\xi(t),\dot c_\xi(t)\big) = 0 \text{ for all } t \in (-\varepsilon,\varepsilon).$$

For $t=0$ this means that $\mathrm{II}(\xi,\xi)=0$. Since ξ is arbitrary, polarization yields $\mathrm{II}\equiv 0$.
"(i)⇒(iv)": We have $\frac{\nabla}{dt}\xi = \frac{\bar\nabla}{dt}\xi$. Hence ξ is parallel in M if and only if ξ is parallel in $\bar M$.
"(iv)⇒(ii)": Let c be a geodesic in M.

$$\begin{aligned} &\Rightarrow \quad \dot c \text{ is parallel in } M.\\ &\overset{\text{(iv)}}{\Rightarrow} \quad \dot c \text{ is parallel in } \bar M.\\ &\Rightarrow \quad c \text{ ist geodesic in } \bar M. \end{aligned}$$

□

Example 4.3.4. Let $M\subset\bar M=\mathbb{R}^n$ be an affine subspace where $\mathbb{R}^n$ is equipped with g_{eucl} or g_{Mink}. Criterion (iii) shows that $M\subset\mathbb{R}^n$ is totally geodesic.

Example 4.3.5. Let $\bar M$ be an arbitrary semi-Riemannian manifold.

(1) All 0-dimensional submanifolds are totally geodesic.

(2) Every submanifold of codimension 0, i.e., every open subset of $\bar M$, is totally geodesic.

(3) Let $M=\{c(t)|t\in I\}$, where $c:I\to\bar M$ is a geodesic. If M is a semi-Riemannian submanifold (has no self-intersection, for instance), then M is totally geodesic.

Remark 4.3.6. Most semi-Riemannian manifolds $\bar M$ do not have totally geodesic submanifolds of dimension $m\in\{2,\dots,\bar m-1\}$.

Theorem 4.3.7. *Let $M\subset\bar M$ be a semi-Riemannian submanifold. Assume that there exists an isometry $\varphi\in\mathrm{Isom}(\bar M)$, such that M is a connected component of*

$$\mathrm{Fix}(\varphi)=\{p\in\bar M\,|\,\varphi(p)=p\}.$$

Then M is totally geodesic.

Proof. We check Criterion (iii) in Theorem 4.3.3. Let $p\in M$ and $\xi\in T_pM$. We first show that

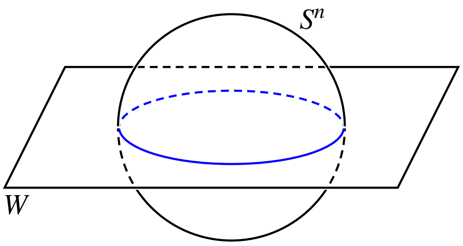
$$d\varphi|_p(\xi)=\xi.$$

Namely, let $\gamma:J\to M$ be a smooth curve with $\gamma(0)=p$ and $\dot\gamma(0)=\xi$. Then

$$d\varphi|_p(\xi)=d\varphi|_p(\dot\gamma(0))=\frac{d}{dt}\underbrace{(\varphi\circ\gamma)}_{\substack{=\gamma,\text{ since}\\ M\subset\mathrm{Fix}(\varphi)}}|_{t=0}=\dot\gamma(0)=\xi. \qquad ✓$$

Now if c is the geodesic in $\bar M$ with $c(0)=p$ and $\dot c(0)=\xi$ then, by Proposition 2.6.20, c lies entirely in $\mathrm{Fix}(\varphi)$. Since $c(I)$ is connected, c remains in M. □

Example 4.3.8. Let $\bar{M} = S^n$. Let $W \subset \mathbb{R}^{n+1}$ be a sub-vector space. Let $A \in \mathrm{O}(n+1)$ be the reflection about W.



$$\Rightarrow \quad \varphi := A|_{S^n} \in \mathrm{Isom}(S^n)$$

$$\Rightarrow \quad \mathrm{Fix}(\varphi) = W \cap S^n \qquad \text{is totally geodesic}$$

Hence all "great subspheres" in S^n are totally geodesic submanifolds. In particular, S^n admits totally geodesic submanifolds of every codimension.

The *Gauß Formula* (Theorem 4.2.14) tells us that if $M \subset \bar{M}$ is totally geodesic, then

$$\begin{aligned} R(\xi,\eta)\zeta &= \tan(\bar{R}(\xi,\eta)\zeta)^{1} && \text{for all } p \in M \text{ and } \xi,\eta,\zeta \in T_pM, \\ K(E) &= \bar{K}(E) && \text{for all non-degenerate planes } E \subset T_pM. \end{aligned}$$

4.4 Hypersurfaces

Definition 4.4.1. A semi-Riemannian submanifold $M \subset \bar{M}$ is called a **semi-Riemannian hypersurface** if $\mathrm{codim} M = 1$.
The **signature** of M is $\varepsilon = +1$ if $(\bar{g}|_p)|_{N_pM \times N_pM}$ is positive definite, and $\varepsilon = -1$ if $(\bar{g}|_p)|_{N_pM \times N_pM}$ is negative definite.

Remark 4.4.2. For $\varepsilon = +1$ we have $\mathrm{Index}(\bar{M},\bar{g}) = \mathrm{Index}(M,g)$ while for $\varepsilon = -1$ we get $\mathrm{Index}(\bar{M},\bar{g}) = \mathrm{Index}(M,g) + 1$.

Notation 4.4.3. For $\xi \in T_pM$ we write

$$|\xi| := \sqrt{|\langle \xi,\xi \rangle|}.$$

Caution! This does not define a norm unless $\langle \cdot,\cdot \rangle$ is definite. In particular, it can occur that $|\xi| = 0$ even if $\xi \neq 0$.

[1] This formula even holds without the tangential projector as a consequence of the Codazzi equation which we did not treat.

Gradient of a differentiable function

Let (M,g) be a semi-Riemannian manifold of dimension n. Let $f : M \to \mathbb{R}$ be differentiable and $p \in M$. Then $df|_p \in T_p^*M$. In coordinates we have

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

Since $g|_p$ is non-degenerate there exists exactly one $\xi \in T_pM$ such that

$$df|_p(\eta) = g|_p(\xi,\eta) \qquad \text{for all } \eta \in T_pM.$$

Write $\xi =: \mathrm{grad} f|_p$. In local coordinates, write $\mathrm{grad} f = \sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i}$. Then we have:

$$\begin{aligned}\frac{\partial f}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \left(\frac{\partial}{\partial x^j}\right) = df\left(\frac{\partial}{\partial x^j}\right) &= g\left(\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\ = \sum_{i=1}^{n} \alpha^i g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= \sum_{i=1}^{n} \alpha^i g_{ij}.\end{aligned}$$

Matrix multiplication with $(g^{ij})_{ij}$ yields $\alpha^i = \sum_{j=1}^{n} g^{ij} \frac{\partial f}{\partial x^j}$, thus

$$\boxed{\mathrm{grad} f = \sum_{i,j=1}^{n} \frac{\partial f}{\partial x^j} g^{ij} \frac{\partial}{\partial x^i}}$$

Lemma 4.4.4. *Let $\bar{M}$ be a semi-Riemannian manifold and $f : \bar{M} \to \mathbb{R}$ smooth and $c \in \mathbb{R}$ be a regular value of f. Then $M := f^{-1}(c) \subset \bar{M}$ is a semi-Riemannian hypersurface of signature ε, if*

$$\langle \mathrm{grad} f, \mathrm{grad} f \rangle \cdot \varepsilon > 0.$$

Moreover, we have $\nu := \frac{\mathrm{grad} f|_p}{|\mathrm{grad} f|_p|} \in N_pM$ and $\langle \nu,\nu \rangle = \varepsilon$.

Proof. Since c is a regular value, M is a hypersurface. The lemma follows once we show

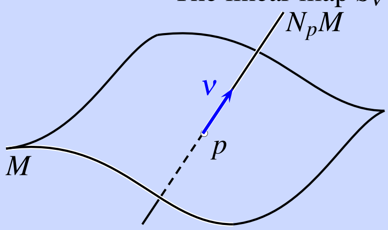
$$\mathrm{grad} f|_p \perp T_pM.$$

Let $\xi \in T_pM$. We choose $\gamma : I \to M$ with $\dot{\gamma}(0) = \xi$ and we obtain:

$$0 = \frac{d}{dt} \underbrace{f(\gamma(t))}_{\equiv c} |_{t=0} = df|_p(\xi) = \langle \mathrm{grad} f|_p, \xi \rangle . \qquad \square$$

Definition 4.4.5. Let $M \subset \bar{M}$ be a semi-Riemannian hypersurface and $p \in M$. Let $\nu \in N_pM$ with $|\nu| = 1$.

The linear map $S_\nu : T_pM \to T_pM$, characterized by



$$\langle S_\nu(\xi), \eta\rangle = \langle \mathrm{II}(\xi,\eta), \nu\rangle \text{ for all } \xi, \eta \in T_pM,$$

is called the **Weingarten map**.

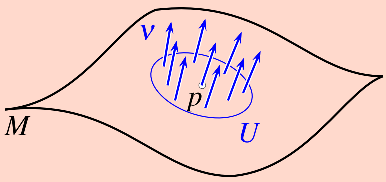
Lemma 4.4.6. *The Weingarten map S_ν is self-adjoint.*

Proof. This is clear because II is symmetric. □

Remark 4.4.7. We have $S_{-\nu} = -S_\nu$. Without specifying the choice of ν, the Weingarten map is only determined up to a sign.

Lemma 4.4.8. *Let $M \subset \bar{M}$ be a semi-Riemannian hypersurface and $p \in M$. Let $U \subset M$ be an open neighborhood of p and $\nu \in C^\infty(U, NM)$ with $|\nu| = 1$. Then we have*

$$S_\nu(\xi) = -\bar{\nabla}_\xi \nu.$$



Proof. For all $\eta \in C^\infty(U, TM)$ we have:

$$\begin{aligned}\langle S_\nu(\xi), \eta\rangle &= \langle \mathrm{II}(\xi,\eta), \nu\rangle = \left\langle \mathrm{nor}(\bar{\nabla}_\xi \eta), \nu\right\rangle = \left\langle \bar{\nabla}_\xi \eta, \nu\right\rangle \\ &= \partial_\xi \underbrace{\langle \eta, \nu\rangle}_{=0} - \left\langle \eta, \bar{\nabla}_\xi \nu\right\rangle = -\left\langle \bar{\nabla}_\xi \nu, \eta\right\rangle.\end{aligned}$$

□

Gauß formula:
Let $M \subset \bar{M}$ be a semi-Riemannian hypersurface with signature ε. Let $\xi, \eta, \zeta \in T_pM$. Then:

$$R(\xi,\eta)\zeta = \tan(\bar{R}(\xi,\eta)\zeta) + \varepsilon\big\{ \langle S_\nu(\eta),\zeta\rangle S_\nu(\xi) - \langle S_\nu(\xi),\zeta\rangle S_\nu(\eta)\big\}.$$

For any non-degenerate plane $E \subset T_pM$ we have

$$K(E) = \bar{K}(E) + \varepsilon \cdot \frac{\langle S_\nu(\xi),\xi\rangle \langle S_\nu(\eta),\eta\rangle - \langle S_\nu(\xi),\eta\rangle^2}{\langle \xi,\xi\rangle \langle \eta,\eta\rangle - \langle \xi,\eta\rangle^2}$$

where ξ, η is a basis of E.

Pseudospheres and pseudo-hyperbolic spaces

Now consider $\bar{M} = \mathbb{R}^{n+1}$ with $\bar{g} = -\sum_{i=0}^{k-1} dx^i \otimes dx^i + \sum_{i=k}^{n} dx^i \otimes dx^i$ in Cartesian coordinates $x^0, \ldots, x^n$. Then $(\bar{M}, \bar{g})$ is a semi-Riemannian manifold of index k. For $k = 0$ we have the Euclidean metric and for $k = 1$ the Minkowski metric. For general k the representing matrix of $\bar{g}$ in Cartesian coordinates is given by

$$(\bar{g}_{ij}) = \begin{pmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

In particular, all $\bar{g}_{ij}$ are constant. Hence all Christoffel symbols vanish in Cartesian coordinates. Therefore the curvature vanishes:

$$\bar{R} \equiv 0, \quad \bar{K} \equiv 0, \quad \overline{\mathrm{ric}} \equiv 0 \quad \text{and} \quad \overline{\mathrm{scal}} \equiv 0.$$

Now consider the function

$$f : \mathbb{R}^{n+1} \to \mathbb{R}, \quad f(x^0, \ldots, x^n) = -\sum_{i=0}^{k-1} (x^i)^2 + \sum_{i=k}^{n} (x^i)^2 = \sum_{i=0}^{n} \varepsilon_i (x^i)^2.$$

For the gradient we get

$$\begin{aligned} \mathrm{grad} f|_x &= \sum_{i,j=0}^{n} \frac{\partial f}{\partial x^i}(x) \underbrace{g^{ij}}_{=\varepsilon_i \delta^{ij}} \frac{\partial}{\partial x^j} \\ &= \sum_{i=0}^{n} \varepsilon_i \frac{\partial f}{\partial x^i}(x) \frac{\partial}{\partial x^i} \\ &= 2\sum_{i=0}^{n} \varepsilon_i \cdot \varepsilon_i x^i \frac{\partial}{\partial x^i} \\ &= 2\sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}. \end{aligned}$$

Thus $\operatorname{grad} f|_x = 0$ if and only if $x = 0$. Consequently, the only critical point of f is $x = 0$ and $0 = f(0)$ is the only critical value of f. If $c \neq 0$ then $M := f^{-1}(c)$ therefore defines a differentiable submanifold of codimension 1. We compute:

$$\begin{aligned}
\langle \operatorname{grad} f|_x, \operatorname{grad} f|_x \rangle &= 4\left\langle \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}, \sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i} \right\rangle \\
&= 4 \sum_{i,j=0}^{n} x^i x^j g_{ij} \\
&= 4 \sum_{j=0}^{n} \varepsilon_j (x^j)^2 \\
&= 4 f(x).
\end{aligned}$$

Hence for $c > 0$ we have that $f^{-1}(c)$ is a semi-Riemannian hypersurface of signature $\varepsilon = +1$, for $c < 0$ it is a semi-Riemannian hypersurface of signature $\varepsilon = -1$.

