On some Rigidity properties of holomorphic maps

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One of the fundamental questions in the theory of several complex variables, going back to the work of H. Poincaré, is how to classify domains up to biholomorphic equivalence. In the complex plane, the classical Riemann mapping theorem asserts that domains posses only topological invariants. As was realized already by Poincaré, there is no analogous statement in higher dimension, and smooth boundaries of domains have infinitely many local biholomorphic invariants.

The starting point of this talk is the following Cartan's Uniqueness Theorem

Theorem (Cartan 1931)

Let $\Omega \subset \mathbb{C}^n$ *be a bounded connected open set containing* 0 *and* $F : \Omega \longrightarrow \Omega$ *be a holomorphic map satisfying*

1. $F(0) = 0$
2. $F'(0) = I$, *where* I *is the identity matrix.*

Then $F(z) = z$.

Sketch of the proof. Suppose by contradiction that F is not the identity map.

1. Since Ω is bounded, there exist $r_1 > 0$ and $r_2 > 0$ such that
$$r_1 B \subset \Omega \subset r_2 B,$$
where B is the unit ball in $\mathbb{C}^n$, that is $B = \{z \in \mathbb{C}^n \mid |z| < 1\}$.
2. Since F is holomorphic, we may write its Taylor expansion in $r_1 B$, using the hypotheses, to get
$$F(z) = z + F_m(z) + \dots,$$
where F_m is the first nonzero homogeneous (polynomial) term of degree m, while the dots stand for possible homogeneous components of F of higher degree.
3. Since $F : \Omega \longrightarrow \Omega$, we may iterate F k times: we denote its kth iteration by F^k.

We then obtain the Taylor expansion of F^k

$$F^k(z) = z + kF_m(z) + \dots.$$

By homogeneity, we obtain in $r_1 B$

$$kF_m(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F^k(e^{i\theta}z)e^{-im\theta}d\theta$$

Therefore,

$$k|F_m(z)| < r_2,$$

Since this is valid for any k, we obtain that $F_m = 0$. Contradiction.

Cartan's Uniqueness Theorem leads to the following theorem

Theorem (Cartan 1931)

Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^n$ be connected open sets containing 0 *satisfying*

1. $e^{i\theta} z \in \Omega_j$ *if* $z \in \Omega_j,\ j = 1, 2,$
2. Ω_1 *is bounded,*

and let $F : \Omega_1 \longrightarrow \Omega_2$ be a biholomorphic map satisfying $F(0) = 0$. *Then F is linear.*

An interesting feature is that this Theorem fails if we omit the hypothesis that Ω_1 is bounded.

Indeed, consider in $\mathbb{C}^2$ the domain Ω given by

$$\Omega = \{(z, w) \in \mathbb{C}^2 \mid |zw| < 1\}.$$

Consider the holomorphic map $F : \Omega \longrightarrow \Omega$ given by

$$F(z, w) = (z(zw + 1), \frac{w}{zw + 1}).$$

One can check that

1. F is a biholomorphic map with inverse

$$F^{-1}(z, w) = (\frac{z}{zw + 1}, w(zw + 1))$$

2. $F(0) = 0$ and $F'(0) = I$.

But F is not linear!

As an application of Cartan's Theorem, we can prove that the Riemann mapping Theorem fails when $n > 1$.

Theorem

If $n > 1$, there is no biholomorphic map of B onto the polydisc $\{z \in \mathbb{C} \mid |z| < 1\}^n$.

Sketch of the proof.

- Suppose that there is a biholomorphic map $F : B \longrightarrow \{z \in \mathbb{C} \mid |z| < 1\}^n$. Let $a = F^{-1}(0)$.
- Consider the wellknown biholomorphism of the ball B

$$\varphi_a(z) = \frac{a - P_a z - (1 - |a|^2)^{\frac{1}{2}} Q_a z}{1 - < z, a >},$$

where $< z, a > = \sum z_j \overline{a_j}$, $P_a z = \dfrac{< z, a >}{< a, a >} a$, $Q_a z = z - P_a z$.

- Then $F \circ \varphi_a : B \longrightarrow \{z \in \mathbb{C} \mid |z| < 1\}^n$ is a biholomorphism with $F \circ \varphi_a(0) = 0$, and hence, by Cartan's Theorem it is linear. But invertible linear maps send B onto ellipsoids.

- Instead of considering holomorphic maps between connected open sets in $\mathbb{C}^n$, we may consider *local* holomorphic maps in $\mathbb{C}^n$ between *boundaries* of connected open sets in $\mathbb{C}^n$, or more generally between *hypersurfaces* in $\mathbb{C}^n$, and address the following question:
- When are these maps uniquely determined by their value and a fixed number of derivatives at a given point of the hypersurface?