Definition 4.4.9. Let $r > 0$. The semi-Riemannian hypersurface

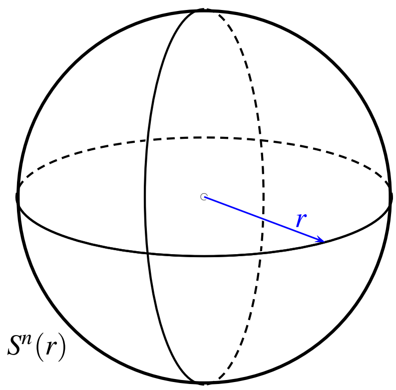
$$S^n_k(r) := f^{-1}(r^2)$$

of $(\mathbb{R}^{n+1}, \bar{g})$ is called the **pseudo-sphere** of radius r and with index k. The semi-Riemannian hypersurface

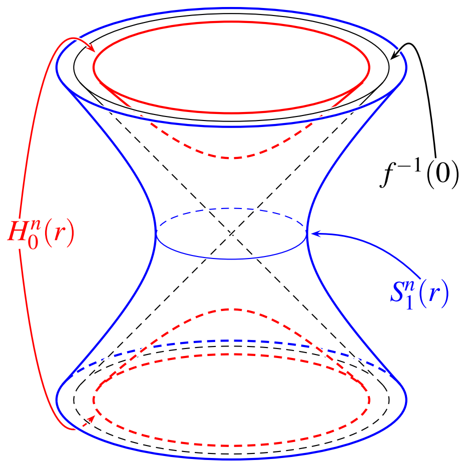
$$H^n_{k-1}(r) := f^{-1}(-r^2),$$

is called the **pseudo-hyperbolic space** of index $k-1$.

Example 4.4.10. Let $k = 0$ and $\bar{g} = g_{\mathrm{eucl}}$. Then $S^n_0(r) = S^n(r)$ is the standard sphere of radius r.



Example 4.4.11. The case $k = 1$ and $\bar{g} = g_{\mathrm{Mink}}$ is also of great interest.



Definition 4.4.12. The hypersurface $H^n := \{x \in H^n_0(1) \,|\, x^0 > 0\}$ together with the induced Riemannian metric g_{hyp} is called the n-dimensional **hyperbolic space**.

Definition 4.4.13. The hypersurface $S^4_1(r)$ together with the induced Lorentzian metric is called **deSitter spacetime** and $H^4_1(r)$ is called **anti-deSitter spacetime**.

Remark 4.4.14. The pseudo-sphere $S^n_k(r)$ is diffeomorphic to $\mathbb{R}^k \times S^{n-k}$ while the pseudo-hyperbolic space $H^n_k(r)$ is diffeomorphic to $S^k \times \mathbb{R}^{n-k}$. See the exercises or [ON83, page 111] for a proof of this fact.

We determine the curvature of these hypersurfaces. For $M = f^{-1}(c)$ with $c \neq 0$ we recall

$$\langle \operatorname{grad} f|_x, \operatorname{grad} f|_x \rangle = 4f(x) = 4c,$$

hence

$$\nu|_x := \frac{\operatorname{grad} f|_x}{\sqrt{|4c|}} = \frac{\operatorname{grad} f|_x}{2r} = \frac{1}{r}\sum_{i=0}^{n} x^i \frac{\partial}{\partial x^i}.$$

By Lemma 4.4.8 we get

$$\boxed{S_\nu = -\frac{1}{r}\mathrm{id}}$$

Now the Gauß formula yields

$$\boxed{R(\xi,\eta)\zeta = \frac{\varepsilon}{r^2}(\langle \eta,\zeta\rangle \xi - \langle \xi,\zeta\rangle \eta)} \quad \text{and} \quad \boxed{K \equiv \frac{\varepsilon}{r^2}}$$

We compute

$$\begin{aligned}
\operatorname{ric}(\xi,\eta) &= \sum_{i=1}^{n} \varepsilon_i \langle R(\xi,e_i)e_i,\eta\rangle \\
&= \frac{\varepsilon}{r^2}\sum_{i=1}^{n} \varepsilon_i \langle \underbrace{\langle e_i,e_i\rangle}_{=\varepsilon_i} \xi - \langle \xi,e_i\rangle e_i, \eta \rangle \\
&= \frac{\varepsilon}{r^2}(n\langle \xi,\eta\rangle - \langle \xi,\eta\rangle),
\end{aligned}$$

thus

$$\boxed{\operatorname{ric} = \frac{\varepsilon(n-1)}{r^2} g} \quad \text{and} \quad \boxed{\operatorname{scal} = \frac{\varepsilon(n-1)n}{r^2}}$$

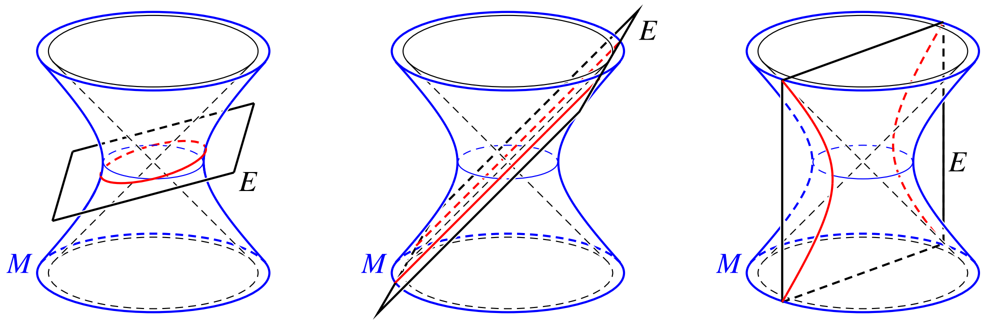
Remark 4.4.15. For the *Einstein tensor* of $S^4_1(r)$ or $H^4_1(r)$ we get

$$G = \operatorname{ric} - \frac{1}{2}\operatorname{scal}\cdot g = \frac{3\varepsilon}{r^2}g - \frac{1}{2}\frac{\varepsilon\cdot 3\cdot 4}{r^2}g = -3\frac{\varepsilon}{r^2}g.$$

Thus deSitter and anti-deSitter spacetime are vacuum solutions of the Einstein field equations with cosmological constant $\Lambda = \frac{3}{r^2}$ and $\Lambda = -\frac{3}{r^2}$, respectively.

Next we determine the geodesics of the pseudo-spheres and pseudo-hyperbolic spaces. Let $p \in M$ where $M = S^n_k(r)$ or $M = H^n_{k-1}(r)$ and let $\xi \in T_pM \subset T_p\mathbb{R}^{n+1} \cong \mathbb{R}^{n+1}$ with $\xi \neq 0$. What is the geodesic c with $c(0) = p$ and $\dot{c}(0) = \xi$?

Note that $p \neq 0$. Then p and ξ, considered as vectors in $\mathbb{R}^{n+1}$, are linearly independent because $\xi \in T_pM$ and $p \in N_pM$. Let $E \subset \mathbb{R}^{n+1}$ be the plane spanned by p and ξ. If ξ is space-like or time-like, then E is non-degenerate for $\bar{g}$. Then the reflection (w.r.t. $\bar{g}$) about E is an isometry of $(\mathbb{R}^{n+1}, \bar{g})$. The restriction of the reflection to M yields an isometry φ of M, see the discussion of isometries below. Now $E \cap M$ is the fixed point set of φ, hence a 1-dimensional totally geodesic submanifold. In other words, if we parametrize the connected component of $E \cap M$ containing p proportionally to arc-length or eigentime, respectively, in such a way that $\dot{c}(0) = \xi$, then it is the geodesic c we are after.



If ξ is light-like, then E is degenerate. But now $E \cap M$ consists of two parallel straight lines. If we take any affine parametrization of the straight line containing p, then we get a geodesic in $(\mathbb{R}^{n+1}, \bar{g})$ which contains p and lies entirely in M. Thus it is also a geodesic in M. When choose the affine parametrization such that $c(0) = p$ and $\dot{c}(0) = \xi$, then we found the right geodesic also in the light-like case.

In order to determine the isometry group of pseudo-spheres and pseudo-hyperbolic spaces we define

$$\mathrm{O}(n+1-k,k) := \{A \in GL(n+1) \mid \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall x, y \in \mathbb{R}^{n+1}\}.$$

Here $\langle x, y \rangle = -\sum_{j=0}^{k-1} x^j y^j + \sum_{j=k}^{n} x^j y^j$. We have $\mathrm{O}(n+1,0) = \mathrm{O}(n+1)$ and $\mathrm{O}(n,1)$ is the *Lorentz group*. For any $A \in \mathrm{O}(n+1-k,k)$ we have

$$A(S^n_k(r)) = S^n_k(r) \quad \text{and} \quad A(H^n_{k-1}(r)) = H^n_{k-1}(r).$$

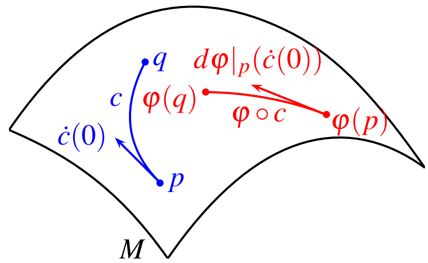
Since the semi-Riemannian metric of M is obtained by restricting $\bar{g}$ to M, the restriction of isometries of $(\mathbb{R}^{n+1}, \bar{g})$ to M are isometries of M. We have constructed an injective group homomorphism

$$\begin{aligned} \mathrm{O}(n+1-k,k) &\to \mathrm{Isom}(M), \\ A &\mapsto A|_M. \end{aligned}$$

Next we show that this homomorphism is also surjective.

Proposition 4.4.16. *Let M be a semi-Riemannian manifold, let $p \in M$ and $\varphi, \psi \in \mathrm{Isom}(M)$ with $\varphi(p) = \psi(p)$ and $d\varphi|_p = d\psi|_p$.*
Then φ and ψ coincide on the set of all points which can be joined with p by a geodesic.

Proof: Let $c : [0,1] \to M$ be a geodesic with $c(0) = p$ and $c(1) = q$. Then $\tilde{c} := \varphi \circ c$ and $\hat{c} := \psi \circ c$ are also geodesics and we have $\tilde{c}(0) = \varphi(p) = \psi(p) = \hat{c}(0)$ and $\dot{\tilde{c}}(0) = d\varphi|_p(\dot{c}(0)) = d\psi|_p(\dot{c}(0)) = \dot{\hat{c}}(0)$. Therefore $\tilde{c} = \hat{c}$. In particular, $\varphi(q) = \tilde{c}(1) = \hat{c}(1) = \psi(q)$.



Corollary 4.4.17. *If all points of M can be joined by geodesics with p, then every isometry φ of M is uniquely determined by $\varphi(p)$ and $d\varphi|_p$.*

Example 4.4.18. Let $M = (\mathbb{R}^n, g_{\mathrm{eucl}})$. We already know

$$\{\text{Euclidean motions}\} \subset \mathrm{Isom}(M),$$

where a Euclidean motion $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ has the form $\varphi(x) = Ax + b$ with $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. We can now use Proposition 4.4.16 to show that there are no further isometries of Euclidean space. Let $\varphi \in \mathrm{Isom}(M)$. Put $b := \varphi(0)$ and $A := d\varphi|_0 \in \mathrm{O}(n)$. Then the Euclidean motion $\tilde{\varphi}(x) := Ax + b$ satisfies $\tilde{\varphi}(0) = b = \varphi(0)$ and $d\tilde{\varphi}|_0 = A = d\varphi|_0$. Since any two points in Euclidean space can be joined by a straight line we can apply Corollary 4.4.17 and conclude $\varphi = \tilde{\varphi}$. This proves

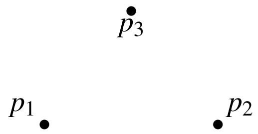
$$\{\text{Euclidean motions}\} = \mathrm{Isom}(M).$$

Similarly one can show

$$\mathrm{Isom}(\mathbb{R}^n, g_{\mathrm{Mink}}) = \text{ Poincaré group}.$$

Remark 4.4.19. The assumption that the points can be joined with p by geodesics is necessary for the statement of Corollary 4.4.17.

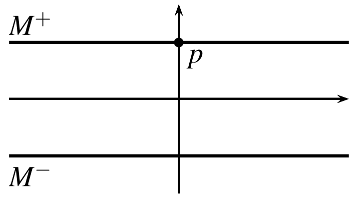
Example 4.4.20. Let $M = \{p_1, p_2, p_3\}$ be a 0-dimensional manifold consisting of 3 points. On a 0-dimensional manifold all tangent spaces are trivial so $g = 0$ is a Riemannian metric. All bijective maps $M \to M$ are isometries. Consider the following two maps:



$$\varphi_1 := \mathrm{id}, \quad \text{and} \quad \varphi_2 := \begin{cases} p_1 \mapsto p_1 \\ p_2 \mapsto p_3 \\ p_3 \mapsto p_2 \end{cases}$$

Then $\varphi_1 \neq \varphi_2$ despite $\varphi_1(p_1) = \varphi_2(p_1)$ and $d\varphi_1|_{p_1} = 0 = d\varphi_2|_{p_1}$.

Example 4.4.21. Here is a 1-dimensional example. Let $M = \{(x,y) \in \mathbb{R}^2 \,|\, |y| = 1\} = M^+ \sqcup M^-$ where $M^\pm = \{(x,y) \in \mathbb{R}^2 \,|\, y = \pm 1\}$. Let $p = (0,1) \in M$. We provide M with the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^2$. Put



$$\varphi_1 := \mathrm{id}, \quad \text{and} \quad \varphi_2(x,y) := \begin{cases} (x,y) \text{ on } M^+ \\ (-x,y) \text{ on } M^- \end{cases}$$

Both φ_1 and φ_2 are isometries. Now $\varphi_1(p) = \varphi_2(p)$ and $d\varphi_1|_p = d\varphi_2|_p$ but $\varphi_1 \neq \varphi_2$.

Lemma 4.4.22. *On $M = S^n_k(r)$ (where $0 \le k \le n-1$), on $M = H^n_k(r)$ (where $1 \le k \le n$) and on $M = H^n(r)$ any two points can be joined by a geodesic.*

Proof. W.l.o.g. we assume $n \ge 2$. Let $p, q \in M$. Since M is connected we can choose a continuous curve $c : [0,1] \to M$ with $c(0) = p$ and $c(1) = q$. W.l.o.g. we assume $c(t) \notin \{p, -p\}$ for all $t \in (0,1)$. Then p and $c(t)$ are linearly independent for all $t \in (0,1)$ and span a unique plane $E(t)$.

For any $t \in (0,1)$ the intersection $M \cap E(t)$ consists of an ellipse or a pair of hyperbolas or a pair of straight lines. For $t \to 0$ the points $c(t)$ converges to p; hence the points p and $c(t)$ lie on the same connected component of $M \cap E(t)$ for sufficiently small t.

For $t \in (0,1)$ we choose $X(t) \in \mathbb{R}^{n+1}$ depending continuously on t, which spans $E(t)$ together with p and which, w.r.t. to the Euclidean skalar product $\langle \cdot, \cdot \rangle_{\mathrm{eukl}}$, satisfies $X(t) \perp p$ and $\|X(t)\|_{\mathrm{eukl}} = 1$. Since S^n is compact there is a sequence $t_i \in (0,1)$ with $t_i \to 1$ such that $X(t_i)$ converges. Put $\lim\limits_{i \to \infty} X(t_i) =: X(1)$. By continuity, $X(1) \perp p$ and $\|X(1)\|_{\mathrm{eukl}} = 1$. Hence p and $X(1)$ are linearly independent and span a plane $E(1)$. By continuity, $p, q \in M \cap E(1)$ and they lie in the same connected component of $M \cap E(1)$. $\square$

Theorem 4.4.23. *Restriction yields isomorphisms*

$$\mathrm{Isom}(S^n_k(r)) \quad \cong \quad O(n+1-k, k) \quad \textit{for } 0 \le k \le n-1$$

$$
\begin{aligned}
\operatorname{Isom}(H^n_k(r)) &\cong O(n-k,k+1) \quad \textit{for } 1\le k\le n\\
\operatorname{Isom}(H^n(r)) &\cong SO(n,1) := \{A\in O(n,1)\,|\,A^0_0>0\}.
\end{aligned}
$$

Proof. Put $M=S^n_k(r)$, $M=H^n_k(r)$ or $M=H^n(r)$ and $G=\mathrm{O}(n+1-k,k)$, $G=\mathrm{O}(n-k,k+1)$, or $G=\mathrm{SO}(n,1)$, respectively. We need to show: Every isometry of M is of the form

$$\varphi = A|_M \text{ with } A\in G.$$

a) We first show that *G acts transitively on M*. This means that for all $p,q\in M$ there exists an $A\in G$ with $Ap=q$.

   Namely: W.l.o.g. let $p=r\cdot e_0=(r,0,\dots,0)^T$. From $\langle q,q\rangle=\pm r^2$ we see that $b_0:=\frac{1}{r}q$ satisfies $\langle b_0,b_0\rangle=\pm1$. We extend b_0 to a generalized eigenbasis $b_0,b_1,\dots,b_n$ of $\mathbb{R}^{n+1}$. Now $A:=(b_0,b_1,\dots,b_n)\in G$ and $Ap=rAe_0=rb_0=q$.

b) Next we show: *For any linear isometry $B:T_{p_0}M\to T_{p_0}M$ where $p_0=re_0$, there exists an $A\in G$ such that $\varphi=A|_M$ satisfies $\varphi(p_0)=p_0$ and $d\varphi|_{p_0}=B$.* Namely:

$$A := \left(\begin{array}{c|c} 1 & 0\dots0 \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right)$$

   does the job.

c) Let now $\varphi\in\operatorname{Isom}(M)$. Put $p_1:=\varphi(p_0)$ where $p_0=re_0$. By a) there exists an $A_1\in G$ with $A_1p_0=p_1$. Hence $\psi:=A_1^{-1}|_M\circ\varphi$ is an isometry of M with $\psi(p_0)=p_0$.

   Moreover, $B:=d\psi|_{p_0}:T_{p_0}M\to T_{p_0}M$ is a linear isometry. By b) there is an $A_2\in G$ such that $\chi:=A_2|_M$ satisfies $d\chi|_{p_0}=B$. Lemma 4.4.22 and Corollary 4.4.17 imply $\chi=\psi$. Thus

$$\varphi = A_1|_M\circ\psi = A_1|_M\circ\chi = A_1|_M\circ A_2|_M = \underbrace{(A_1\circ A_2)}_{\in G}|_M. \qquad \square$$

4.5 Trigonometry in spaces of constant curvature

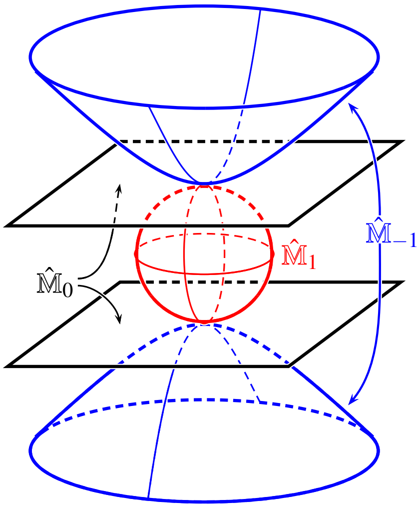
We want to extend the classical trigonometry of the Euclidean plane to 2-dimensional model spaces of constant curvature. This means that we investigate length- and angular relations in geodesic triangles.

Notation 4.5.1. The **model space** $\mathbb{M}^n_\kappa$ is defined as

$$\mathbb{M}^n_\kappa := \begin{cases} S^n(\frac{1}{\sqrt{\kappa}}) & \text{if } \kappa > 0, \\ \mathbb{R}^n & \text{if } \kappa = 0, \\ H^n\left(\frac{1}{\sqrt{|\kappa|}}\right) & \text{if } \kappa < 0. \end{cases}$$

Thus $\mathbb{M}^n_\kappa$ is an n-dimensional Riemannian manifold with the constant sectional curvature κ.

Remark 4.5.2. Since for any given three points there exists a two-dimensional totally geodesic submanifold of $\mathbb{M}^n_\kappa$ isometric to $\mathbb{M}^2_\kappa$ which contains these points, it suffices to consider the case $n = 2$.



Define the bilinear form on $\mathbb{R}^3$

$$\langle x,y\rangle_\kappa := x^0y^0 + \kappa(x^1y^1 + x^2y^2).$$

Set $\hat{\mathbb{M}}_\kappa := \{x \in \mathbb{R}^3 \mid \langle x,x\rangle_\kappa = 1\}$ and put

$$\mathbb{M}_\kappa := \begin{cases} \hat{\mathbb{M}}_\kappa & \text{if } \kappa > 0, \\ \{x \in \hat{\mathbb{M}}_\kappa \mid x^0 > 0\} & \text{if } \kappa \le 0. \end{cases}$$

In the case $\kappa \neq 0$, the metric $\frac{1}{\kappa}\langle\cdot,\cdot\rangle_\kappa$ on $\mathbb{R}^3$ induces a Riemannian metric on $\mathbb{M}_\kappa$. In particular,

$$\mathbb{M}_\kappa = \begin{cases} S^2 & \text{if } \kappa = 1, \\ H^2 & \text{if } \kappa = -1. \end{cases}$$

In the case $\kappa = 0$, every bilinear form on $\mathbb{R}^3$ of the form $\lambda \cdot x^0y^0 + x^1y^1 + x^2y^2$ induces the same Euclidean metric on $\mathbb{M}_0$, independent of $\lambda \in \mathbb{R}$. We choose $\lambda = 0$ and in the case $\kappa = 0$ we make the definition:

$$\frac{1}{\kappa}\langle x,y\rangle_\kappa := x^1y^1 + x^2y^2.$$

Lemma 4.5.3. *For every $\kappa \in \mathbb{R}$, the isometry group of $\mathbb{M}_\kappa$ contains the subgroup*

$$G_\kappa := \{\varphi \mid \varphi = A|_{\mathbb{M}_\kappa} \text{ where } A \in GL(3) \text{ with } \langle Ax,Ay\rangle_\kappa = \langle x,y\rangle_\kappa,$$
$$\frac{1}{\kappa}\langle Ax,Ay\rangle_\kappa = \frac{1}{\kappa}\langle x,y\rangle_\kappa\ \forall x,y \in \mathbb{R}^3 \text{ and } A(\mathbb{M}_\kappa) = \mathbb{M}_\kappa\}.$$

Remark 4.5.4. In the case $\kappa \neq 0$ the conditions $\langle Ax,Ay\rangle_\kappa = \langle x,y\rangle_\kappa$ and $\frac{1}{\kappa}\langle Ax,Ay\rangle_\kappa = \frac{1}{\kappa}\langle x,y\rangle_\kappa$

are of course equivalent and we could omit one of them. But in the case $\kappa = 0$ we need both of them.
From $\langle Ax, Ay\rangle_\kappa = \langle x, y\rangle_\kappa$ it already follows that $A(\hat{\mathbb{M}}_\kappa) = \hat{\mathbb{M}}_\kappa$. In the case $\kappa \leq 0$, A could possibly exchange the two connected components of $\hat{\mathbb{M}}_\kappa$. This is ruled out by the condition $A(\mathbb{M}_\kappa) = \mathbb{M}_\kappa$. In the case $\kappa > 0$ we could omit this condition.

Proof of the Lemma. Let $A \in G_\kappa$. Since $\varphi = A|_{\mathbb{M}_\kappa}$ is the restriction of the linear map A, we get that for $p \in \mathbb{M}_\kappa$ the differential $d\varphi(p) : T_p\mathbb{M}_\kappa \to T_{\varphi(p)}\mathbb{M}_\kappa$ also is the restriction of A,

$$d\varphi(p) = A|_{T_p\mathbb{M}_\kappa}.$$

Here, the tangent spaces of $\mathbb{M}_\kappa$ are viewed as subvector spaces of $\mathbb{R}^3$. Since A respects the bilinear form $\frac{1}{\kappa}\langle\cdot,\cdot\rangle_\kappa$, the differential $d\varphi(p)$ is a linear isometry for every $p \in \mathbb{M}_\kappa$. Thus φ is an isometry of Riemannian manifolds. ◻

Remark 4.5.5. Indeed, we have $\mathrm{Isom}(\mathbb{M}_\kappa) = G_\kappa$ but we will not need this fact.

For $\kappa = 1$ we have

$$G_\kappa \cong \{A \in GL(3) \,|\, \langle Ax, Ay\rangle = \langle x, y\rangle \quad \forall x, y \in \mathbb{R}^3\} = \mathrm{O}(3)$$

the group of orthogonal transformations. For $\kappa = -1$, G_κ is the group of **time-orientation preserving Lorentz transformations**.

In case $\kappa = 0$, we have:

$$\begin{aligned} G_0 &= \{A \in GL(3) \,|\, \langle Ax, Ay\rangle_0 = \langle x, y\rangle_0, \tfrac{1}{0}\langle Ax, Ay\rangle_0 = \tfrac{1}{0}\langle x, y\rangle_0 \forall x, y \in \mathbb{R}^3, A\mathbb{M}_0 = \mathbb{M}_0\} \\ &= \left\{ A = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline b^1 & \multicolumn{2}{c}{\multirow{2}{*}{B}} \\ b^2 & & \end{array} \right) \middle| \, b^1, b^2 \in \mathbb{R}, B \in \mathrm{O}(2) \right\} \end{aligned}$$

This holds true since for any $A \in G_0$,

$$x^0 y^0 = (Ax)^0 (Ay)^0 = (A^0_0 x^0 + A^0_1 x^1 + A^0_2 x^2)(A^0_0 y^0 + A^0_1 y^1 + A^0_2 y^2)$$

Thus

$$\begin{cases} \text{For } x = y = e_0\text{: } 1 = (A^0_0)^2 \quad \Rightarrow \quad A^0_0 = \pm 1 & \overset{\mathbf{A}(\mathbb{M}_0)=\mathbb{M}_0}{\Rightarrow} \quad A^0_0 = 1. \\ \text{For } x = y = e_1\text{: } 0 = (A^0_1)^2 & \Rightarrow \quad A^0_1 = 0. \\ \text{For } x = y = e_2\text{: } 0 = (A^0_2)^2 & \Rightarrow \quad A^0_2 = 0. \end{cases}$$

For $\hat{x}, \hat{y} \in \mathbb{R}^2$ we have with $x = (0, \hat{x})^\top$ and $y = (0, \hat{y})^\top$:

$$\langle \hat{x}, \hat{y}\rangle_{\mathbb{R}^2} = \tfrac{1}{0}\langle x, y\rangle_0 = \tfrac{1}{0}\langle Ax, Ay\rangle_0 = \tfrac{1}{0}\left\langle \begin{pmatrix} 0 \\ B\hat{x} \end{pmatrix}, \begin{pmatrix} 0 \\ B\hat{y} \end{pmatrix} \right\rangle_0 = \langle B\hat{x}, B\hat{y}\rangle_{\mathbb{R}^2}.$$

Hence $B \in \mathrm{O}(2)$ and therefore

$$G_0 \subset \left\{ \mathrm{A} = \left(\begin{array}{c|c} 1 & 0 \\ \hline b & \mathrm{B} \end{array} \right) \middle| b \in \mathbb{R}^2, \mathrm{B} \in \mathrm{O}(2) \right\}.$$

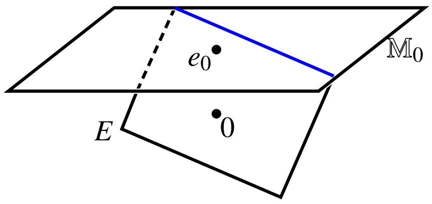
The other inclusion "'$\supset$'" follows by a direct computation.
We now analyze, how G_0 acts, if we identify $\mathbb{M}_0$ with $\mathbb{R}^2$.

$$\begin{array}{ccccccc} \mathbb{R}^2 & \to & \mathbb{M}_0 & \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ b & B \end{smallmatrix}\right)} & \mathbb{M}_0 & \to & \mathbb{R}^2 \\ \hat{x} & \mapsto & \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} & \mapsto & \begin{pmatrix} 1 & 0 \\ b & B \end{pmatrix} \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} = \begin{pmatrix} 1 \\ b + B\hat{x} \end{pmatrix} & \mapsto & b + B\hat{x} \end{array}$$

Hence the group G_0 acts like the group of Euclidean motions.
As seen in the last paragraph, the geodesics in $\mathbb{M}_\kappa$, viewed as a set of points, equal the sets of the form

$$\mathbb{M}_\kappa \cap E,$$

where $E \subset \mathbb{R}^3$ is a two-dimensional subvector space.