- This question is not valid for $n = 1$, as one can see immediately:

Consider the polynomial

$$p(x) = \sum_{j=0}^{d} a_j x^j, \ \ a_j \in \mathbb{R}, a_1 \neq 0.$$

Then $p(z) = \sum_{j=0}^{d} a_j z^j$ is a local biholomorphism sending a piece of $\mathbb{R}$ onto $\mathbb{R}$ for any choice of a_j. There is no fixed number of derivatives to get unicity of p.

- A first example in $\mathbb{C}^2 = \mathbb{C}^{1+1}$.

It is well-known that local biholomorphisms of the quadric

$$M = \{(z,w) \in \mathbb{C}^2 \mid \operatorname{Im} w =< z, z >= |z|^2\}$$

at $p = 0$ are given by

$$(z,w) \longrightarrow \frac{(c(z+aw), |c|^2 w)}{1 - 2i\bar{a}z - (r + i|a|^2)w}, \; c \in \mathbb{C}^*, \; a \in \mathbb{C}, r \in \mathbb{R}.$$

Here c and a are determined by their first derivatives, while r is determined by the second derivative.

Let $M \subset \mathbb{C}^{n+1}$ be a smooth hypersurface given near p, given by

$$\operatorname{Im} w = P(z, \bar{z}) + o(|z|^d, \operatorname{Re} w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where P is a real homogeneous polynomial without pluriharmonic terms of degree $d \geq 2$. We are interested to the case $d = 2$. In that case, M is given by

$$\operatorname{Im} w =< Az, z > + o(|z|^2, \operatorname{Re} w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where A is a hermitian matrix.

- We say that M is Levi nondegenerate if $\det A \neq 0$.

In $\mathbb{C}^2$, Levi nondegenerate hypersurfaces are then smooth (or real analytic) perturbations of our previous quadric example

$$M = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w =< z, z >\}.$$

The Chern-Moser Theorem asserts that the same phenomena as for the quadric $M = \{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = < z, z >\}$ holds for Levi nondegenerate hypersurfaces in $\mathbb{C}^2$, or more generally for M given by

$$\operatorname{Im} w = < Az, z > + o(|z|^2, \operatorname{Re} w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

with $\det A \neq 0$.

Theorem (Chern-Moser 1974)

If a real hypersurface $M \subset \mathbb{C}^{n+1}$ is smooth Levi nondegenerate at a point $p \in M$, then its (local) automorphisms, that is, its (local) biholomorphisms sending M onto M, are uniquely determined by their jets of order two at p, that is, by their value, by their first and second derivatives at p.

- Question: Can we remove the condition on the Levi form?

The answer is NO:

The so-called Levi flat hypersurface given by $\operatorname{Im} w = 0$ in some local coordinates $z = (z, w) \in \mathbb{C}^2$ is invariant under all biholomorphic maps of $\mathbb{C}^2$ of the form

$$H(z, w) = (z + F(w), w),$$

where F is any entire holomorphic function on $\mathbb{C}$. These maps are not determined by their jets at 0 of any finite order.

A remarkable fact is that the information about the number of derivatives needed is encoded in the model hypersurface $M_{<Az,z>}$ given by

$$\operatorname{Im} w = < Az, z >,$$

and more particularly, in its Lie algebra $\operatorname{hol}(M_{<Az,z>}, 0)$ the set of all germs of real-analytic infinitesimal CR automorphisms of $M_{<Az,z>}$ at 0.

- $\operatorname{hol}(M_{<Az,z>}, 0)$ is the set of germs of holomorphic vector fields in $\mathbb{C}^{n+1}$ whose real parts are tangent to $M_{<Az,z>}$.
- Examples of such holomorphic vector fields:

$$\frac{\partial}{\partial w}, \quad \frac{1}{2}(\sum_j z_j \frac{\partial}{\partial z_j}) + w \frac{\partial}{\partial w}.$$

Indeed, for instance,

$$(\operatorname{Re} \frac{\partial}{\partial w})(\operatorname{Im} w - < Az, z >) = 0.$$

More precisely, by the classical Chern-Moser theory developed in "Real hypersurfaces in complex manifolds, Acta Math, 1974", the kernel of the so-called Chern-Moser operator is in one to one correspondence with the Lie algebra $\mathrm{hol}(M_{<Az,z>}, 0)$ and gives a precise description of the derivatives needed to characterize the local biholomorphisms of M given by

$$\mathrm{Im}\, w = < Az, z > + o(|z|^2, \mathrm{Re}\, w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

with $\det A \neq 0$.