Lemma 4.5.6. *The geodesics parametrized by arc-length $\gamma : \mathbb{R} \to \mathbb{M}_\kappa$ with $\gamma(0) = e_0$ are then given by*

$$\gamma(r) = \begin{pmatrix} \mathfrak{c}_\kappa(r) \\ \mathfrak{s}_\kappa(r) \cdot \sin(\varphi) \\ \mathfrak{s}_\kappa(r) \cdot \cos(\varphi) \end{pmatrix}$$

where $\varphi \in \mathbb{R}$ is fixed.

Proof. The curve γ stays in $\hat{\mathbb{M}}_\kappa$ because

$$\langle \gamma(r), \gamma(r) \rangle_\kappa = \mathfrak{c}_\kappa(r)^2 + \kappa\big(\mathfrak{s}_\kappa(r)^2 \sin(\varphi)^2 + \mathfrak{s}_\kappa^2 \cos(\varphi)^2\big) = \mathfrak{c}_\kappa(r)^2 + \kappa\, \mathfrak{s}_\kappa(r)^2 = 1.$$

Since $\gamma(0) = e_0 \in \mathbb{M}_\kappa$ and γ is continuous, γ remains in $\mathbb{M}_\kappa$. Moreover, γ lies in the plane E, which is spanned by e_0 and $(0, \sin(\varphi), \cos(\varphi))^\top$. Hence γ is contained in $\mathbb{M}_\kappa \cap E$. In addition, γ is parametrized by arc-length because

$$\begin{aligned} \tfrac{1}{\kappa} \langle \dot{\gamma}(r), \dot{\gamma}(r) \rangle_\kappa &= \tfrac{1}{\kappa} \left\langle \begin{pmatrix} -\kappa\, \mathfrak{s}_\kappa(r) \\ \mathfrak{c}_\kappa(r) \sin(\varphi) \\ \mathfrak{c}_\kappa(r) \cos(\varphi) \end{pmatrix}, \begin{pmatrix} -\kappa\, \mathfrak{s}_\kappa(r) \\ \mathfrak{c}_\kappa(r) \sin(\varphi) \\ \mathfrak{c}_\kappa(r) \cos(\varphi) \end{pmatrix} \right\rangle_\kappa \\ &= \tfrac{1}{\kappa}\Big(\kappa^2 \mathfrak{s}_\kappa(r)^2 + \kappa\big(\mathfrak{c}_\kappa(r)^2 \sin(\varphi)^2 + \mathfrak{c}_\kappa(r)^2 \cos(\varphi)^2\big)\Big) \\ &= \kappa\, \mathfrak{s}_\kappa(r)^2 + \mathfrak{c}_\kappa(r)^2 \\ &= 1. \end{aligned}$$

□

The generalized sine and cosine functions allow us to explicitly write down many isometries in G_κ.

Example 4.5.7. Rotations about the e_0-Axis are isometries,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix} \in G_\kappa$$

for any φ and any $\kappa \in \mathbb{R}$. Using $\kappa\,\mathfrak{s}_\kappa^2 + \mathfrak{c}_\kappa^2 = 1$ one easily checks that

$$\begin{pmatrix} \mathfrak{c}_\kappa(r) & -\kappa\,\mathfrak{s}_\kappa(r) & 0 \\ \mathfrak{s}_\kappa(r) & \mathfrak{c}_\kappa(r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G_\kappa$$

for all $r \in \mathbb{R}$. In the case $\kappa = 1$ this is a rotation about the e_2-axis. For $\kappa = 0$ this is the identity, hence uninteresting. In the case $\kappa = -1$ such isometries are called *Lorentz boosts*. Similarly, one sees that

$$L_r := \begin{pmatrix} \mathfrak{c}_\kappa(r) & 0 & \kappa\,\mathfrak{s}_\kappa(r) \\ 0 & 1 & 0 \\ \mathfrak{s}_\kappa(r) & 0 & -\mathfrak{c}_\kappa(r) \end{pmatrix} \in G_\kappa.$$

Before using these isometries we observe that

$$L_r e_0 = \begin{pmatrix} \mathfrak{c}_\kappa(r) \\ 0 \\ \mathfrak{s}_\kappa(r) \end{pmatrix}$$

and

$$\begin{aligned} L_r \begin{pmatrix} \mathfrak{c}_\kappa(r) \\ 0 \\ \mathfrak{s}_\kappa(r) \end{pmatrix} &= \begin{pmatrix} \mathfrak{c}_\kappa(r) & 0 & \kappa\mathfrak{s}_\kappa(r) \\ 0 & 1 & 0 \\ \mathfrak{s}_\kappa(r) & 0 & -\mathfrak{c}_\kappa(r) \end{pmatrix} \begin{pmatrix} \mathfrak{c}_\kappa(r) \\ 0 \\ \mathfrak{s}_\kappa(r) \end{pmatrix} \\ &= \begin{pmatrix} \mathfrak{c}_\kappa(r)^2 + \kappa\mathfrak{s}_\kappa(r)^2 \\ 0 \\ \mathfrak{s}_\kappa(r)\mathfrak{c}_\kappa(r) - \mathfrak{c}_\kappa(r)\mathfrak{s}_\kappa(r) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_0. \end{aligned}$$

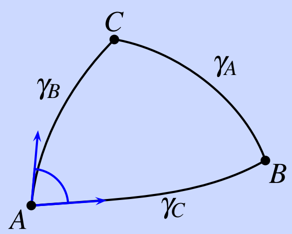
Thus L_r interchanges the points e_0 and $(\mathfrak{c}_\kappa(r), 0, \mathfrak{s}_\kappa(r))^\top$.

Definition 4.5.8.

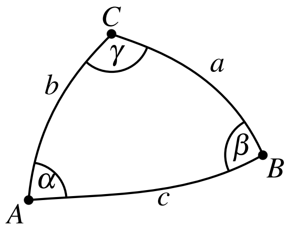
Let M be a Riemannian manifold. A **geodesic triangle** is a 6-tupel

$$(A, B, C, \gamma_A, \gamma_B, \gamma_C),$$

where $A, B, C \in M$ are pairwise disjoint points, γ_A, γ_B and γ_C geodesic segments with endpoints B and C, C and A or A and B, respectively.



The points A, B and C are the **vertices**, the geodesic segments γ_A, γ_B and γ_C are the **sides** of the geodesic triangle. The **angle** at a vertex is defined to be the angle of the tangent vectors of the sides at that vertex.



Let $(A,B,C,\gamma_A,\gamma_B,\gamma_C)$ a geodesic triangle in $\mathbb{M}_\kappa$. The sides have the lengths a, b and c, respectively, and the angles are denoted by α, β and γ, respectively.

Here the length of a geodesic segment γ is defined as the length of the parameter interval $\times$ the norm of $\dot{\gamma}$, which is constant. A more general definition of the length of a differentiable curve in a Riemannian manifold will be introduced later. Since the isometry group of $\mathbb{M}_\kappa$ acts transitively, we can assume w.l.o.g. that

$$A = e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Applying an isometry of the form

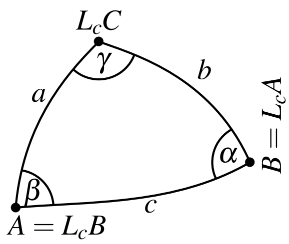
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) \\ 0 & \sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

(rotation about the e_0-axis) we can rotate B in the e_0-e_2-plane without moving $A = e_0$. The formula from Lemma 4.5.6 for the geodesic γ_C with $\varphi = 0$ and $r = c$ then tells us

$$B = \begin{pmatrix} \mathfrak{c}_\kappa(c) \\ 0 \\ \mathfrak{s}_\kappa(c) \end{pmatrix}$$

Lemma 4.5.6 for the geodesic γ_B with $\varphi = \alpha$ and $r = b$ yields

$$C = \begin{pmatrix} \mathfrak{c}_\kappa(b) \\ \mathfrak{s}_\kappa(b)\sin(\alpha) \\ \mathfrak{s}_\kappa(b)\cos(\alpha) \end{pmatrix}.$$



Hence the isometry L_c interchanges the points A and B and we obtain a new geodesic triangle. On the one hand one can compute L_cC similarly as C itself and one obtains

$$L_cC = \begin{pmatrix} \mathfrak{c}_\kappa(a) \\ \mathfrak{s}_\kappa(a)\sin(\beta) \\ \mathfrak{s}_\kappa(a)\cos(\beta) \end{pmatrix}.$$

On the other hand

$$L_cC = \begin{pmatrix} \mathfrak{c}_\kappa(c) & 0 & \kappa\mathfrak{s}_\kappa(c) \\ 0 & 1 & 0 \\ \mathfrak{s}_\kappa(c) & 0 & -\mathfrak{c}_\kappa(c) \end{pmatrix} \begin{pmatrix} \mathfrak{c}_\kappa(b) \\ \mathfrak{s}_\kappa(b)\sin(\alpha) \\ \mathfrak{s}_\kappa(b)\cos(\alpha) \end{pmatrix} = \begin{pmatrix} \mathfrak{c}_\kappa(c)\mathfrak{c}_\kappa(b) + \kappa\mathfrak{s}_\kappa(c)\mathfrak{s}_\kappa(b)\cos(\alpha) \\ \mathfrak{s}_\kappa(b)\sin(\alpha) \\ \mathfrak{s}_\kappa(c)\mathfrak{c}_\kappa(b) - \mathfrak{c}_\kappa(c)\mathfrak{s}_\kappa(b)\cos(\alpha) \end{pmatrix}$$

Thus we obtain the equations:

$$\begin{aligned} \mathfrak{c}_\kappa(a) &= \mathfrak{c}_\kappa(c)\mathfrak{c}_\kappa(b) + \kappa\,\mathfrak{s}_\kappa(c)\mathfrak{s}_\kappa(b)\cos(\alpha) && \text{(Law of Cosines)} \quad (1) \\ \mathfrak{s}_\kappa(a)\sin(\beta) &= \mathfrak{s}_\kappa(b)\sin(\alpha) \\ \frac{\mathfrak{s}_\kappa(a)}{\sin(\alpha)} &= \frac{\mathfrak{s}_\kappa(b)}{\sin(\beta)} && \text{(Law of Sines)} \quad (2) \\ \mathfrak{s}_\kappa(a)\cos(\beta) &= \mathfrak{s}_\kappa(c)\mathfrak{c}_\kappa(b) - \mathfrak{c}_\kappa(c)\mathfrak{s}_\kappa(b)\cos(\alpha) && (3) \end{aligned}$$

Equation (3) with the roles of B and C interchanged yields

$$\mathfrak{s}_\kappa(a)\cos(\gamma) = \mathfrak{s}_\kappa(b)\mathfrak{c}_\kappa(c) - \mathfrak{c}_\kappa(b)\mathfrak{s}_\kappa(c)\cos(\alpha) \tag{4}$$

Equation (3) $\cdot\cos(\alpha)$ – (2) $\cdot\sin(\alpha)^2\cdot\sin(\beta)$ then yields

$$\begin{aligned} &\mathfrak{s}_\kappa(a)\cos(\beta)\cos(\alpha) - \mathfrak{s}_\kappa(a)\sin(\beta)\sin(\alpha) \\ &\qquad = \mathfrak{s}_\kappa(c)\mathfrak{c}_\kappa(b)\cos(\alpha) - \mathfrak{c}_\kappa(c)\mathfrak{s}_\kappa(b)\cos(\alpha)^2 - \mathfrak{s}_\kappa(b)\sin(\alpha)^2 \end{aligned}$$

Hence

$$\begin{aligned} &\mathfrak{s}_\kappa(a)(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\ &\overset{(4)}{=} \mathfrak{s}_\kappa(b)\mathfrak{c}_\kappa(c) - \mathfrak{s}_\kappa(a)\cos(\gamma) - \mathfrak{s}_\kappa(b)\mathfrak{c}_\kappa(c)\cos(\alpha)^2 - \mathfrak{s}_\kappa(b)\sin(\alpha)^2 \\ &= \mathfrak{s}_\kappa(b)\mathfrak{c}_\kappa(c)\sin(\alpha)^2 - \mathfrak{s}_\kappa(a)\cos(\gamma) - \mathfrak{s}_\kappa(b)\sin(\alpha)^2 \\ &\overset{(2)}{=} \mathfrak{s}_\kappa(a)\mathfrak{c}_\kappa(c)\sin(\alpha)\sin(\beta) - \mathfrak{s}_\kappa(a)\cos(\gamma) - \mathfrak{s}_\kappa(a)\sin(\alpha)\sin(\beta) \end{aligned}$$

and thus $\cos(\alpha)\cos(\beta) = \mathfrak{c}_\kappa(c)\sin(\alpha)\sin(\beta) - \cos(\gamma)$, hence

$$\cos(\gamma) = \mathfrak{c}_\kappa(c)\sin(\alpha)\sin(\beta) - \cos(\alpha)\cos(\beta) \qquad \text{(Cosine Rule for Angles).}$$

We have proved

Theorem 4.5.9. *Let $\kappa \in \mathbb{R}$. For a geodesic triangle in $\mathbb{M}_\kappa$ with the side lengths a, b, c and the angles α, β, γ we have*

(1) Law of Sines:
$$\frac{\mathfrak{s}_\kappa(a)}{\sin(\alpha)} = \frac{\mathfrak{s}_\kappa(b)}{\sin(\beta)} = \frac{\mathfrak{s}_\kappa(c)}{\sin(\gamma)}.$$

(2) Law of Cosines (Cosine Rule for Sides):

$$\begin{aligned}
\mathfrak{c}_\kappa(a) &= \mathfrak{c}_\kappa(b)\mathfrak{c}_\kappa(c) + \kappa\mathfrak{s}_\kappa(b)\mathfrak{s}_\kappa(c)\cdot\cos(\alpha),\\
\mathfrak{c}_\kappa(b) &= \mathfrak{c}_\kappa(a)\mathfrak{c}_\kappa(c) + \kappa\mathfrak{s}_\kappa(a)\mathfrak{s}_\kappa(c)\cdot\cos(\beta),\\
\mathfrak{c}_\kappa(c) &= \mathfrak{c}_\kappa(a)\mathfrak{c}_\kappa(b) + \kappa\mathfrak{s}_\kappa(a)\mathfrak{s}_\kappa(b)\cdot\cos(\gamma).
\end{aligned}$$

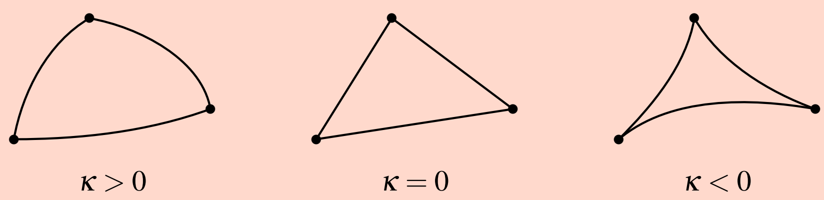
(3) Cosine Rule for Angles:

$$\begin{aligned}
\cos(\alpha) &= \mathfrak{c}_\kappa(a)\sin(\beta)\sin(\gamma) - \cos(\beta)\cos(\gamma),\\
\cos(\beta) &= \mathfrak{c}_\kappa(b)\sin(\alpha)\sin(\gamma) - \cos(\alpha)\cos(\gamma),\\
\cos(\gamma) &= \mathfrak{c}_\kappa(c)\sin(\alpha)\sin(\beta) - \cos(\alpha)\cos(\beta).
\end{aligned}$$

Now we analyze the sum of angles in the model space of constant curvature.

Theorem 4.5.10. *Let $\kappa\in\mathbb{R}$. For the sum of angles $\alpha+\beta+\gamma$ of a geodesic triangle in $\mathbb{M}_\kappa$ with the inner angles $0<\alpha,\beta,\gamma<\pi$ we have*

$$\alpha+\beta+\gamma\begin{cases} >\pi, & \text{if } \kappa>0\\ =\pi, & \text{if } \kappa=0\\ <\pi, & \text{if } \kappa<0\end{cases}$$



Proof. W.l.o.g. we assume that $\alpha\ge\beta$. For this proof we will use the notation "$\lesseqgtr$" for "$<$", if $\kappa>0$, for "$=$", if $\kappa=0$, and for "$>$", if $\kappa<0$. We have $-\kappa\lesseqgtr 0$, for instance.
If is $\kappa>0$, then $\mathbb{M}_\kappa$ is the sphere of radius $\frac{1}{\sqrt{\kappa}}$. Thus in this case the side lengths have to be $<\frac{2\pi}{\sqrt{\kappa}}$. In the case $\kappa\le 0$, we do not have any bounds on the side lengths. We use the convention $\frac{1}{\sqrt{\kappa}}=\infty$, if $\kappa\le 0$. With this convention we have in all cases

$$\mathfrak{c}_\kappa\lesseqgtr 1$$

in the interval $(0,\frac{2\pi}{\sqrt{\kappa}})$. Since sin is positive on $(0,\pi)$ the Cosine Rule for Angles yields

$$\cos(\alpha) = \mathfrak{c}_\kappa(a)\sin(\beta)\sin(\gamma) - \cos(\beta)\cos(\gamma)$$

$$\begin{aligned}
&\lesseqgtr \sin(\beta)\sin(\gamma)-\cos(\beta)\cos(\gamma) \\
&= -\cos(\beta+\gamma) \\
&= \cos(\pi-(\beta+\gamma)) \\
&= \cos(\beta+\gamma-\pi).
\end{aligned}$$

Since $0<\beta,\gamma<\pi$ we have $-\pi<\pi-(\beta+\gamma)<\pi$.

First case: $\pi-(\beta+\gamma)\geq 0$.
Since cos is strictly monotonically decreasing on $[0,\pi]$, the relation $\cos(\alpha)\lesseqgtr\cos(\pi-(\beta+\gamma))$ yields $\pi-(\beta+\gamma)\lesseqgtr\alpha$ and thus $\pi\lesseqgtr\alpha+\beta+\gamma$. This is what we wanted to show.

Second Case: $\pi-(\beta+\gamma)<0$.
If $\kappa>0$, we obtain $\pi<\beta+\gamma<\alpha+\beta+\gamma$ directly, which proves the claim. Hence, let $\kappa\leq 0$. Then from $\cos(\alpha)\geq\cos(\beta+\gamma-\pi)$ we may deduce that $\alpha\leq\beta+\gamma-\pi$. Since $\alpha\geq\beta$ and $\gamma<\pi$ this implies

$$\alpha<\alpha+\pi-\pi=\alpha,$$

giving a contradiction. □

Remark 4.5.11. Since the inner angles are $<\pi$, we always have for the sum of angles in a geodesic triangle $\alpha+\beta+\gamma<3\pi$. It is easy to see that for $\mathbb{M}_\kappa$ with $\kappa>0$ the sum of angles of a geodesic triangle can take all values in $(\pi,3\pi)$. For $\mathbb{M}_\kappa$ with $\kappa<0$ all values of the interval $(0,\pi)$ occur.

5 Riemannian Geometry

From now on we concentrate on Riemannian geometry, that is, on semi-Riemannian manifolds whose metric is positive definite and hence defines a Euclidean scalar product on each tangent space. One special feature of the Riemannian case is that each connected Riemannian manifold naturally becomes a metric space.

5.1 The Riemannian distance function

General Assumption. Let M be a connected Riemannian manifold and let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric.

Definition 5.1.1. Let $c : [a,b] \to M$ be a continuous piecewise C^1-curve. Then we call

$$L[c] := \int_a^b \|\dot{c}(t)\| \, dt$$

the **length** of c.

Remark 5.1.2. The length does not depend on the parametrization of the curve. Namely, if $\varphi : [a,b] \to [\alpha, \beta]$ is a parameter transformation, then we have

$$\begin{aligned} L[c \circ \varphi] &= \int_a^b \left\| \frac{d}{dt}(c \circ \varphi)(t) \right\| dt \\ &= \int_a^b \|\dot{c}(\varphi(t))\| \cdot |\dot{\varphi}(t)| \, dt \\ \underset{s=\varphi(t)}{\text{Substitution}} \quad &= \int_\alpha^\beta \|\dot{c}(s)\| \, ds \\ &= L[c]. \end{aligned}$$

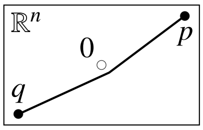
Definition 5.1.3. Let $p,q \in M$. Then we call

$$d(p,q) = \inf\{L[c] \,|\, c : [a,b] \to M \text{ piecewise } C^1\text{-curve with } c(a) = p, c(b) = q\}$$

the **Riemannian distance** of p and q.

Remark 5.1.4. The infimum is, in general, not a minimum. In other words, there need not exist a shortest curve connecting p and q.

Example 5.1.5. $M = \mathbb{R}^n \setminus \{0\}$ and $p = -q$. We have $d(p,q) = 2\|p\|$, but every curve c from p to q has length $L[c] > 2\|p\|$.



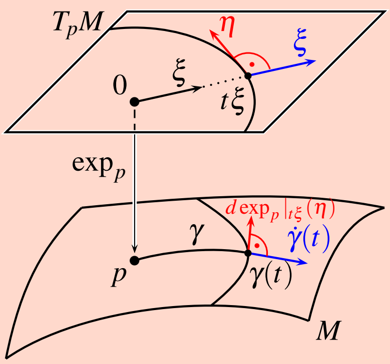
Theorem 5.1.6 (Gauß lemma). *Let $p \in M$ and $\xi \in T_pM$. The geodesic $\gamma(t) = \exp_p(t\xi)$ is supposed to be defined on $[0,b]$.*
Then $\exp_p$ is defined on an open neighborhood of $\{t\xi \,|\, 0 \le t \le b\} \subset T_pM$ and we have

(i) $d\exp_p|_{t\xi}(\xi) = \dot{\gamma}(t)$.

(ii) *For $\eta \in T_{t\xi}T_pM \cong T_pM$ we have*

$$\left\langle d\exp_p|_{t\xi}(\eta), \dot{\gamma}(t) \right\rangle = \langle \eta, \xi \rangle .$$

In particular, $d\exp_p|_{t\xi}(\eta) \perp \dot{\gamma}(t)$, if $\eta \perp \xi$.



Proof. (i) We compute $d\exp_p|_{t\xi}(\xi) = \frac{d}{ds}\exp_p(t\xi + s\xi)|_{s=0} = \frac{d}{ds}\gamma(t+s)|_{s=0} = \dot\gamma(t)$.
(ii) By (i) it suffices to consider the case $\eta \perp \xi$. Let J be the Jacobi field along γ with $J(0) = 0$ and $\frac{\nabla}{dt}J(0) = \eta$. Proposition 3.4.13 yields

$$d\exp_p|_{t\xi}(\eta) = \frac{J(t)}{t} \quad \text{for } t > 0.$$

Since both J and $\frac{\nabla}{dt}J$ are perpendicular to $\dot\gamma$ at $t = 0$, this holds for all t. We conclude

$$\left\langle d\exp_p|_{t\xi}(\eta), \dot\gamma(t)\right\rangle = \left\langle \frac{J(t)}{t}, \dot\gamma(t)\right\rangle = 0 = \langle \eta, \xi\rangle. \qquad \square$$

We now consider the diffeomorphism

$$\Phi : T_pM \setminus \{0\} \longrightarrow (0,\infty) \times S^{n-1}, \quad x = t\cdot y \mapsto (t,y) = (\|x\|, \frac{x}{\|x\|}),$$

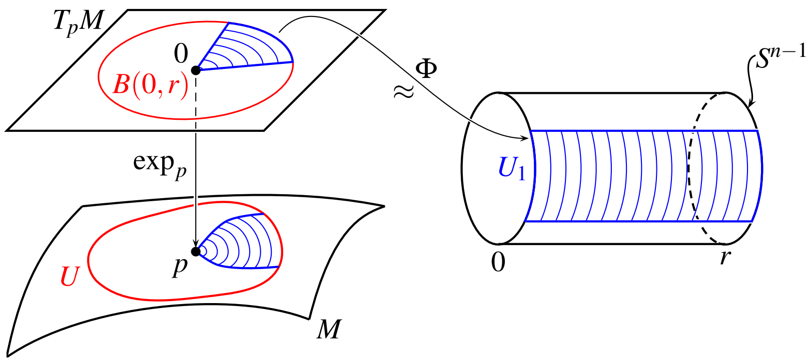
where $S^{n-1} \subset T_pM$ is the unit sphere in the tangent space. There exists an $r > 0$, such that $\exp_p$ maps $B(0,r) \subset T_pM$ diffeomorphically onto a neighborhood U of p in M. Then the map

$$(0,r) \times S^{n-1} \to U \setminus \{p\}, \quad (t,y) \mapsto \exp_p(ty),$$

is a diffeomorphism. Now let $y^2,\dots,y^n$ be local coordinates on an open set $U_1 \subset S^{n-1}$. Then the coordinates given by the diffeomorphism

$$\exp_p(ty) \mapsto (t, y^2,\dots,y^n),$$

are called *geodesic polar coordinates*.