We return to the general case where $M \subset \mathbb{C}^{n+1}$ is a smooth hypersurface given near p, given by

$$\operatorname{Im} w = P(z, \bar{z}) + o(|z|^d, \operatorname{Re} w), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where P is a real homogeneous polynomial without pluriharmonic terms of degree $d \geq 2$.

An analogous statement to the Chern-Moser Theorem has been obtained in "Chern-Moser operators and polynomial models in CR geometry, Advances in Mathematics, 2014", by Kolar, Zaitsev and Meylan.

- One works with the polynomial P of degree $d > 2$ instead of an hermitian form, the nondegeneracy condition on the form beeing replaced by a nondegeneracy condition on the polynomial P.
- Roughly speaking, one obtains that the number of derivatives needed to determine uniquely the biholomorphism is $d - 1$.

Theorem (Kolar, M.,Zaitsev 2014)

The automorphisms of M at p are uniquely determined by their jets of weighted order 2.

- Important work on these problems of "finite jet determination" has been done. We mention for instance the work of Baouendi, Beloshapka, Bertrand, Blanc-Centi, Ebenfelt, Huang, Kim, Kolar, Kossovsky, Lamel, Mir, Rothschild, Zaitsev.

Up to now, we have been talking on hypersurfaces, that is submanifolds of codimension 1. We may ask the same question for biholomorphisms F of submanifolds $M \subset \mathbb{C}^N$ of codimension $d > 1$ of the form

$$\begin{cases} \operatorname{Im} w_1 = < A_1 z, z > + \dots \\ \quad \vdots \\ \operatorname{Im} w_d = < A_d z, z > + \dots \end{cases}$$

- Under analogous nondegeneracy conditions on the hermitian matrices A_j, an analogous Theorem to Chern-Moser's Theorem has been proved in 1990, asserting that biholomorphisms sending M onto M are uniquely determined by their value, their first and their second derivatives at 0.

But very recently, Blanc-Centi and M. found an error in the proof and the second author found a counterexample that shows that more derivatives are needed to determine such biholorphisms.

- Here is the counterexample: Let $M \subseteq \mathbb{C}^9$ be the real submanifold of (real) codimension 5 through 0 given in the coordinates $(z, w) = (z_1, \ldots, z_4, w_1, \ldots, w_5) \in \mathbb{C}^9$, by

$$\begin{cases} \operatorname{Im} w_1 = P_1(z, \bar{z}) = z_1\overline{z_2} + z_2\overline{z_1} \\ \operatorname{Im} w_2 = P_2(z, \bar{z}) = -iz_1\overline{z_2} + iz_2\overline{z_1} \\ \operatorname{Im} w_3 = P_3(z, \bar{z}) = z_3\overline{z_2} + z_4\overline{z_1} + z_2\overline{z_3} + z_1\overline{z_4} \\ \operatorname{Im} w_4 = P_4(z, \bar{z}) = z_1\overline{z_1} \\ \operatorname{Im} w_5 = P_5(z, \bar{z}) = z_2\overline{z_2} \end{cases}$$

The matrices corresponding to the $P_i's$ are

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The following holomorphic vectors fields are in $hol(M, 0)$, the set of germs of real-analytic infinitesimal CR automorphisms at 0.

1. $X := i(z_1 \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_4})$
2. $Y := i(-iz_1 \frac{\partial}{\partial z_3} + iz_2 \frac{\partial}{\partial z_4})$
3. $Z := i(z_1 \frac{\partial}{\partial z_4})$
4. $U := i(z_2 \frac{\partial}{\partial z_3})$

Theorem (M.2020)

Let $Y_0 = -Y, \quad Y_1 = 2Y, \quad Z_1 = -2Z, \quad U_1 = -2U.$
The holomorphic vector field T *defined by*

$$T = \frac{1}{2}{w_1}^2 Y_0 + \frac{1}{2}{w_2}^2 Y + w_1 w_2 X + w_2 w_5 Z_1 + w_2 w_4 U_1 + w_4 w_5 Y_1$$

is in $hol(M, 0)$.
Hence $2-$*jet determination does not hold for germs of biholomorphisms sending* M *to* M.

Very recently, Tumanov proved $2-$jet determination when the submanifold M is strongly pseudoconvex, that is satisfies some "positivity" condition.

Part of our work with Florian Bertrand is to understand more generally the conditions to put on the submanifolds in order that such biholomorphisms are determined by their value, their first and their second derivatives at 0.

- THANK YOU!