The Gauß lemma says that in such coordinates the Riemannian metric takes the form

$$(g_{ij}) = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array}\right)$$

Corollary 5.1.7. *Let $r > 0$ so small that $\exp_p|_{\bar{B}(0,r)}$ is a diffeomorphism onto its image. Let $c : [a,b] \to M$ be a piecewise C^1-curve with $c(a) = p$ and $c(b) \notin \exp_p(B(0,r))$. Then $L[c] \geq r$.*

Proof. Let $\beta \in (a,b)$ be minimal such that $c(\beta) \in \partial \exp_p(B(0,r)) = \exp_p(S^{n-1}(r))$. Let $\alpha \in [a,\beta)$ maximal such that $c(\alpha) = p$. Now it is ensured that for $\tau \in (\alpha,\beta)$ the curve $c(\tau)$ lies in $\exp_p(B(0,r)) \setminus \{p\}$. For $\tau \in (\alpha,\beta]$ we write

$$\tilde{c}(\tau) \;:=\; {\exp_p}^{-1}(c(\tau)) \;=\; t(\tau) \cdot y(\tau)$$

where $t(\tau) := \|\tilde{c}(\tau)\| \in (0,r]$ and $y(\tau) := \frac{\tilde{c}(\tau)}{\|\tilde{c}(\tau)\|} \in S^{n-1}$. Let $\tilde{\xi}$ be the unit vector field on $T_pM \setminus \{0\}$ which corresponds to $\tilde{\xi}(x) = \frac{x}{\|x\|}$ under the canonical isomorphism $T_xT_pM \cong T_pM$. Using the diffeomorphism $\exp_p$ we transport this vector field to the manifold, that is, on $\exp_p(\bar{B}(0,r)) \setminus \{p\}$ we set

$$\xi(q) := d\exp_p\left(\tilde{\xi}\left({\exp_p}^{-1}(q)\right)\right).$$

The first part of the Gauß lemma implies $\|\xi\| \equiv 1$. Because of

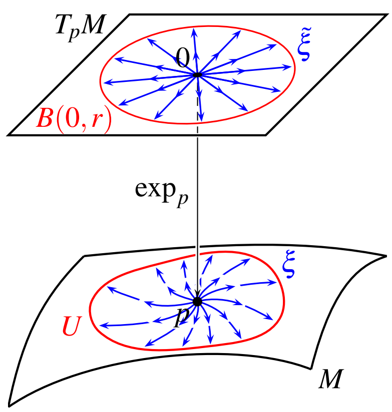
$$\frac{d}{d\tau}\tilde{c}(\tau) = \frac{dt}{d\tau} \cdot \underbrace{y(\tau)}_{=\tilde{\xi}(\tilde{c}(\tau))} + t(\tau) \cdot \underbrace{\frac{dy}{d\tau}(\tau)}_{\perp \tilde{\xi}(\tilde{c}(\tau))}$$

part (ii) of the Gauß lemma yields

$$\langle \xi(c(\tau)), \dot{c}(\tau) \rangle = \left\langle d\exp_p\left(\tilde{\xi}(\tilde{c}(\tau))\right), d\exp_p\left(\dot{\tilde{c}}(\tau)\right) \right\rangle = \left\langle \tilde{\xi}(\tilde{c}(\tau)), \dot{\tilde{c}}(\tau) \right\rangle = \frac{dt}{d\tau}.$$

Thus we get

$$\begin{aligned}
L[c] &\geq L[c|_{[\alpha,\beta]}] \\
&= \int_\alpha^\beta \|\dot{c}(\tau)\|\, d\tau \\
\text{Cauchy-Schwarz inequality} \quad &\geq \int_\alpha^\beta \langle \xi(c(\tau)), \dot{c}(\tau) \rangle\, d\tau \\
&= \int_\alpha^\beta \frac{dt}{d\tau}\, d\tau \\
&= t(\beta) - t(\alpha) \\
&= r - 0 = r.
\end{aligned}$$

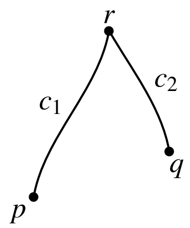


□

Theorem 5.1.8. *(M,d) is a metric space.*

Proof. a) Obviously we have $d(p,q) \geq 0$ and $d(p,p) = 0$ because the constant curve has length 0. Now let $p \neq q$. We have to show $d(p,q) > 0$. Choose $r > 0$ such that $\exp_p|_{B(0,r)}$ is a diffeomorphism and $q \notin \exp_p(B(0,r))$. Then by Corollary 5.1.7 every curve from p to q has length r at least. Hence $d(p,q) \geq r > 0$.
b) Symmetry $d(p,q) = d(q,p)$ is clear. Simply traverse the curves in the opposite direction.
c) It remains to show the triangle inequality $d(p,q) \leq d(p,r) + d(r,q)$.

Let $\varepsilon > 0$. Choose a continuous piecewise C^1-curves c_1 from p to r with $L[c_1] \leq d(p,r) + \varepsilon$ and c_2 from r to q with $L[c_2] \leq d(r,q) + \varepsilon$. Now concatenate c_1 and c_2 to a continuous piecewise C^1-curve c from p to q. Then we have



$$d(p,q) \leq L[c] = L[c_1] + L[c_2] \leq d(p,r) + \varepsilon + d(r,q) + \varepsilon.$$

Taking the limit $\varepsilon \searrow 0$ yields the assertion. □

Notation 5.1.9. For $p \in M$ and $r > 0$ set

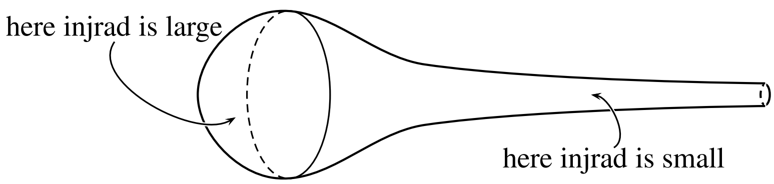
$$B(p,r) := \{q \in M \,|\, d(p,q) < r\},$$
$$\bar{B}(p,r) := \{q \in M \,|\, d(p,q) \leq r\},$$
$$S(p,r) := \{q \in M \,|\, d(p,q) = r\}.$$

Definition 5.1.10. For $p \in M$

$$\mathrm{injrad}(p) := \sup\{r \,|\, \exp_p|_{B(0,r)} : B(0,r) \to \exp_p\big(B(0,r)\big) \text{ is diffeomorphism}\}$$

is called the **injectivity radius** of M at p.

Example 5.1.11. The injectivity radius depends on p.



Remark 5.1.12. For $0 < r < \mathrm{injrad}(p)$ we have $\exp_p(B(0,r)) = B(p,r)$. Namely:

"$\subset$": Let $q = \exp_p(\xi)$ with $\|\xi\| < r$. Then $t \mapsto \exp_p(t\xi)$, $t \in [0,1]$, is a curve from p to q with length $\|\xi\| < r$. Hence $d(q,p) < r$, i.e., $q \in B(p,r)$.

"$\supset$": Corollary 5.1.7.

Theorem 5.1.13. *The metric d induces the original topology on M.*

Proof. For the moment we denote the open subsets w.r.t. d of M as "d-open". We have to show: d-open $=$ open.
a) *Claim:* Every d-open set is open.
Let $U \subset M$ be d-open. For every $p \in U$ there exists a $r(p) > 0$, such that $B(p,r(p)) \subset U$. W.l.o.g. let $r(p) < \mathrm{injrad}(p)$. Then $B(p,r(p)) = \exp_p(\underbrace{B(0,r(p))}_{\text{open in } T_pM})$ is the diffeomorphic image of an open subset of T_pM. Hence it is open itself. Therefore $U = \bigcup_{p\in M} B\big(p,r(p)\big)$ is the union of open subsets of M and thus open.
b) *Claim:* Every open set is d-open. The proof is similar. □

Corollary 5.1.14. *The map $d : M \times M \to \mathbb{R}$ is continuous.*

Remark 5.1.15. If $\Phi \in \mathrm{Isom}(M)$. Then we have $L[\Phi \circ c] = L[c]$ and thus also $d\big(\Phi(p),\Phi(q)\big) = d(p,q)$.

Recall that $E[c] = \frac{1}{2}\int_a^b \|\dot{c}(t)\|^2\, dt$ is the *energy* of c.

Proposition 5.1.16. *Let M be a Riemannian manifold and $c : [a,b] \to M$ be a continuous, piecewise C^1-curve. Then we have*

$$L[c]^2 \le 2(b-a)\cdot E[c].$$

Equality holds if and only if c is parametrized proportionally to arc-length.

Proof. With the Cauchy-Schwarz inequality for the L^2-scalar product we obtain:

$$L[c]^2 = \left(\int_a^b \|\dot{c}(t)\|\cdot 1\, dt\right)^2 \le \int_a^b \|\dot{c}(t)\|^2\, dt \cdot \int_a^b 1^2\, dt = 2E[c]\cdot(b-a).$$

Equality holds if and only if $\|\dot{c}\|$ and 1are linearly dependent (as functions) . This means that $\|\dot{c}\|$ is constant, i.e., that c is parametrized proportionally to arc-length. □

Corollary 5.1.17. *A curve c minimizes the energy in the set of all continuous piecewise C^1-curves connecting p and q if and only if c minimizes the length and is parametrized proportionally to arc-length.*

Remark 5.1.18. By Corollary 2.6.10 energy minimizing curves are geodesics.

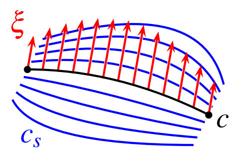
Corollary 5.1.19. *Every shortest curve c from p to q with $\dot{c}(t) \neq 0$ for all t is a geodesic up to parametrization. It is a geodesic if and only if it is parametrized proportionally to arc-length.*

Caution! The converse is not true. Not every geodesic is a shortest curve connecting its end-points.

Example 5.1.20. Great circles on S^n are geodesics connecting points to themselves. But the only shortest curves connecting a point to itself are constant curves which have length 0.

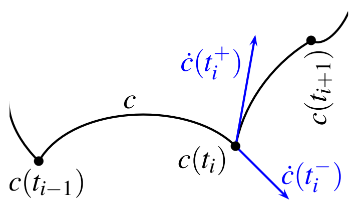
Definition 5.1.21. A geodesic $\gamma\colon [a,b] \to M$ with $L[\gamma] = d\big(\gamma(a), \gamma(b)\big)$ is called **minimal**.

5.2 The second variation of the energy



We recall: If c_s is a C^2-variation of $c\colon [a,b] \to M$ with variational field ξ, then the first variation formula (Theorem 2.6.5) says:

$$\tfrac{d}{ds} E[c_s]|_{s=0} = -\int_a^b \left\langle \xi, \tfrac{\nabla}{dt}\dot{c} \right\rangle dt + \langle \xi, \dot{c} \rangle |_a^b.$$



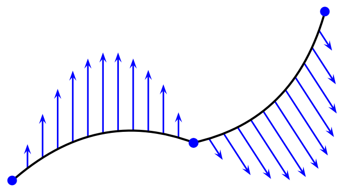
If c_s is continuous and only piecewise C^2, that is, there exists a partition $a = t_0 < t_1 < \cdots < t_N = b$, such that $(s,t) \mapsto c_s(t)$ is continuous on $(-\varepsilon, \varepsilon) \times [a,b]$ and C^2 on $(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]$, then we have

$$\frac{d}{ds} E[c_s]|_{s=0} \quad = \quad \frac{d}{ds} \sum_{i=1}^{N} E[c_s|_{[t_{i-1}, t_i]}]|_{s=0}$$

$$= \sum_{i=1}^{N}\left(-\int_{t_{i-1}}^{t_i}\left\langle \xi, \frac{\nabla}{dt}\dot c\right\rangle dt + \langle \xi(t_i), \dot c(t_i^-)\rangle - \langle \xi(t_{i-1}), \dot c(t_i^+)\rangle\right)$$
$$= -\int_a^b \left\langle \xi, \frac{\nabla}{dt}\dot c\right\rangle dt + \langle \xi(b), \dot c(b^-)\rangle - \langle \xi(a), \dot c(a^+)\rangle + \sum_{i=1}^{N}\langle \xi(t_i), \dot c(t_i^-) - \dot c(t_i^+)\rangle$$

Question: If c is a continuous and only piecewise C^2-curve with $\frac{d}{ds}E[c_s]|_{s=0} = 0$ for all continuous, piecewise C^2-variations c_s with fixed endpoints, does c then have to be a geodesic (and thus in particular C^∞)?

Answer: Yes. *Namely:* First of all, consider only such variations with $\xi(t_i) = 0$ for all $i \in \{0, \dots, N\}$, then it follows as in the proof of Corollary 2.6.10 that $\frac{\nabla}{dt}\dot c \equiv 0$ on every $[t_{i-1}, t_i]$ for $i = 1, \dots, N$.



$\Rightarrow$ The curve c is piecewise a geodesic.

If $\dot c(t_i^-) \neq \dot c(t_i^+)$ for an $i \in \{1, \dots, N-1\}$ then we can choose an $\eta \in T_{c(t_i)}M$ with

$$\langle \eta, \dot c(t_i^-) - \dot c(t_i^+)\rangle > 0.$$

Now continue η via parallel transport along c. Choose a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(t_i) = 1$ and $\varphi \equiv 0$ on $\mathbb{R} \setminus (t_{i-1}, t_{i+1})$. Set $\xi(t) := \varphi(t)\eta(t)$. Then we have $\xi(a) = \xi(b) = 0$ and thus

$$0 = \frac{d}{ds}E[c_s]|_{s=0} = \langle \xi(t_i), \dot c(t_i^-) - \dot c(t_i^+)\rangle = \langle \eta, \dot c(t_i^-) - \dot c(t_i^+)\rangle > 0.$$

This is a contradiction. We summarize:

Theorem 5.2.1. *Let M be a semi-Riemannian manifold and $c : [a,b] \to M$ be a continuous, piecewise C^2-curve. Then for every continuous piecewise C^2-variation c_s of c with variational field ξ we have*

$$\frac{d}{ds}E[c_s]\Big|_{s=0} = -\int_a^b \left\langle \xi, \frac{\nabla}{dt}\dot c\right\rangle dt + \langle \xi, \dot c\rangle\Big|_a^b + \sum_{i=1}^{N-1}\langle \xi(t_i), \dot c(t_i^-) - \dot c(t_i^+)\rangle,$$

where $a = t_0 < t_1 < \dots < t_N = b$ is a partition for which both c and c_s are C^2 on the intervals $[t_{i-1}, t_i]$, $i = 1, \dots N$.
The curve c is a geodesic if and only if for all such variations with fixed endpoints we have

$$\frac{d}{ds}E[c_s]\Big|_{s=0} = 0.$$

To investigate the minima of the energy, we have to consider the second variation of the energy.

Theorem 5.2.2 (Second Variation of the energy). *Let M be a semi-Riemannian manifold. Let $c : [a,b] \to M$ be a geodesic. Let c_s be a C^3-variation of c with variational field ξ and fixed endpoints. Then we have*

$$\frac{d^2}{ds^2} E[c_s]\bigg|_{s=0} = \int_a^b \left(\left\langle \frac{\nabla}{dt}\xi, \frac{\nabla}{dt}\xi \right\rangle - \langle R(\xi,\dot{c})\dot{c},\xi\rangle \right) dt.$$

Proof. In the proof of Theorem 2.6.5 we have already shown that

$$\frac{d}{ds} E[c_s] = \int_a^b \left\langle \frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}, \frac{\partial c_s}{\partial t} \right\rangle dt$$

holds for all s, not just for $s = 0$. Therefore

$$\begin{aligned}
\frac{d^2}{ds^2} E[c_s]\bigg|_{s=0} &= \int_a^b \left(\left\langle \frac{\nabla}{\partial s}\frac{\nabla}{\partial t}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \dot{c} \right\rangle + \left\langle \frac{\nabla}{dt}\xi, \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial t}\bigg|_{s=0} \right\rangle \right) dt \\
&= \int_a^b \left\langle \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \dot{c} \right\rangle dt + \int_a^b \langle R(\xi,\dot{c})\xi,\dot{c}\rangle\, dt + \int_a^b \left\langle \frac{\nabla}{dt}\xi, \frac{\nabla}{dt}\xi \right\rangle dt.
\end{aligned}$$

The assertion follows from

$$\begin{aligned}
\int_a^b \left\langle \frac{\nabla}{\partial t}\frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \dot{c} \right\rangle dt &= \int_a^b \left(\frac{\partial}{\partial t} \left\langle \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \dot{c} \right\rangle - \left\langle \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \underbrace{\frac{\nabla}{dt}\dot{c}}_{=0} \right\rangle \right) dt \\
&= \left\langle \frac{\nabla}{\partial s}\frac{\partial c_s}{\partial s}\bigg|_{s=0}, \dot{c} \right\rangle \bigg|_a^b = 0,
\end{aligned}$$

because c_s is a variation with fixed endpoints. □

5.3 Completeness

General Assumption. Throughout this section let M be a connected Riemannian manifold.

Definition 5.3.1. Let $p \in M$. Then M is called **geodesically complete** at p if $\exp_p$ is defined on all of T_pM, i.e., if all geodesics through p are defined on all of $\mathbb{R}$.

Theorem 5.3.2 (Hopf-Rinow). *Let M be a connected Riemannian manifold and $p \in M$. Then the following assertions are equivalent:*

(1) M is geodesically complete at p.

(2) M ist geodesically complete at all $q \in M$.

(3) The closed balls $\bar{B}(p,r)$ are compact for all $r > 0$.

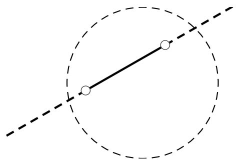
(4) The closed balls $\bar{B}(q,r)$ are compact for all $r > 0$ and all $q \in M$.

(5) (M,d) is complete as a metric space, i.e., all d-Cauchy sequences converge.

All of these conditions imply in addition

(6) Every point q can be joined with p by a minimal geodesic.

Remark 5.3.3. Assertion (6) is weaker than (1) through (5).



Example 5.3.4. Let $M = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ with the Euclidean metric. Then M satisfies (6), but not (1)–(5).

Definition 5.3.5. If the equivalent conditions (1)–(5) in Theorem 5.3.2 hold, then one calls M a **complete Riemannian manifold**.

Corollary 5.3.6. *Every compact connected Riemannian manifold is complete.*

Proof of Corollary 5.3.6. We check condition (3) in the Hopf-Rinow theorem. Indeed, $\bar{B}(p,r) \subset M$ is a closed subset of the compact space M and thus compact itself. □

Proof of Theorem 5.3.2. We will prove this theorem in five steps. The structure of the proof is as follows:

$$\begin{tikzcd}
\textbf{(5)} \arrow[r, Rightarrow, "\textbf{(a)}"] & \textbf{(2)} \arrow[r, Rightarrow, "\text{trivial}"] & \underbrace{\textbf{(1)} \overset{\textbf{(e)}}{\Longrightarrow} \textbf{(6)}} \arrow[d, Rightarrow, "\textbf{(d)}"] \\
& \textbf{(4)} \arrow[ul, Rightarrow, "\textbf{(b)}"] & \textbf{(3)} \arrow[l, Rightarrow, "\textbf{(c)}"]
\end{tikzcd}$$

a) Let $\gamma : (\alpha, \beta) \to M$ be a geodesic with maximal domain of definition. W.l.o.g. we assume that γ is parametrized by arc-length.

We assume $\beta < \infty$ (the case $\alpha > -\infty$ is analogous). Then we have for a sequence $t_i \in (\alpha, \beta)$ with $t_i \overset{i\to\infty}{\longrightarrow} \beta$, that

$$d\big(\gamma(t_i), \gamma(t_j)\big) \le L[\gamma|_{[t_i,t_j]}] = |t_i - t_j|.$$

Hence $(\gamma(t_i))_{i\in\mathbb{N}}$ is a d-Cauchy sequence. Since (M,d) is complete there exists a $q \in M$ with $\gamma(t_i) \overset{i\to\infty}{\longrightarrow} q$.

1. Claim: *The limit point q does not depend on the choice of the sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \overset{i\to\infty}{\longrightarrow} \beta$.*

Proof. If $(t_i')_{i\in\mathbb{N}}$ is another such sequence with $q' = \lim_{i\to\infty} \gamma(t_i')$, then also $(t_i'')_{i\in\mathbb{N}}$ is such a sequence, where

$$t_i'' := \begin{cases} t_j, \; i = 2j \\ t_j', \; i = 2j+1 \end{cases}$$

The sequence $(\gamma(t_i''))_{i\in\mathbb{N}}$ is a d-Cauchy sequence with accumulation points q and q'. We thus have $q = q'$. This proves the first claim. ✓

Thus we obtain a continuous continuation $\bar{\gamma} : (\alpha, \beta] \to M$ of γ by

$$\bar{\gamma}(t) = \begin{cases} \gamma(t), \, t \in (\alpha, \beta) \\ \;\; q \;\;\, , \, t = \beta \end{cases}$$

2. Claim: *The velocity field $\dot{\gamma}$ also has a continuous extension to $(\alpha, \beta]$.*

Proof. Let $x : U \to V$ be a chart of M around q with $x(q) = 0$. Choose $r > 0$ such that $\bar{B}(0,r) \subset V$. Since $\bar{B}(0,r)$ is compact, there exist constants $C_1, C_2, C_4 > 0$ with

- $\left|\Gamma_{ij}^k(y)\right| \le C_1$ for all $y \in \bar{B}(0,r)$.
- $\|a\|_{\max} \le C_2 \left\| \sum_{j=1}^n a^j \frac{\partial}{\partial x^j}\big(x^{-1}(y)\big) \right\|_g$ for all $a = (a^1, \ldots, a^n) \in \mathbb{R}^n$ and $y \in \bar{B}(0,r)$.
- $\left| \frac{\partial \Gamma_{ij}^k}{\partial x^l}(y) \right| \le C_4$ for all $y \in \bar{B}(0,r)$.

Choose $\varepsilon > 0$ small enough so that $\gamma(t) \in x^{-1}(\bar{B}(0,r))$ for $t \in (\beta - \varepsilon, \beta)$. Write $\gamma^k := x^k \circ \gamma$ and $a^k := \dot{\gamma}^k$. By the equations for geodesics we obtain:

$$\dot{a}^k = \ddot{\gamma}^k = -\sum_{i,j=1}^n \Gamma_{ij}^k(\gamma^1, \ldots, \gamma^n) \cdot \dot{\gamma}^i \dot{\gamma}^j = -\sum_{i,j=1}^n \Gamma_{ij}^k(\gamma^1, \ldots, \gamma^n)\, a^i a^j$$

and hence

$$\left|\dot{a}^k\right| \le n^2 \cdot C_1 \cdot \|a\|_{\max}^2.$$

This implies

$$\left\|\dot{a}\right\|_{\max} \le n^2 C_1 \cdot \|a\|_{\max}^2 \;\le\; n^2 C_1 \cdot {C_2}^2 \underbrace{{\|\dot{\gamma}\|_g}^2}_{=1} \;=\; n^2 C_1 {C_2}^2 \;=:\; C_3.$$

We get

$$\left\|a(t_i)-a(t_j)\right\|_{\max} = \left\|\int_{t_i}^{t_j} \dot{a}(t)\,dt\right\|_{\max} \le \left|\int_{t_i}^{t_j} \|\dot{a}(t)\|_{\max}\,dt\right| \le C_3\left|t_i-t_j\right|.$$

Thus the $a(t_i)$ form a Cauchy sequence in $\mathbb{R}^n$ and hence converge to some $A\in\mathbb{R}^n$. As before A is independent of the special choice of the sequence $(t_i)_{i\in\mathbb{N}}$ with $t_i \xrightarrow{i\to\infty} \beta$. Thus we obtain a continuous extension of a by

$$\bar{a}(t) := \begin{cases} a(t), & t\in(\beta-\varepsilon,\beta) \\ A, & t=\beta \end{cases}$$

Hence the velocity field $\dot{\gamma}$ is extended continuously to $t=\beta$. This shows that the extension $\bar{\gamma}$ of γ is C^1. ✓

Differentiation of the geodesics equations yields

$$\ddot{a}^k = -\sum_{i,j=1}^{n}\left(\sum_{l=1}^{n}\frac{\partial\Gamma_{ij}^k}{\partial x^l}a^l a^i a^j + 2\Gamma_{ij}^k \dot{a}^i a^j\right)$$

This implies

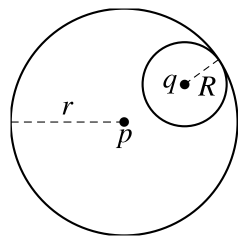
$$\begin{aligned}\|\ddot{a}\|_{\max} &\le n^3 C_4 \|a\|_{\max}^3 + 2n^2 C_1 \|\dot{a}\|_{\max}\|a\|_{\max} \\ &\le n^3 C_4 {C_2}^3 + 2n^2 C_1 C_3 C_2 \\ &=: C_5\end{aligned}$$

As before we see that $(\dot{a}(t_i))_{i\in\mathbb{N}}$ forms a d-Cauchy sequence in $\mathbb{R}^n$. This shows that the extension $\bar{\gamma}$ is even a C^2-curve. By continuity it satisfies the geodesic equation also at $t=\beta$. Now let $\hat{\gamma}:(\beta-\delta,\beta+\delta)\to M$ be the geodesic with $\hat{\gamma}(\beta)=\bar{\gamma}(\beta)$ and $\dot{\hat{\gamma}}(\beta)=\dot{\bar{\gamma}}(\beta)$. Since geodesics are uniquely determined by their initial values, $\hat{\gamma}$ and $\bar{\gamma}$ coincide on their common domain of definition. This yields a continuation of γ as a geodesic on $(\alpha,\beta+\delta)$. This contradicts the maximality of β and thus shows (2).

b) Let all closed balls in M be compact. Let $(p_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in M. Since Cauchy sequences are bounded, there exists a $R>0$ such that $p_i\in\bar{B}(p,R)$ for all $i\in\mathbb{N}$. Since $\bar{B}(p,R)$ is compact, the Cauchy sequence $(p_i)_{i\in\mathbb{N}}$ has an accumulation point $q\in\bar{B}(p,R)$. Since accumulation points of Cauchy sequences are unique, $(p_i)_{i\in\mathbb{N}}$ converges to q.

c) Let all $\bar{B}(p,r)$ be compact for all $r>0$. Let $q\in M$ and let $R>0$. Set $r:=R+d(p,q)$. Then

$$\bar{B}(q,R)\subset\bar{B}(p,r),$$



because for $x\in\bar{B}(q,R)$ we have

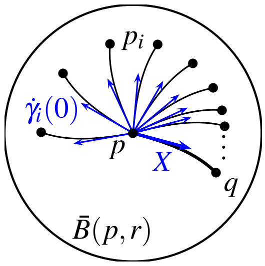
$$d(x,p)\le d(x,q)+d(q,p)\le R+d(q,p)=r.$$

Hence $\bar{B}(q,R)$ is a closed subset of the compact set $\bar{B}(p,r)$ and therefore it is compact itself.

d) Let $(p_i)_{i\in\mathbb{N}}$ be a sequence in $\bar{B}(p,r)$. We have to show that $(p_i)_{i\in\mathbb{N}}$ possesses a convergent subsequence.

By (6) there exist minimal geodesics γ_i with $\gamma_i(0) = p$ and $\gamma_i(t_i) = p_i$ for suitable t_i.

W.l.o.g. let γ_i be parametrized by to arc-length. Then $t_i = L[\gamma_i] = d(p,p_i) \le r$.



The $\dot{\gamma}_i(0)$ are unit vectors in T_pM. Since $S^{n-1}(1) \subset T_pM$ is compact we have, after passing to a suitable subsequence,

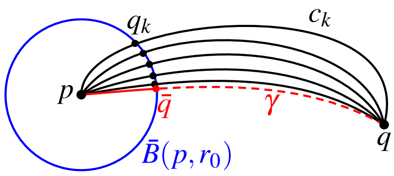
$$\dot{\gamma}_i(0) \overset{i\to\infty}{\longrightarrow} X \in Sn-1(1) \subset T_pM.$$

The t_i lie in the compact interval $[0,r]$. After passing to a subsequence again, $t_i \overset{i\to\infty}{\longrightarrow} T \in [0,r]$ converges too. Set $q := \exp_p(T\cdot X)$. This definition is possible because of (1). We now have

$$\lim_{i\to\infty} p_i = \lim_{i\to\infty} \exp_p\left(t_i \cdot \dot{\gamma}_i(0)\right) = \exp_p\left(\lim_{i\to\infty} t_i\dot{\gamma}_i(0)\right) = \exp_p(TX) = q.$$

This proves (3).

e) Let $q \in M$. We already know, that we can find minimal geodesics from p to q, if $q \in B(p, \mathrm{injrad}(p))$.



Let c_k be continuous piecewise C^1-curves from p to q with $L[c_k] = d(p,q) + \varepsilon_k$ with $\varepsilon_k \searrow 0$.
We assume $q \notin B(p, \mathrm{injrad}(p))$ because otherwise we are finished. Choose $0 < r_0 < \mathrm{injrad}(p)$. Then

$$S(p,r_0) = \exp_p\left(S^{n-1}(r_0)\right)$$

is compact. Let q_k be the first intersection point of c_k with $S(p,r_0)$. After passing to a subsequence, q_k possesses a limit $\bar{q} \in S(p,r_0)$. We have

$$\begin{aligned}
& d(p,q) \le d(p,q_k) + d(q_k,q) \le L[c_k] \le d(p,q) + \varepsilon_k \\
\overset{k\to\infty}{\Rightarrow}\quad & d(p,q) \le d(p,\bar{q}) + d(\bar{q},q) \le d(p,q) \\
\Rightarrow\quad & d(p,q) = d(p,\bar{q}) + d(\bar{q},q)
\end{aligned}$$

Let γ be the unique minimal geodesic that connects p with $\bar{q}$. We parametrize γ by arc-length. With (1) we can extend γ to $[0, d(p,q)]$.

It remains to show that $\gamma : [0, d(p,q)] \to M$ is a minimal geodesic from p to q. Set

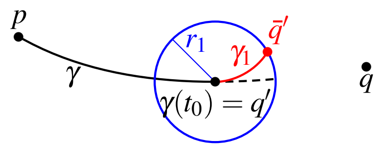
$$I := \left\{ t \in [0, d(p,q)] \,|\, d(p,\gamma(t)) = t \text{ and } d(p,\gamma(t)) + d(\gamma(t),q) = d(p,q) \right\}.$$

We have seen that $[0,r_0] \subset I$. Set $t_0 := \sup(I)$. We have to show that $t_0 = d(p,q)$ because then

$$d\big(\gamma(t_0),q\big) = d(p,q) - d\big(\gamma(t_0),p\big) = d(p,q) - t_0 = 0,$$

which implies $\gamma(t_0) = q$ and that γ is a minimal geodesic from p to q.

We therefore assume that $t_0 < d(p,q)$ from which we have to derive a contradiction. Set $q' := \gamma(t_0)$. Choose $0 < r_1 < d(p,q) - t_0$ such that $B(q', r_1)$ is a normal coordinate neighborhood. As above, there exists a $\bar{q}' \in \partial B(q', r_1)$ with



$$d(q', \bar{q}') + d(\bar{q}', q) = d(q', q).$$

Now let γ_1 be a minimal geodesic, parametrized by arc-length, with $\gamma_1(t_0) = q'$ and $\gamma(t_0 + r_1) = \bar{q}'$.

$$\begin{aligned} \Rightarrow \qquad d(p,\bar{q}') &\leq d(p,q') + d(q',\bar{q}') \\ &= d(p,q') + d(q',q) - d(\bar{q}',q) \\ &= d(p,q) - d(q',q) + d(q',q) - d(\bar{q}',q) \\ &= d(p,q) - d(\bar{q}',q) \\ &\leq d(p,\bar{q}') \end{aligned}$$

$$\Rightarrow \qquad d(p,\bar{q}') = d(p,q') + d(q',\bar{q}')$$

$\Rightarrow$ The curve $\gamma|_{[0,t_0]} \cup \gamma_1|_{[t_0, t_0+r_1]}$ is a shortest one.

$\Rightarrow$ $t_0 + r_1 \in I$. This contradicts the maximality of t_0. We have proved (6). $\square$

5.4 The Bonnet-Myers theorem

Definition 5.4.1. Let M be a connected Riemannian manifold. Then we call

$$\mathrm{diam}(M) := \sup\{d(p,q) \mid p,q \in M\} \in (0,\infty]$$

the **diameter** of M.

Example 5.4.2. For $M = S^n$ equipped with the standard metric $g = g_{\mathrm{std}}$ we have $\mathrm{diam}(S^n) = \pi$. For $M = \mathbb{R}^n$ with the Euclidean metric $g = g_{\mathrm{eucl}}$ and for hyperbolic space $M = H^n$ with $g = g_{\mathrm{hyp}}$ we have $\mathrm{diam}(\mathbb{R}^n) = \mathrm{diam}(H^n) = \infty$.

Remark 5.4.3. If M is complete then

$$\mathrm{diam}(M) < \infty \quad \Leftrightarrow \quad M \text{ is compact.}$$

Namely:

"$\Leftarrow$": M is compact $\Rightarrow$ $M \times M$ is compact $\Rightarrow$ $d: M \times M \to \mathbb{R}$ is bounded and attains its maximum $C \Rightarrow \mathrm{diam}(M) = C < \infty$.

"$\Rightarrow$": If $\operatorname{diam}(M) =: R < \infty$, then for arbitrary $p \in M$ we have $M = \bar{B}(p,R)$. Hence M is compact by the Hopf-Rinow Theorem 5.3.2.

Theorem 5.4.4 (Bonnet-Myers). *Let M be a complete connected Riemannian manifold of dimension n. Assume there exists a $\kappa > 0$ such that*

$$\mathrm{ric} \geq \kappa(n-1)g.$$

This means that $\mathrm{ric}(\xi,\xi) \geq \kappa(n-1)g(\xi,\xi)$ for all $\xi \in TM$. Then M is compact and we have:

$$\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{\kappa}}.$$

Example 5.4.5. (1) Let $M = S^n$ with $g = \alpha^2 \cdot g_{\mathrm{std}}$ where α is a positive constant. Then we have

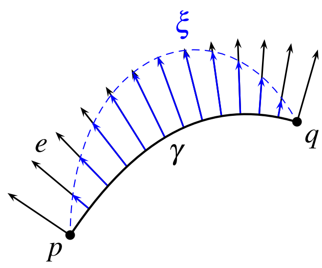
$$\begin{aligned} &\operatorname{diam}(M) = \alpha\pi, \quad K \equiv \frac{1}{\alpha^2}, \quad \mathrm{ric} \equiv \frac{n-1}{\alpha^2}g \\ \Rightarrow\quad &\operatorname{diam}(M) = \frac{\pi}{\sqrt{\kappa}} \quad \text{and } \mathrm{ric} = \kappa(n-1)g \text{ with } \kappa = \frac{1}{\alpha^2}. \end{aligned}$$

This shows that the estimate in the Bonnet-Myers theorem is optimal and cannot be improved.

(2) Now let $M = \mathbb{R}\mathrm{P}^n$ with $g = g_{\mathrm{std}}$. Since $\mathbb{R}\mathrm{P}^n$ is locally isometric to S^n, we have as for the sphere $\mathrm{ric} = (n-1)g$. But $\operatorname{diam}(\mathbb{R}\mathrm{P}^n) = \frac{\pi}{2}$. Here we find $\operatorname{diam}(M) < \frac{\pi}{\sqrt{\kappa}}$ where $\kappa = 1$.

Proof of Theorem 5.4.4. Let $p,q \in M$ with $p \neq q$. Set $\delta := d(p,q)$. Since M is complete, there exists a minimal geodesic from p to q by the Hopf-Rinow theorem. W.l.o.g. let $\gamma : [0,\delta] \to M$ be parametrized by arc-length with $\gamma(0) = p$ and $\gamma(\delta) = q$.
Let $e \in T_pM$ with $e \perp \dot{\gamma}(0)$ and $\|e\| = 1$. Let $e(t)$ be the vector field along γ obtained from e by parallel transport. Set

$$\xi(t) := \sin\left(\frac{\pi}{\delta}t\right) \cdot e(t).$$



Let $\gamma_s(t)$ be a variation of γ with fixed endpoints and variational field ξ, for example

$$\gamma_s(t) = \exp_{c(t)}(s \cdot \xi(t)).$$

Since γ is a minimal geodesic, we have

$$0 = \frac{d}{ds}E[\gamma_s]|_{s=0}$$

and

$$\begin{aligned}
0 &\le \frac{d^2}{ds^2} E[\gamma_s]|_{s=0} \\
&= \int_0^\delta \left(\left\| \frac{\nabla}{dt} \xi \right\|^2 - \langle R(\xi, \dot\gamma)\dot\gamma, \xi \rangle \right) dt \\
&= \int_0^\delta \left(\left\| \frac{\pi}{\delta} \cos\left(\frac{\pi}{\delta} t\right) e(t) \right\|^2 - \sin\left(\frac{\pi}{\delta} t\right)^2 \langle R(e, \dot\gamma)\dot\gamma, e \rangle \right) dt \\
&= \int_0^\delta \left(\frac{\pi^2}{\delta^2} \cos\left(\frac{\pi}{\delta} t\right)^2 \cdot 1 - \sin\left(\frac{\pi}{\delta} t\right)^2 K(e, \dot\gamma) \right) dt.
\end{aligned}$$

If $e_1, \ldots, e_{n-1}$ is a orthonormal basis of $\dot\gamma(0)^\perp$, we obtain with $e = e_i$ and summation over i:

$$\begin{aligned}
0 &\le \int_0^\delta \left((n-1) \frac{\pi^2}{\delta^2} \cos\left(\frac{\pi}{\delta} t\right)^2 - \sin\left(\frac{\pi}{\delta} t\right)^2 \underbrace{\mathrm{ric}(\dot\gamma, \dot\gamma)}_{\ge (n-1)\kappa \cdot 1} \right) dt \\
&\le (n-1) \int_0^\delta \left(\frac{\pi^2}{\delta^2} \cos\left(\frac{\pi}{\delta} t\right)^2 - \sin\left(\frac{\pi}{\delta} t\right)^2 \cdot \kappa \right) dt \\
&= (n-1) \cdot \frac{1}{2} \frac{\pi^2 - \kappa\delta^2}{\delta}.
\end{aligned}$$

Therefore $0 \le \pi^2 - \kappa\delta^2$ and hence $\delta \le \frac{\pi}{\sqrt{\kappa}}$. Since this holds for all choices of p and q we conclude

$$\mathrm{diam}(M) \le \frac{\pi}{\sqrt{\kappa}}.$$

By Remark 5.4.3, M is compact. □

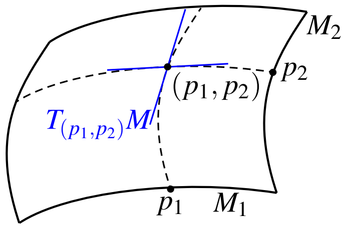
The theorem tells us that the larger the Ricci curvature of a Riemannian manifold, the smaller the manifold.

Note that the following general implications hold:

$$K \ge \kappa \quad \Rightarrow \quad \mathrm{ric} \ge (n-1)\,\kappa \cdot g \quad \Rightarrow \quad \mathrm{scal} \ge n(n-1)\,\kappa. \tag{1}$$

Thus the Bonnet-Myers theorem also holds if the sectional curvature is bounded from below by a postive constant, $K \ge \kappa > 0$. Does the Bonnet-Myers theorem also hold under the weaker condition $\mathrm{scal} \ge n(n-1)\kappa$?

The answer is "no" as we see by the following counterexample. If M_1 and M_2 are Riemannian manifolds and if $M := M_1 \times M_2$ carries the product metric, then



$$g_M(\underbrace{\xi_1+\xi_2}_{\substack{\in T_{p_1}M_1\oplus T_{p_2}M_2\\ =T_{(p_1,p_2)}M}}, \eta_1+\eta_2) = g_{M_1}(\xi_1,\eta_1) + g_{M_2}(\xi_2,\eta_2).$$

$$\begin{aligned}
\Rightarrow \quad R^M(\xi_1+\xi_2,\eta_1+\eta_2) &= \left(\begin{array}{c|c} R^{M_1}(\xi_1,\eta_1) & 0 \\ \hline 0 & R^{M_2}(\xi_2,\eta_2) \end{array}\right) \\
\Rightarrow \quad \mathrm{ric}^M &= \left(\begin{array}{c|c} \mathrm{ric}^{M_1} & 0 \\ \hline 0 & \mathrm{ric}^{M_2} \end{array}\right) \\
\Rightarrow \quad \mathrm{scal}^M &= \mathrm{scal}^{M_1} + \mathrm{scal}^{M_2}.
\end{aligned}$$

For $n \geq 3$ we obtain with $M = S^{n-1} \times \mathbb{R}$ that

$$\mathrm{scal}^M = (n-1)(n-2) + 0 = (n-1)(n-2),$$

but $\mathrm{diam}(M) = \infty$. Thus the Bonnet-Myers theorem does not hold under the weaker condition $\mathrm{scal} \geq n(n-1)\kappa$ if $n \geq 3$.
For $n = 2$ on the other hand, the three conditions in (1) are equivalent.

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